ECSE 6520: Estimation and Detection Theory

Minimum Variance Unbiased Estimation

Class Notes - 7

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Contents

1 Bias-Variance Trade off

- The MSE of and estimator $\hat{\theta}$ can be broken down into two components, the bias of $\hat{\theta}$ and the variance of $\hat{\theta}$. In particular

$$MSE(\hat{\theta}) = \|Bias(\hat{\theta})\|^2 + tr(Cov(\hat{\theta})).$$

- This means when designing an estimator, you can trade off bias and variance. Decreasing the bias of an estimator will increase the variance, while increasing the bias will decrease variance.

- How practical is the MSE as a design criterion? MSE estimate may depend on unknown parameters. *An alternative is the minimum variance unbiased estimator.*

2 Minimum Variance Unbiased Estimation

- In general minimum MSE has non-zero bias and variance. However, in many situations, only the bias depends on the unknown parameters.

- This suggests the following alternative:
  - Constrain the estimator to be unbiased and minimize the variance.
  - Equivalently, minimize the MSE among all unbiased estimators.

Definition 1 ***Minimum Variance Unbiased Estimator (MVUE) -*** *$\hat{\theta}$ is called a (uniformly) Minimum Variance Unbiased Estimator (MVUE) estimate of θ if*

- *$E[\hat{\theta}] = \theta$ for all $\theta \in \Theta$*

- *If $E[\acute{\hat{\theta}}] = \theta$ for all $\theta \in \Theta$, then $Var(\hat{\theta}) \leq Var(\acute{\hat{\theta}})$ for all $\theta \in \Theta$.*

- Note that the MVUE criterion requires all estimators to be optimal for all values of $\theta \in \Theta$.

Existence of MVUE

- The MVUE does not always exist. In fact, there may not exist an unbiased estimator.

3 Methods of Finding MVUEs

There is no systematic way of finding MVUE. There are three potential ways of finding MVUE.

- Restrict the possible class of estimators to be linear.

- Calculate the Cramer-Rao lower bound (we will cover this later) and see if some estimator achieves the bound. Note that the Cramer-Rao lower bound gives a necessary and sufficient condition for the existence of an efficient estimator. However, MVUE's are not necessarily efficient.

- Apply the Rao-Blackwell theorem with a complete sufficient statistic. Rao-Blackwell theorem, when applied in conjunction with a complete sufficient statistics, gives another way to find MVUEs that applies even when CRLB is not defined.

Recall the Rao-Blackwell theorem:

Theorem 1 ***Rao-Blackwell Theorem -*** *Let $f_\theta(x)$ be the pdf or the pmf of the random variable X. Let $T = \tau(X)$ be a sufficient statistic for θ. Let $\hat{\theta}_1(x)$ be an estimator of θ and define the mean square error*

$$MSE(\hat{\theta}_1) = E[\|\hat{\theta}_1(x) - \theta\|^2].$$

Define

$$\hat{\theta}_2(x) = E[\hat{\theta}_1(x)|T = \tau(x)].$$

Then

$$MSE(\hat{\theta}_2) \leq MSE(\hat{\theta}_1)$$

with equality if and only if $\hat{\theta}_1 = \hat{\theta}_2$ *with probability 1.*

We can apply this theorem to reduce the variance of an unbiased estimator -

- Let $\hat{\theta}_1$ be an unbiased estimator for θ and let $T = \tau(X)$ be a sufficient statistic for θ. Apply Rao-Blackwell theorem:

- Consider the new estimator

$$\hat{\theta}_2(x) = g(\tau(x)) = E[\hat{\theta}_1(x)|T = \tau(x)].$$

- Then, $E[\hat{\theta}_2(x)] = E[\hat{\theta}_1(x)] = \theta$ and $\hat{\theta}_2(x)$ is an unbiased estimator.

- Then by Rao-Blackwell theorem, we can conclude that

$$Var(\hat{\theta}_2(x)) \leq Var(\hat{\theta}_1(x)).$$

- Thus, if $\hat{\theta}_1$ is any unbiased estimator, then smoothing $\hat{\theta}_1$ with respect to a sufficient statistics decreases the variance while preserving unbiasedness.

- Thus, we can restrict our search for the MVUE to functions of sufficient statistics.

The Rao-Blackwell theorem describes how to reduce the variance of an estimator. But, how do we know that we have an MVUE? Answer: $T = \tau(X)$ should be a complete sufficient statistics.

Theorem 2 ***Lehmann-Scheffe -*** *If* T *is complete, there is at most one unbiased estimator that is a function of* T*.*

- Find a complete sufficient statistic $T = \tau(X)$.

- Find any unbiased estimator $\acute{\hat{\theta}}$ and set

$$\hat{\theta} = E[\acute{\hat{\theta}}|T = \tau(x)]$$

- Find a function g such that

$$\hat{\theta} = g(\tau(x))$$

is unbiased.

- Recall that the sufficient statistics arising from exponential family are complete. However, for this family MVUEs can also be found via CRLB.

- Rao-Blackwell theorem's strength comes from the fact that even if CRLB is not defined, it can produce MVUEs.

4 Application

- Assume that we transmit a sequence of radar/sonar waveforms s_i, $i = 1, ..., N$ to probe an environment to check if there is any target. Assume that the signals scatted from a point target are attenuated by the reflectivity, α, of the target, where α is a non-random, scalar constant. Furthermore, assume that the scattered signal at the receiver is contaminated by additive Gaussian white noise with zero mean and known variance σ^2.

- Our objective is to estimate the reflectivity of the target α using the MVUE criterion.

- We set up the following model for our received signal y_i, $i = 1, ..., N$.

$$y_i = \omega_i + \alpha s_i \qquad i = 1, ..., N$$

where ω_i are i.i.d. and $\sim \mathcal{N}(0, \sigma^2)$, σ^2 is known and s_i, $i = 1, ..., N$ are deterministic and known.

- Define $s = [s_1, ..., s_N]^T$ and $y = [y_1, ..., y_N]^T$.

- We are going to use the method suggested by Rao-Blackwell and Lehmann-Scheffe theorems in deriving the MVUE for α.

$$\begin{aligned} f_\theta(y) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\{-\frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i - \alpha s_i)^2\} \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} y_i^2 + -\frac{\alpha}{\sigma^2}\sum_{i=1}^{N} s_i y_i + -\frac{\alpha^2}{2\sigma^2}\sum_{i=1}^{N} s_i^2\} \end{aligned}$$

- $\tau(y) = \sum_{i=1}^{N} s_i y_i$ is a complete sufficient statistic for α. Why?

- Let $\hat{\alpha}_1 = \frac{y_1}{s_1}$. Then, $\hat{\alpha}_1$ is unbiased. Why?

- Define

$$\hat{\alpha}_2 = g(\tau(y)) = E[\hat{\alpha}_1 \mid \tau(y)].$$

- We need to determine $g(\tau(y))$. What is the conditional pdf of $\hat{\alpha_1}$ given $\tau(y)$? Conditional Gauss ... Why?

- Thus, conditional mean of $\hat{\alpha}_1$ given $\tau(y)$ is

$$\hat{\alpha}_2 = g(\tau(y)) = \frac{\sum_{i=1}^{N} s_i y_i}{\sum_{i=1}^{N} s_i^2}.$$

Let's consider the case where both α and σ^2 is unknown.

- Define $\theta = [\alpha,\ \sigma^2]^T$.

- From the pdf of y above

$$\tau(y) = [\tau_1(y), \tau_2(y)] = [\sum_{i=1}^{N} s_i y_i,\ \sum_{i=1}^{N} y_i^2]$$

is a complete sufficient statistic for θ. Why?

- We need to determine an unbiased estimate of θ first. $\hat{\alpha}_1$ defined above is an unbiased estimator of α. Since $\hat{\alpha}_2$ does not involve σ^2, $\hat{\alpha}_2$ is still an MVUE for α.

- Show that

$$E[\tau_2(y) - \frac{\tau_1^2(y)}{\sum_{i=1}^N s_i^2}] = (N-1)\sigma^2.$$

- Thus, we have an unbiased estimator of σ^2 as a function of the complete sufficient statistic $\tau(y)$. As a result

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{(N-1)}[\tau_2(y) - \frac{\tau_1^2(y)}{\sum_{i=1}^N s_i^2}] \\ &= \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \hat{\alpha}s_i)^2\end{aligned}$$

  is MVUE for σ^2. Why?