ECSE 6520: Estimation and Detection Theory

Wiener Filtering

Class Notes - 12

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1 Description

Wiener filtering is the linear MMSE recovery of signals in additive noise under the assumption of *wide sense stationarity*. Consider the following problem:

$$x[n] = s[n] + \omega[n]$$

where

$x[n-k], \quad k = 0, ..., p$ are observations.

$x[n]$ is wide sense stationary (WSS) with autocorrelation

$$r_{xx}[k] = E[x[n]x[n+k]].$$

$x[n]$ and $s[n]$ are jointly WSS with cross correlation

$$r_{xs}[k] = E[x[n]s[n+k]].$$

Assumptions -

- $s[n]$ and $\omega[n]$ are zero mean.
- Signal prediction if $\tau > 0$.
- Smoothing if $\tau < 0$.

2 FIR Wiener Filtering

Problem Statement -

Estimate $s[n+\tau]$ where $\tau = ... -1, 0, 1, ...,$ under the above assumptions using

$$\hat{\theta} = \hat{s}[n+\tau] = \sum_{k=0}^{p-1} h[k]x[n-k].$$

Depending on the value of τ the problem is referred to as:

- Filtering if $\tau = 0$.

- Signal prediction if $\tau > 0$.

- Smoothing if $\tau < 0$.

Note that smoothing requires the filter h to be non-causal, i.e., estimation of present signal requires knowledge of future observations. Filtering and prediction, on the other hand, are causal.

2.1 Wiener-Hopf Equation

- For simplicity, we focus to the case where $\tau = 0$. However, other cases are also addressed in a similar fashion.

- From the LMMSE theory we know that the Wiener-Hopf equation is given by

$$R_{XX}H = R_{\theta X}$$

  where $\theta = s[n+\tau]$.

- Under the WSS assumption, the ij element of R_{XX} is given by $r_{xx}[|i-j|]$ and the ijth element of $R_{\theta X}$ is given by $r_{sx}[|i-j|]$.

- Since R_{XX} and $R_{\theta X}$ are both Toeplitz, so is H. Therefore, H can be represented as a linear time-invariant filter. Thus, an alternative representation of the Wiener-Hopf equation is given by

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[p-1] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[p-1] \end{bmatrix} = \begin{bmatrix} r_{xs}[0] \\ r_{xs}[1] \\ \vdots \\ r_{xs}[p-1] \end{bmatrix}$$

- Since R_{XX} is symmetric, Toeplitz and positive-definite, thus a unique solution for the Wiener-Hopf equation exists. Furthermore, R_{XX} can be efficiently inverted by the Generalized *Levinson-Durbin algorithm.*

3 IIR Wiener Filtering

- So far, we consider the estimation of $s[n+\tau]$ when only finitely may observations $x[n],\ x[n-1],\ ...\ x[n-p+1]$ are available which results in a Wiener filter with finitely many non-zero taps.

- We will now consider infinite impulse response Wiener filtering. In particular, we consider

  1. The causal IIR Wiener filter

  $$\hat{s}[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

  2. The non-causal IIR Wiener filter

  $$\hat{s}[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

3.1 Non-causal IIR Wiener Filter

We will first consider the non-causal IIR Wiener filter.

- Given $x[n], \quad n = ..., -1, 0, 1, ...$, we want to determine the linear time-invariant filter $h[k], \quad k = ..., -1, 0, 1, ...$ such that

  $$E[(s[n] - \sum_{k=-\infty}^{\infty} h[k]x[n-k])^2]$$

  is minimized.

- Using the orthogonality relationship, we can show that the filter h satisfies the following equation:

$$\sum_{k=-\infty}^{\infty} h[k]r_{xx}[n-k] = r_{xs}[n], \quad for\ all\ n = ..., -1, 0, 1, ...$$

- This is the Wiener-Hopf equation for the IIR Wiener smoother.

- To solve the Wiener-Hopf equation, we will use the Discrete-Time Fourier Transform (DTFT), since DTFT diagonalizes the linear time-invariant filters.

- Let $S_{xx}(f)$ be the spectral density function of x and $S_{xs}(f)$ be the cross-spectral density function of x and s. Then,

$$r_{xs} = h * r_{xx} \quad \Leftrightarrow \quad S_{xs}(f) = S_{xx}(f)H(f)$$

where $H(f)$ is the discrete-time Fourier transform of the h. Thus,

$$H(f) = \frac{S_{xs}(f)}{S_{xx}(f)}.$$

- If the noise process $\omega[n]$ and the signal $s[n]$ are uncorrelated, the IIR Wiener filter becomes

$$H(f) = \frac{S_{ss}(f)}{S_{ss}(f) + S_{\omega\omega}(f)}$$

where $S_{\omega\omega}(f)$ is the power spectral density function of the WSS noise process $\omega[n]$.

3.2 Causal IIR Wiener Filter

- We consider the case where we estimate the present signal $s[n]$ based on data from present and infinite past,i.e.,

$$\hat{s}[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

- We apply the ortogonality principle to arrive at the Wiener-Hopf equation:

$$\sum_{k=-0}^{\infty} h[k]r_{xx}[n-k] = r_{xs}[n], \quad n = 0, 1, ...$$

- However, this time we can not express the Wiener-Hopf equation as a convolution and make use of the DTFT to solve it since the Wiener-Hopf equation holds only for non-negative integers.

- It turns out that

$$H(z) = \frac{1}{G(z)} \left[\frac{P_{xs}(z)}{G(z^{-1})} \right]_+$$

where

$$\begin{aligned} P_{xx} &= \sum_{k=-\infty}^{\infty} r_{xx}[k] z^{-k} \\ &= G(z)G(z^{-1}) \end{aligned}$$

where $G(z)$ is the minimum phase causal part and

$$[Y(z)]_+ = \sum_{k=0}^{\infty} y[k] z^{-k}$$

is the z-transform of the causal part.