Rensselaer

ECSE 4520 - Communication Engineering

Exam 2 - Solutions

Problem 1) Multiple choice with four questions. For each of the following statements, choose the single BEST option by circling it.

A) [7 points] An advantage of an ergotic wide-sense stationary random process is:

1. It is always the compound result of many random variables, thus it models electromagnetic noise well.
   Not this one. This was how we described white Gaussian noise using the Central Limit Theorem.
2. It is best suitable for modeling the random phase of a sinusoid.
   Not this one. We said the Uniform PDF was best used to describe the phase.
3. To estimate its mean, we need only take its time average, and not its sample average.
   Yes, this was mentioned in class and in the text.

B) [7 points] Regarding white Gaussian noise that is fed into a Linear Time-Invariant system:

1. It preserves its stochastic properties, so the output is also Gaussian.
   Yes, we mentioned this in class as an attractive aspect of Gaussian.
2. The output has the same mean as the input.
   Not this one. Consider an LTI system with a tiny gain, so the average cannot be preserved.
3. The output has the same variance as the input.
   Not this one.
4. All of the above.
   Not this one.

C) [7 points] Additive Gaussian noise may be represented with its in-phase and q-phase components when we analyze SNR because

1. It is white, so its spectrum is a constant, therefore any carrier may be used without loss of generality.
   Not this one. in-phase and q-phase assume a bandpass signal, so it's the noise after the noise reduction filter, which is centered around the signal's carrier.
2. It is colored and bandpass, since the received signal was fed through a noise reducing filter.
   Yes, this one.
3. It is white, but the inner product with any *signal* component is zero unless the correct carrier is used for the representation.
   It is not white after the noise reduction filter.

D) [7 points] In probability, if you have a collection of independent random variables, X_i, each with its own PDF, $f_i(X_i)$, its own mean, μ_i, and its own variance, σ_i^2, the result that

$$\frac{\sum_{i=1}^{N} X_i - \sum_{i=1}^{N} \mu_i}{\sqrt{\sum_{i=1}^{N} \sigma_i^2}}$$

has a limiting (as $N \to \infty$) CDF which approaches a Normal Distribution is called

1. The Central Limit Theorem
   Yes, this one.
2. The Strong Law of Large Numbers
   Not this one. This is about how a sample average converges to the mean.
3. The Weak Law of Large Numbers
   Not this one. This is about how a sample average converges to the mean.
4. The Ergotic Theorem
   No.

Problem 2) [32 points] A DSB-SC AM modulated signal with power-spectral density shown in Figure 2(a) is corrupted with additive noise that has power-spectral density $N_0/2$ within the passband of the signal. The received signal plus noise is demodulated and passed through a lowpass filter (shown in Figure 2(b)). Determine the SNR at the output of the lowpass filter.

ANSWER: The noise power content of the received signal $r(t) = u(t) + n(t)$ is

$$P_n = \int_{-\infty}^{\infty} \mathcal{S}_n(f)df = \frac{N_0}{2} \times 4W = 2N_0W$$

If we write $n(t)$ as

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

then,

$$\begin{aligned} n(t)\cos(2\pi f_c t) &= n_c(t)\cos^2(2\pi f_c t) - n_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}n_c(t) + \frac{1}{2}n_c(t)\cos(2\pi 2 f_c t) - n_s(t)\sin(2\pi 2 f_c t) \end{aligned}$$

The noise signal at the output of the LPF is $\frac{1}{2}n_c(t)$ with power content

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n = \frac{N_0W}{2}$$

If the DSB modulated signal is $u(t) = m(t)\cos(2\pi f_c t)$, then its autocorrelation function is $\bar{R}_u(\tau) = \frac{1}{2}R_M(\tau)\cos(2\pi f_c \tau)$ and its power

$$P_u = \bar{R}_u(0) = \frac{1}{2}R_M(0) = \int_{-\infty}^{\infty} \mathcal{S}_u(f)df = 2WP_0$$

From this relation we find $R_M(0) = 4WP_0$. The signal at the output of the LPF is $y(t) = \frac{1}{2}m(t)$ with power content

$$P_{s,o} = \frac{1}{4}E[m^2(t)] = \frac{1}{4}R_M(0) = WP_0$$

Hence, the SNR at the output of the demodulator is

$$\textbf{SNR} = \frac{P_{s,o}}{P_{n,o}} = \frac{WP_0}{\frac{N_0W}{2}} = \frac{2P_0}{N_0}$$

Problem 3) [40 points] A communication channel is characterized by 90 dB attenuation and additive white noise with power-spectral density $N_0/2 = .5 \times 10^{-14}$ W/Hz. The bandwidth of the message signal is 1.5 MHz and its amplitude is uniformly distributed on the interval $[-1, 1]$. If we require that the SNR after demodulation be 30 dB, **in the case of DSB-SC AM modulation**, find the necessary transmitted power.

ANSWER: This was done in class for USSB.

First we determine the baseband signal to noise ratio $(\frac{S}{N})_b$. With $W = 1.5 \times 10^6$, we obtain

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = \frac{P_R}{2 \times 0.5 \times 10^{-14} \times 1.5 \times 10^6} = \frac{P_R 10^8}{1.5}$$

Since the channel attenuation is 90 db, then

$$10 \log \frac{P_T}{P_R} = 90 \Longrightarrow P_R = 10^{-9} P_T$$

Hence,

$$\left(\frac{S}{N}\right)_b = \frac{P_R 10^8}{1.5} = \frac{10^8 \times 10^{-9} P_T}{1.5} = \frac{P_T}{15}$$

For DSB-SC modulation

$$\left(\frac{S}{N}\right)_{o,\mathrm{DSB-SC}} = \left(\frac{S}{N}\right)_b = \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 15 \textbf{ KWatts}$$

Here are some identities you may find useful:

$a \star \frac{1}{\pi t} = 0,$ where a is a constant and $\star$ means the operator convolution.

$\cos(\omega t) \star \frac{1}{\pi t} = \sin(\omega t)$.

$\sin(\omega t) \star \frac{1}{\pi t} = -\cos(\omega t)$.

$F[\Pi(t)] = sinc(f),$ where $F[\cdot]$ means the Fourier transform and $\Pi(\cdot)$ is the single square wave.

$F[sinc(t)] = \Pi(f)$

$F[\Lambda(t)] = sinc^2(f),$ where $\Lambda(\cdot)$ is the single triangle wave.

$F[sinc^2(t)] = \Lambda(f)$

$F[sgn(t)] = \frac{1}{j\pi f},$ where $sgn(\cdot)$ is the sign function.

Differentiation Property of the Fourier Transform:

$$F\left[\frac{d}{dt}x(t)\right] = j2\pi f X(f)$$

Integration Property of the Fourier Transform:

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f) ,$$ where $\delta(\cdot)$ is the delta function.

Moments Property of the Fourier Transform:

$$\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)|_{f=0}$$

$$\int \frac{1}{1+x^2}dx = \arctan(x)$$