ECSE 6520: Detection and Estimation Theory

Homework 5

Due March 17th, 2014

Read Textbook Chp. 7, pages 157-214 and Chp. 10 pages 309-337.

1. Let $\mathbf{X} = [X_1, \ldots, X_2]^T$ be a vector i.i.d. r.v.s. X_i which are uniformly distributed over the interval (θ_1, θ_2), $\theta_1 < \theta_2$. Find the maximum likelihood estimator of θ.

2. Let $W_i, i = 1, \ldots, n$, be a set of zero mean i.i.d. Gaussian random variables with variance σ_2^2. Let a be a zero mean Gaussian random variable with variance σ_a^2 which is independent of W_i. The objective is to estimate the value of a given the observation

$$X_i = a + W_i, \quad i = 1, \ldots, n$$

   a . Find the MMSE estimator of a. How does this estimator compare to the MAP and MMSE estimators of a?

   b . Compute the MSE of the MMSE estimator (Hint: express error as a sum of two independent r.v.'s to simplify algebra). What happens to the MSE as $n \to \infty$ or as SNR= $\sigma_a^2/\sigma_w^2 \to \infty$?

3. Let Z be a single observation having density function

$$p_\theta(z) = 2\theta z + 1 - \theta, \quad 0 \leq z \leq 1$$

   where $-1 \leq \theta \leq 1$.

   a . Assuming that θ is a nonrandom parameter, find and plot the maximum likelihood estimator of θ as a function Z.

   b . Is the ML estimator unbiased? If so does it achieve CR bound?

   c . Now assume θ is a random variable with uniform prior density: $p_\theta(\theta) = 1/2, \theta \in [-1, 1]$. Find and plot the MMSE estimator of θ as a function of Z.

   d . Compute the conditional bias $E[\hat{\theta}|\theta] - \theta$ and the conditional MSE $E[(\hat{\theta} - \theta)^2|\theta]$ given θ for each of the estimators of part (a) and (c). Plot the two conditional MSE functions obtained and compare the MSE's of the two estimators. Does one estimator perform uniformly better than the other?

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. variables with *standard Pareto density*:

$$f(x;\theta) = \begin{cases} \theta c^{\theta} x^{-(\theta+1)}, & \text{if } x \geq c \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $c > 0$ is known and $\theta > 0$ is unknown.

a . Is $f(x;\theta)$ a member of the exponential family? Why or why not?

b . Find a one dimensional sufficient statistic for θ given $X_1, X_2, \ldots, X_n$.

c . Find the Fisher information and state CR bound for unbiased estimators of θ.

d . Derive the maximum likelihood estimator $\hat{\theta}$ of θ.

e . Is your estimator efficient?