Waveform Design for Distributed Aperture using Gram-Schmidt Orthogonalization

Can Evren Yarman, Trond Varslot, Birsen Yazıcı
Deparment of Electrical, Computer and Systems Engineering
Rensselaer Polytechnic Institute
Troy, NY 12180
Email: yarman, trond, yazici@ecse.rpi.edu

Margaret Cheney
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180
Email: cheney@rpi.edu

***Abstract*— In this work, we consider a distributed aperture radar system and present a method for clutter rejecting waveforms and reflectivity function reconstruction. This work generalizes the monostatic radar waveform design method for range-doppler imaging, developed in [1], [2] to distributed aperture radar systems. The designed waveforms also lead to a filtered backprojection type reconstruction of the reflectivity function which can be efficiently implemented in a parallel fashion.**

I. Introduction

In this work, we generalize the waveform design strategy presented in [1], [2], which was developed for wideband range-doppler imaging using monostatic radar in the presence of clutter, to design waveforms and develop a reconstruction method for distributed aperture which utilizes Gram-Schmidt orthogonalization (GSO).

II. Scattered Field

We model the antenna as a time-varying current density $j_{tr}(t,\mathbf{x})$ over an aperture. This is appropriate for a wide variety of antennas [3], [4], [5].

We assume that the electromagnetic waves emitted from the antenna travels in a known background and ignore polarization effects. Under these assumptions, the field emanating from the antenna satisfies the scalar wave equation

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2)u^{in}(t,\mathbf{x}) = -j_{tr}(t,\mathbf{x}), \qquad (1)$$

where $c_0(\mathbf{x})$ is the speed of light in the background.

Let $g_0(t,\mathbf{x};\sigma,\mathbf{y})$ be the Green's function of (2)

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2)g_0(t,\mathbf{x};\sigma;\mathbf{y}) = \delta(t-\sigma)\delta(x-y). \qquad (2)$$

Then the incident field is given by

$$u^{in}(t,\mathbf{x}) = -\int g_0(t,\mathbf{x};\sigma,\mathbf{y})j_{tr}(\sigma,\mathbf{y})d\sigma\, d\mathbf{y}. \qquad (3)$$

The model we use for wave propagation, including the source, is

$$(\nabla^2 - c^{-2}(\mathbf{x})\partial_t^2)u(t,\mathbf{x}) = -j_{tr}(t,\mathbf{x}). \qquad (4)$$

where $c(\mathbf{x})$ is the speed of light in the distorted medium. We write $u = u^{in} + u^{sc}$ in (4) and use (2) to obtain

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2)u^{sc}(t,\mathbf{x}) = T(\mathbf{x})\partial_t^2 u(t,\mathbf{x}), \qquad (5)$$

where

$$T(\mathbf{x}) = \frac{1}{c^2(\mathbf{x})} - \frac{1}{c_0^2(\mathbf{x})}. \qquad (6)$$

The *reflectivity function* T contains all the information about how the scattering medium differs from given back ground. It is T, or at least its discontinuities and other singularities, that we want to recover.

We make the *single-scattering* approximation to the scattered field u^{sc} by replacing the full field u on the right side of (2) by the incident field u^{in}. Solution of the resulting differential equation leads to

$$u^{sc}(t,\mathbf{x}) \approx -\int g_0(t,\mathbf{x};\sigma,\mathbf{z})T(\mathbf{z})\partial_\sigma^2 u^{in}(\sigma,\mathbf{z})d\sigma d\mathbf{z}. \qquad (7)$$

For the incident field (3), (7) becomes

$$u^{sc}(t,\mathbf{x}) \approx \int g_0(t,\mathbf{x};\sigma,\mathbf{z})T(\mathbf{z}) \times \partial_\sigma^2 \left(\int g_0(\sigma,\mathbf{z};\tau,\mathbf{y}) j_{tr}(\tau,\mathbf{y}) d\tau d\mathbf{y} \right) d\sigma d\mathbf{z} \quad (8)$$

III. Measurement Model

For the rest of the paper, we will assume that the antenna is small compared with the distance to the scatterers in the far filed and focus to the model $j_{tr}(t,\mathbf{x}) = p(t)j(\mathbf{x})\delta(\mathbf{x}-\mathbf{z}_j)$, where p is the transmitted waveform, referred to as the pulse, and $j(\mathbf{x})$ is the waveguide.

For transmitter location $\mathbf{z}_j$, $j \geq 0$, receiver location $\mathbf{x}_i$, $i \geq 0$, and receiver antenna beam pattern $j_{rc}(t,\mathbf{x}_i)$, we model the measurements as

$$e(t,\mathbf{x}_i,\mathbf{z}_j) = [u^{sc}(\,.\,,\mathbf{x}_i,\mathbf{z}_j) *_t j_{rc}(\,.\,,\mathbf{x}_i)](t), \quad (9)$$

where $*_t$ denotes convolution in t.

Let $\lambda_{ij} = (\mathbf{x}_i,\mathbf{z}_j)$. Define

$$\kappa(\mathbf{z};t;\lambda_{ij};\tau) = \int g_0(\tau',\mathbf{x};\sigma,\mathbf{z})\partial_\sigma^2 g_0(\sigma,\mathbf{z};\tau,\mathbf{z}_j) \times j(\mathbf{z}_j) j_{rc}(t-\tau',\mathbf{x}_i)\, d\sigma\, d\tau', \quad (10)$$

and linear operators $\mathcal{H}^{(\lambda_{ij})}$ and $\mathcal{G}^{(\lambda_{ij})}$ acting on T and p by

$$[\mathcal{H}^{(\lambda_{ij})}(T)](t,\tau) = \int T(\mathbf{z})\kappa(\mathbf{z};t;\lambda_{ij};\tau)d\mathbf{z} = \langle T(\,.\,), \kappa(\,.\,;t;\lambda_{ij};\tau)\rangle, \quad (11)$$

and

$$[\mathcal{G}^{(\lambda_{ij})}p](t,\mathbf{z}) = \int \kappa(\mathbf{z};t;\lambda_{ij};\tau)p(\tau)d\tau = \langle \kappa(\mathbf{z};t;\lambda_{ij};\,.\,), p(\,.\,)\rangle, \quad (12)$$

respectively. Then, (9) defines a bilinear integral operator acting on T and p with kernel κ as follows:

$$e(t,\lambda_{ij}) = \int T(\mathbf{z})\kappa(\mathbf{z};t;\lambda_{ij};\tau)p(\tau)d\mathbf{z}d\tau. \quad (13)$$

Using the notation $\mathcal{H}^{(\lambda_{ij})}$ and $\mathcal{G}^{(\lambda_{ij})}$, we rewrite (13) as

$$e(t,\lambda_{ij}) = \int [\mathcal{H}^{(\lambda_{ij})}(T)](t,\tau)p(\tau)d\tau = \left\langle [\mathcal{H}^{(\lambda_{ij})}(T)](t,\,.\,), p(\,.\,) \right\rangle, \quad (14)$$

and

$$e(t,\lambda_{ij}) = \int T(\mathbf{z})[\mathcal{G}^{(\lambda_{ij})}p](t,\mathbf{z})d\mathbf{z} = \left\langle [\mathcal{G}^{(\lambda_{ij})}p](t,\,.\,), T(\,.\,) \right\rangle. \quad (15)$$

IV. Clutter Suppression Filter W

In the presence of additive clutter C, we model measurements $e_{\rm c}$ by replacing the target T by $T+C$ in e:

$$e_{\rm c}(t,\lambda_{ij}) = \left\langle [\mathcal{H}^{(\lambda_{ij})}(T+C)](t,\,.\,), p(\,.\,) \right\rangle \quad (16)$$

Let W be a linear integral operator acting on the waveform space with kernel $\eta(t)$. We define the *clutter suppression filter* $W^{(\lambda_{ij})}$ as the W that minimizes

$$\Delta_W = \frac{E\left[\int \left| \left\langle [\mathcal{H}^{(\lambda_{ij})}(T+C)](t,\,.\,), Wp(\,.\,) \right\rangle - \left\langle [\mathcal{H}^{(\lambda_{ij})}(T)](t,\,.\,), p(\,.\,) \right\rangle \right|^2 dt \right]}{\|p\|^2}, \quad (17)$$

for any waveform p, i.e.

$$W^{(\lambda_{ij})} = \min_W \Delta_W, \qquad \forall p. \quad (18)$$

Let b_n be an orthonormal basis for the waveform space. Define P to be the operator whose matrix elements are given by $P_{mn} = p_n\overline{p_m}$ where $p_n = \langle p, b_n\rangle/\|p\|$, alias $\frac{p(t)\overline{p(t')}}{\|p\|^2} = \sum_{m,n} P_{mn} b_n(t) b_m(t')$. Assuming T and C are mutually uncorrelated and C has zero mean, (18) can be equivalently expressed by

$$W^{(\lambda_{ij})} = \min_W \operatorname{tr}\left[\left((W-I)^* \mathcal{K}_T^{(\lambda_{ij})}(W-I) + W^* \mathcal{K}_C^{(\lambda_{ij})} W \right) P \right], \quad \forall P, \quad (19)$$

where tr is the trace operator, and $\mathcal{K}_T^{(\lambda_{ij})}$ and $\mathcal{K}_C^{(\lambda_{ij})}$ are positive definite operators given by

$$\mathcal{K}_T^{(\lambda_{ij})} = E\left[\mathcal{H}^{(\lambda_{ij})}(T)^* \mathcal{H}^{(\lambda_{ij})}(T) \right], \quad (20)$$

$$\mathcal{K}_C^{(\lambda_{ij})} = E\left[\mathcal{H}^{(\lambda_{ij})}(C)^* \mathcal{H}^{(\lambda_{ij})}(C) \right]. \quad (21)$$

Note that the cross correlation terms vanishes,

$$E[\mathcal{H}^{(\lambda_{ij})}(C)^*\mathcal{H}^{(\lambda_{ij})}(T)] = E[\mathcal{H}^{(\lambda_{ij})}(T)^*\mathcal{H}^{(\lambda_{ij})}(C)] = 0, \quad (22)$$

since $E[T(z)C(z')] = 0$. Let $R_T(\boldsymbol{z},\boldsymbol{z}') = E[T(\boldsymbol{z})T^*(\boldsymbol{z}')]$ and $R_C(\boldsymbol{z},\boldsymbol{z}') = E[C(\boldsymbol{z})C^*(\boldsymbol{z}')]$ be the autocorrelation of T and C, respectively. We can write $\mathcal{K}_T^{(\lambda_{ij})}$ and $\mathcal{K}_C^{(\lambda_{ij})}$ in terms of R_T and R_C as follows:

$$\mathcal{K}_T^{(\lambda_{ij})}(t,\tau) = \langle\kappa|R_T|\kappa^*\rangle(t,\tau) := \int \kappa(\boldsymbol{z};t;\lambda_{ij},\tau)R_T(\boldsymbol{z},\boldsymbol{z}')\kappa^*(\boldsymbol{z}';t;\lambda_{ij};\tau)d\boldsymbol{z}\,d\boldsymbol{z}', \quad (23)$$

and

$$\mathcal{K}_C^{(\lambda_{ij})}(t,\tau) = \langle\kappa|R_C|\kappa^*\rangle(t,\tau) := \int \kappa(\boldsymbol{z};t;\lambda_{ij},\tau)R_C(\boldsymbol{z},\boldsymbol{z}')\kappa^*(\boldsymbol{z}';t;\lambda_{ij};\tau)d\boldsymbol{z}\,d\boldsymbol{z}'. \quad (24)$$

We determine $W^{(\lambda_{ij})}$ by equating the variational derivative of (19) with respect to W to 0:

$$\mathrm{tr}\left[\left(\left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]W^{(\lambda_{ij})} - \mathcal{K}_T^{(\lambda_{ij})}\right)P\right] = 0. \quad (25)$$

In order for (25) to hold for any P, $W^{(\lambda_{ij})}$ must be

$$W^{(\lambda_{ij})} = \left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]^{-1}\mathcal{K}_T^{(\lambda_{ij})} = I - \left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]^{-1}\mathcal{K}_C^{(\lambda_{ij})}, \quad (26)$$

where $^{-1}$ denotes pseudo- or approximate-inverse.

Note that minimization of (19) with respect to all P is equivalent to minimizing $E\|\mathcal{H}(T+C)W - \mathcal{H}(T)\|_{HS}^2$ with some additional assumptions (i.e. finite transmit and receive durations), where $\|A\|_{HS} = \sqrt{\mathrm{tr}[A^*A]}$ denotes the Hilbert Schmidt norm, i.e.

$$W^{(\lambda_{ij})} = \min_W E\|\mathcal{H}(T+C)W - \mathcal{H}(T)\|_{HS}^2 = \min_W \mathrm{tr}\left[(W-I)^*\mathcal{K}_T(W-I) + W^*\mathcal{K}_C W\right],$$

which again leads to the clutter suppression filter in (26).

V. Designing the Transmitted Waveforms

We want to design waveform $p_m^{(\lambda_{ij})}$'s such that given the clutter suppression filter $W^{(\lambda_{ij})}$

$$\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}) = \langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}p_m^{(\lambda_{ij})}](.\,,\mathbf{z}), p_n^{(\lambda_{ij})}(.\,)\rangle \quad (27)$$

form an orthonormal basis with respect to the inner product over $\mathbf{z}$, i.e.

$$\langle\widehat{\Lambda}_{mn}^{(\lambda_{ij})}, \widehat{\Lambda}_{kl}^{(\lambda_{i'j'})}\rangle = \delta_{mk}\delta_{nl}\delta_{ii'}\delta_{jj'}, \quad (28)$$

where δ_{mn} is the Kronecker delta function, which is equal to one when $m=n$ and zero otherwise.Thus the best approximation to $T(\mathbf{z})$ will be

$$\widetilde{T}(\mathbf{z}) \approx \sum_{m,n,\lambda_{ij}} \alpha_{mn}^{(\lambda_{ij})}\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}), \quad (29)$$

where

$$\alpha_{mn}^{(\lambda_{ij})} = \langle T, \widehat{\Lambda}_{mn}^{(\lambda_{ij})}\rangle. \quad (30)$$

For ease of exposition, we will first present the main ideas of our discussion for a singe transmitter-receiver pair case and then generalize the main ideas to distributed aperture.

A. For a Single Transmitter-Receiver Pair

Assuming that we have a single transmitter-receiver pair, given an orthonormal set $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}$ the best approximation to $T(\mathbf{z})$ will be

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z}) \approx \sum_{m,n} \alpha_{mn}^{(\lambda_{ij})}\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}). \quad (31)$$

Substituting (27) in (30),

$$\alpha_{mn}^{(\lambda_{ij})} = \int T(\mathbf{z})\langle[\mathcal{G}^{(\lambda_{ij})}p_m^{(\lambda_{ij})}](.\,,\mathbf{z}), p_n^{(\lambda_{ij})}(.\,)\rangle d\mathbf{z} = \langle e_{cm}^{(\lambda_{ij})}(.\,,\lambda_{ij}), p_n^{(\lambda_{ij})}(.\,)\rangle, \quad (32)$$

where $e_{cm}^{(\lambda_{ij})}$ denotes the scattered field from $T+C$ due to transmitted waveform $W^{(\lambda_{ij})}p_m^{(\lambda_{ij})}$. Thus, by (32), (31) becomes

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z}) \approx \sum_{m,n}\langle e_{cm}^{(\lambda_{ij})}(.\,,\lambda_{ij}), p_n^{(\lambda_{ij})}(.\,)\rangle\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}). \quad (33)$$

We can treat (33) as a filtered backprojection type reconstruction formula [4], [5], where the filtering takes in two steps. First step is in transmission, where we filtering over the scene $T+C$ by the transmitting the waveforms $W^{(\lambda_{ij})}p_m$. Second step is matched filtering of the measurements with p_n in receive. Finally, we backproject the matched filtered measurements with $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$.

1) Minimum Mean Square Error Waveforms: Let $\{q_m^{(\lambda_{ij})}\}_{m\geq 0}$ be the eigenfunctions of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$,

$$[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](t,\mathbf{z}) = Q_m^{(\lambda_{ij})}(\mathbf{z})q_m^{(\lambda_{ij})}(t), \tag{34}$$

ordered such that corresponding eigenvalues are descending. Then

$$\begin{aligned}&\langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](\,.\,,\mathbf{z}),q_n^{(\lambda_{ij})}(\,.\,)\rangle\\ &=\delta_{mn}\langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](\,.\,,\mathbf{z}),q_m^{(\lambda_{ij})}(\,.\,)\rangle.\end{aligned} \tag{35}$$

Define

$$Q_m^{(\lambda_{ij})}(\mathbf{z})=\langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](\,.\,,\mathbf{z}),q_m^{(\lambda_{ij})}(\,.\,)\rangle, \tag{36}$$

and

$$\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z}) = Q_m^{(\lambda_{ij})}(\mathbf{z})/\|Q_m^{(\lambda_{ij})}\|. \tag{37}$$

If the transmitted waveforms were $p_m^{(\lambda_{ij})} = q_m^{(\lambda_{ij})}$, then matching the echo with $p_m^{(\lambda_{ij})}$ means projection of $T+C$ onto $\widehat{Q}_m^{(\lambda_{ij})}$. However $\{\widehat{Q}_m^{(\lambda_{ij})}\}_{m\geq 0}$ does not necessarily form an orthonormal set in the image domain. In this regard, we want to design our waveforms from $\{q_m^{(\lambda_{ij})}\}_{m\geq 0}$ such that the corresponding $\{\widehat{\Lambda}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m\geq 0}$ form an orthonormal set. Consequently, due to the linearity of the measurement model, designing transmitted waveforms is equivalent to form an orthonormal set $\{\widehat{\Lambda}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m\geq 0}$ from $\{\widehat{Q}_m^{(\lambda_{ij})}\}_{m\geq 0}$, which we will present in the next section. Usage of eigenfunctions $\{q_m\}$ simplifies the decomposition (31) as

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z}) \approx \sum_m \alpha_{mm}^{(\lambda_{ij})}\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z}), \tag{38}$$

where

$$\alpha_{mm}^{(\lambda_{ij})} = \langle T,\widehat{\Lambda}_{mm}^{(\lambda_{ij})}\rangle = \langle e_m(\,.\,,\lambda_{ij}),p_m^{(\lambda_{ij})}(\,.\,)\rangle. \tag{39}$$

If we were to transmit a single pulse $p_0^{(\lambda_{ij})}$, then the minimum mean square error (MMSE) $E[\|T - \widetilde{T}^{(\lambda_{ij})}\|_2^2]$ is achieved when $p_0^{(\lambda_{ij})}$ is the eigenfunction of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$ corresponding to the largest eigenvalue, i.e. $p_0^{(\lambda_{ij})} = q_0^{(\lambda_{ij})}$. Here $\|f\|_2^2 = \int |f(\mathbf{z})|^2 d\mathbf{z}$. Similarly, if N pulses were transmitted then, the MMSE is achieved when $p_m^{(\lambda_{ij})}$ are chosen as the eigenfunctions of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$ corresponding to the N largest eigenvalues, i.e. $p_i^{(\lambda_{ij})} = q_i^{(\lambda_{ij})}$ for $i=0,\ldots,N-1$.

2) Construction of $\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z})$ and $p_m^{(\lambda_{ij})}(t)$: We will form $\{\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z})\}_{m\geq 0}$ by performing the Gram-Schmidt orthonormalization (GSO) process over $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m\geq 0}$. Since the GSO depends on the choice of the order of $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m>0}$, in the light of the previous section, in order to obtain MMSE, we will choose $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m>0}$ with the increasing order in m. Let $\widehat{\Lambda}_{00}^{(\lambda_{ij})}(\mathbf{z}) = \widehat{Q}_0^{(\lambda_{ij})}(\mathbf{z})$. Then, by GSO, we form

$$\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z}) = \sum_{k=0}^{m-1}\beta_{mk}^{(\lambda_{ij})}\widehat{\Lambda}_{kk}^{(\lambda_{ij})}(\mathbf{z}) + \beta_{mm}^{(\lambda_{ij})}\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z}), \tag{40}$$

for some constants $\beta_{mn}^{(\lambda_{ij})}$ and $\gamma_{mn}^{(\lambda_{ij})}$ obtained by GSO.

Due to the linearity of GSO, we can determine the transmitted waveforms along with the GSO of $\widehat{Q}_m^{(\lambda_{ij})}$ as follows. Let $\{p_m^{(\lambda_{ij})}\}_{m\geq 0}$ be the transmitted waveforms. In order to obtain $\widehat{\Lambda}_{mm}^{(\lambda_{ij})}$ after matching the echo by $p_m^{(\lambda_{ij})}$, by GSO of $\widehat{Q}_m^{(\lambda_{ij})}$,

$$p_0^{(\lambda_{ij})}(t) = q_0^{(\lambda_{ij})}(t)/\sqrt{\|Q_0^{(\lambda_{ij})}\|}, \tag{41}$$

$$p_m^{(\lambda_{ij})}(t) = \sum_{k=0}^{m-1}\frac{q_k^{(\lambda_{ij})}(t)\sqrt{\gamma_{mk}^{(\lambda_{ij})}}}{\sqrt{\|Q_k^{(\lambda_{ij})}\|}} + \frac{q_m^{(\lambda_{ij})}(t)\sqrt{\beta_{mm}^{(\lambda_{ij})}}}{\sqrt{\|Q_m^{(\lambda_{ij})}\|}}, \tag{42}$$

where

$$\gamma_{mk}^{(\lambda_{ij})} = \sum_{n=k}^{m-1}\beta_{mn}^{(\lambda_{ij})}\gamma_{nk}^{(\lambda_{ij})}. \tag{43}$$

3) In the Absence of the Eigenfunctions: It is not always possible to find the eigenfunctions of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$ and only a set of pulses $s_m(t)$ can be transmitted. Then one can still use the

GSO to form a basis $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}$ from

$$S_{mn}^{(\lambda_{ij})}(\mathbf{z}) = \langle [\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} s_m](\,.\,,\mathbf{z}), s_n(\,.\,)\rangle. \quad (44)$$

Thus for each $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$, one will obtain a pulse $p_{mn}^{(\lambda_{ij})}(t)$. This will square the number of waveforms to be transmitted. Consequently, the computational burden is also square when compared the reconstruction using the waveforms determined by the eigenfunctions of $\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})}$.

B. For Distributed Aperture

In the case of distributed aperture, it is enough to generalize the GSO over $\{S_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$ for some order on $\{(\lambda_{ij}, m)\}$.

For example let the ordering on $\{(\lambda_{ij}, m)\}$ be defined by $(\lambda_{i'j'}, m') < (\lambda_{ij}, m)$ if $i' < i$, or if $i' = i$ and $j' < j$, or $\lambda_{i'j'} = \lambda_{ij}$ and $m' < m$. Then given $\{S_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$, determine $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$ by GSO such that $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$ is orthogonal to all $\widehat{\Lambda}_{m'n'}^{(\lambda_{i'j'})}(\mathbf{z})$, and hence to all $S_{m'n'}^{(\lambda_{i'j'})}(\mathbf{z})$, for $(\lambda_{i'j'}, m') < (\lambda_{ij}, m)$, with the initial condition $\widehat{\Lambda}_{00}^{(\lambda_{00})}(\mathbf{z}) = S_{00}^{(\lambda_{00})}(\mathbf{z})/\|S_{00}^{(\lambda_{00})}\|$.

VI. From Group Theoretic Point of View

The proposed method is inspired from statistical signal processing over groups developed in [1], [2]. Under the group theoretic perspective (30) and (31) corresponds to the Fourier and inverse Fourier transforms. In this regard, $\widehat{S}_{mn}^{(\lambda_{ij})}(\mathbf{z}) = S_{mn}^{(\lambda_{ij})}(\mathbf{z})/\|S_{mn}^{(\lambda_{ij})}\|$ can be treated as the matrix elements of the irreducible unitary representations, where mn denotes the matrix entry and $\lambda_{ij} = (\mathbf{x}_i, \mathbf{z}_j)$ is the corresponding irreducible subspace. In this setting GSO provides the square root of the linear operator known as the discrepancy operator for non-commutative groups, which forms the symmetric Fourier basis $\{\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$ from $\{\widehat{S}_{mn}^{(\lambda_{ij})}\}$. Finally $W^{(\lambda_{ij})}$ corresponds to the Wiener filter. [1], [2]

VII. Summary

The presented reconstruction method can be summarized by the following diagram:

$$\begin{tikzcd}
T(\mathbf{z}) \arrow[r, "W^{(\lambda_{ij})} p_{mn}^{(\lambda_{ij})}(t)", "1"'] & e(t, \lambda_{ij}) \arrow[d, "2", "\langle \cdot, p_{mn}^{(\lambda_{ij})}\rangle"'] \\
\alpha_{mn}^{(\lambda_{ij})} \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}) \arrow[u, "\sum_{m,n,\lambda_{ij}}", "4"'] & \alpha_{mn}^{(\lambda_{ij})} \arrow[l, "3", "\times \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})"']
\end{tikzcd}$$

The first step is transmission of the $W^{(\lambda_{ij})}$ filtered waveforms $\{p_{mn}^{(\lambda_{ij})}(t)\}$. The second step is the matching of the received echo with the transmitted pulse. The third step is the backprojection of the matched signal. Finally, the image is formed by summing over each backprojected image.

REMARK - Transmission vs Receive: In stead of transmitting the waveforms $\{p_{mn}^{(\lambda_{ij})}(t)\}$'s one can still transmit the waveforms $\{s_m(t)\}$. Since $W^{(\lambda_{ij})}$ is linear and Step 1 is assumed to be linear, the measurements due to transmitting $\{p_{mn}^{(\lambda_{ij})}(t)\}$ can be obtained by weighted summation of the measurements made by transmitting $\{s_m(t)\}$. However, this induces more computation on the receiver side. The weighted summation can also be handled in Steps 2,3 and 4, due to the linearity of the steps.

Acknowledgment

We are grateful to Air Force Office of Scientific Research (AFOSR) and the Defense Advanced Research Projects Agency (DARPA) for supporting this work under the agreements FA9550-04-1-0223, FA9550-06-1-0017 and FA8750-05-2-0285.

References

[1] B. Yazıcı and G. Xie, "Wideband extended range-doppler imaging and waveform design in the presence of clutter and noise," *IEEE Trans. Inf. Theory*, vol. 52, pp. 4563–4580, 2006.

[2] B. Yazici and C. Yarman, *Deconvolution over Groups in Image Reconstruction*, 2006, vol. 141, pp. 257–300.

[3] M. Cheney and B. Borden, "Microlocal structure of inverse synthetic aperture radar data," *Inverse Problems*, vol. 19, pp. 173–194, 2003.

[4] C. J. Nolan and M. Cheney, "Synthetic aperture inversion," *Inverse Problems*, vol. 18, pp. 221–236, 2002.

[5] C. Nolan and M. Cheney, "Synthetic aperture inversion for arbitrary flight paths and non-flat topography," *IEEE Transactions on Image Processing*, vol. 12, pp. 1035–1043, 2003.