ECSE 6520: Estimation and Detection Theory

Detection Theory

Class Notes - 14

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1 Detection Theory

Detection theory is the application of hypothesis testing in the context of signal processing problems. Detection can be viewed as equivalent to estimating a parameter from a finite parameter space, as such, detection can be viewed as a special subclass of estimation problems. In this part of the course, we will cover

- Bayesian approach to detection
- Frequentist approach to detection, i.e., Neyman-Pearson approach
- Signal detection in Gaussian noise
- Uniformly most powerful tests
- Bayes factors and Generalized Likelihood Ratio Tests (GLRT)
- Constant False Alarm Rate (CFAR) detectors

2 Hypothesis Testing

2.1 Notation and Terminology -

In hypothesis testing, we have the following general setup:

- X is a measured random variable, random vector or process.
- $x \in \mathcal{X}$ is a realization of X.
- $\theta \in \Theta$ are unknown parameters.
- Θ denotes the parameter space.
- $f_\theta(x)$ or $f(x;\theta)$ is pdf of X (a known function).

Definition 1 ***Binary Hypothesis Testing -*** *We assume two distinct hypothesis on* θ

$$\theta \in \Theta_0 \quad or \quad \theta \in \Theta_1$$

where Θ_0 *and* Θ_1 *partition* Θ *into two disjoint regions:*

$$\Theta_0 \cap \Theta_1 = \varnothing \quad \Theta_0 \cup \Theta_1 = \Theta.$$

Given the data $X = [x_1, x_2, ..., x_N]^T$*, the process of deciding whether* $\theta \in \Theta_0$ *or* $\theta \in \Theta_1$ *is called the binary hypothesis testing.*

- **Null hypothesis -** An interesting event did not happen usually denoted by $\mathcal{H}_0$.
- **Alternative hypothesis -** Alternative of the null hypotheses usually denoted by $\mathcal{H}_1$.
- We define the two hypothesis as follows:

$$\mathcal{H}_0 : \quad \theta \in \Theta_0$$
$$\mathcal{H}_1 : \quad \theta \in \Theta_1$$

or equivalently,

$$\mathcal{H}_0 : \quad X \sim f_\theta(x) \quad \theta \in \Theta_0$$
$$\mathcal{H}_1 : \quad X \sim f_\theta(x) \quad \theta \in \Theta_1$$

- **M-ary hypothesis testing -** Θ is decomposed into $M > 2$ disjoint sets:

$$\Theta = \Theta_1 \cup \Theta_1 \cup ... \cup \Theta_M.$$

The process of deciding which set θ belongs to is called the *M-ary hypothesis testing.*

Definition 2 ***Simple/Composite Hypothesis -*** *If a hypothesis $\mathcal{H}$ specifies a unique distribution for x, then the hypothesis $\mathcal{H}$ is said to be* simple. *If $\mathcal{H}$ specifies a class of possible distributions for x, then the hypothesis $\mathcal{H}$ is said to be* composite.

- In other words, if θ can only take two values and Θ_0 and Θ_1 are singleton sets, the hypothesis are said to be simple.
- Simple hypothesis are much easier to deal with than the composite hypothesis.

Definition 3 ***One-sided and Two-sided Tests -***

- ***One-sided test -***

$$\begin{array}{ll} \mathcal{H}_0: & \theta \leq \theta_0 \\ \mathcal{H}_1: & \theta > \theta_0 \end{array}, \quad \theta \in \mathbb{R}.$$

- ***Two-sided test -***

$$\begin{array}{ll} \mathcal{H}_0: & \theta = \theta_0 \\ \mathcal{H}_1: & \theta \neq \theta_0 \end{array}, \quad \theta \in \mathbb{R}^n.$$

2.2 Decision Rules

Definition 4 ***Test function (Decision rule) -*** *In hypothesis testing, the goal is to construct a decision rule, which is a mapping T from the measurement space $\mathcal{X}$ to the set of hypothesis $\{\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_M\}$*

$$T : \mathcal{X} \rightarrow \{\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_M\}$$

assigning an observation $x \in \mathcal{X}$ to a hypothesis. This decision rule is called the test statistic or test function.

- A decision rule or the test function partitions the input space into decision regions.

$$R_k = \{x \in \mathcal{X} \mid T(x) = \mathcal{H}_k\}.$$

- **Thresholding rule for binary hypothesis testing-** $\Gamma(x)$ is a scalar test statistic.

$$\Gamma(x) \quad > \quad \gamma \quad \Rightarrow \quad \textit{Declare } \mathcal{H}_1$$

$$\Gamma(x) \quad < \quad \gamma \quad \Rightarrow \quad \textit{Declare } \mathcal{H}_0.$$

- **Detection probability -** $P_D = P(\mathcal{H}_1|\mathcal{H}_1)$.

- **Miss probability -** $P_M = 1 - P_D = P(\mathcal{H}_0|\mathcal{H}_1)$.

- **False alarm probability -** $P_F = P(\mathcal{H}_1|\mathcal{H}_0)$.

- **Rejection probability -** $P_R = P(\mathcal{H}_0|\mathcal{H}_0)$.