Homework 1 - Solutions

Problem 1) The *vast* majority of the Fourier transform problems you will encounter in this course for homework as well as exams need not be solved by explicitly doing the Fourier integral. *In most cases,* the problems can be solved in a couple of lines by using a property (such as linearity or time shift) applied to a basic transform pair.

Basic pairs are given in your text on page 38 in Table 2.1. You will be expected to know these pairs by heart. You will be expected to know the properties by heart. The material should have become second nature to you when you learned "signals and systems" and you can hopefully relearn it quickly.

Though I expect you to know the properties by heart (and most certainly the common and important ones) I may consider handing out a table with the last four properties described in Section 2.2.2 for the exam. The others should be second nature to you. If they are not, I suggest you do more problems, since that will help the material stick.

1) Using the Fourier transform pair (line 13 of Table 2.1)

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

We can write

$$\frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

Notice that this has the format we want, but in the frequency domain, not in the time domain. This suggests that we use the "duality property" of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$$

We need only use linearity to pull out that constant:

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

We define $\alpha = 2\pi$, so as to get rid of the term on the left of the denominator. Then we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2) First use linearity to treat each $\Pi(\cdot)$ separately.

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3) + \Pi(t+3)] \\
&= \mathcal{F}[\Pi(t-3)] + \mathcal{F}[\Pi(t+3)]
\end{aligned}$$

Then use the time shift property along with the transform pair provided in Table 2.1 (line 7)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3)] + \mathcal{F}[\Pi(t+3)] \\
&= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3}
\end{aligned}$$

Finally, use the well known identity $cos(v) = \frac{e^{jv}+e^{-jv}}{2}$ to obtain:

$$\begin{aligned}\mathcal{F}[x(t)] &= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3}\\ &= 2\text{sinc}(f)\cos(2\pi 3f)\end{aligned}$$

3) Here, you'll use "time shifting", "scaling", and "linearity" applied to the transform pair in line 9 of Table 2.1:

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[\Lambda(2t+3) + \Lambda(3t-2)]\\ &= \mathcal{F}[\Lambda(2(t+\frac{3}{2})) + \Lambda(3(t-\frac{2}{3})]\\ &= \frac{1}{2}\text{sinc}^2(\frac{f}{2})e^{j\pi f3} + \frac{1}{3}\text{sinc}^2(\frac{f}{3})e^{-j2\pi f\frac{2}{3}}\end{aligned}$$

4) With this problem, we have $\text{sinc}^2(t)$ (line 10 of Table 2.1) and we have $\text{sinc}(t)$ (line 8 of Table 2.1), but we don't have $\text{sinc}^3(t)$. We do know that

$$\text{sinc}^3(t) = \text{sinc}^2(t) \cdot \text{sinc}(t)$$

And we know that multiplication in the time domain translates into convolution in the frequency domain, so

$$\mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t) \cdot \text{sinc}(t)] = \Lambda(f) \star \Pi(f)$$

Now the convolution would simplify easily because $\Pi(\cdot)$ is just equal to 1 for $|t| < 1/2$ (don't worry about the exact points $t = \pm 1/2$ because their contribution is negligible for the integral). So,

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

Now, we can solve this integral in pieces. Let's use the short-hand $T(f) = \mathcal{F}[\text{sinc}^3(t)]$. So now we know that $T(f) = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$, so we can solve

$$\begin{aligned}
&\text{For} \quad f \le -\frac{3}{2} \Longrightarrow T(f) = 0\\
&\text{For} \quad -\frac{3}{2} < f \le -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}\\
&\text{For} \quad -\frac{1}{2} < f \le \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv\\
&\qquad = (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}\\
&\text{For} \quad \frac{1}{2} < f \le \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}\\
&\text{For} \quad \frac{3}{2} < f \Longrightarrow T(f) = 0
\end{aligned}$$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

5) We can recall that $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, so $t\ \text{sinc}(t) = \frac{\sin(\pi t)}{\pi}$, and then use linearity and the transform pair on line 6 of Table 2.1 with $f_0 = 1/2$.

$$\mathcal{F}[t\ \text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f + \frac{1}{2}) - \delta(f - \frac{1}{2})\right]$$

6) Now, this property isn't explicitly mentioned in Section 2.2.2, but we're reminded of what we need by the "moments property". The moments property here describes it as a way to get the moments of $x(\cdot)$ by setting $f = 0$. But, what if we don't set $f = 0$? You may be reminded of the derivation for this property:

$$\begin{aligned} \tfrac{d}{df}X(f) &= \tfrac{d}{df}\textstyle\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= \textstyle\int_{-\infty}^{\infty} x(t)\tfrac{d}{df}e^{-j2\pi ft}dt \qquad \text{Mathematicians, forgive us!} \\ &= \textstyle\int_{-\infty}^{\infty} x(t)(-j2\pi t)e^{-j2\pi ft}dt \\ &= -j2\pi\textstyle\int_{-\infty}^{\infty} t\ x(t)e^{-j2\pi ft}dt \\ &= -j2\pi\mathcal{F}[t\ x(t)] \end{aligned}$$

So we conclude that

$$\mathcal{F}[t\ x(t)] = \frac{j}{2\pi}\frac{d}{df}X(f)\ .$$

We apply that to the transform pair of line 5 of Table 2.1 and you get:

$$\begin{aligned} \mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)\right) \\ &= \frac{j}{4\pi}\left(\delta'(f - f_0) + \delta'(f + f_0)\right) \end{aligned}$$

This type of manipulation is not uncommon, and you get quite creative at shuffling common properties around. This becomes increasingly needed in the real world, as the measured signals from some sources are approximated rather awkwardly by novel variations of known signals.
Leaving the solution in terms of $\delta'(\cdot)$ is perfectly fine, you need not take it any further.

7) Here we use the fact that multiplication in the time domain is convolution in the frequency domain. We apply that to the transform pairs on lines 5 and 13 of Table 2.1. Since the transform for $\cos(\cdot)$ are deltas, the convolution is easy:

$$\mathcal{F}[e^{-\alpha|t|}\cos(\beta t)] = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2 + (2\pi(f - \frac{\beta}{2\pi}))^2} + \frac{2\alpha}{\alpha^2 + (2\pi(f + \frac{\beta}{2\pi}))^2}\right]$$

8) Here we see the same problem of "7)", but with a t in front of it. Well, we already have the transform for $e^{-\alpha|t|}\cos(\beta t)$, which we now know is $\frac{1}{2}\left[\frac{2\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{2\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right]$. From problem "6)" we know that $\mathcal{F}[t\ x(t)] = \frac{j}{2\pi}\frac{d}{df}X(f)$. So, now we use both of these ideas together:

$$\begin{aligned}
\mathcal{F}[te^{-\alpha|t|}\cos(\beta t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right) \\
&= -j\left[\frac{2\alpha\pi(f-\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2\right)^2}+\frac{2\alpha\pi(f+\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2\right)^2}\right]
\end{aligned}$$

Problem 2) We start just writing the Fourier transform out (*be careful to use a different dummy variable for the convolution than for the Fourier transform, or you'll get terribly confused*):

$$\mathcal{F}[x(t)\star y(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau\right]e^{-j2\pi ft}dt$$

Note: we used t for the transform and τ for the convolution.

$$= \int_{-\infty}^{\infty}x(\tau)\left[\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi f(t-\tau)}\,dt\right]e^{-j2\pi f\tau}d\tau$$

Here, we just switched the orders of the integrals and we pulled out all of the τ stuff from the inner integral.

Now with the change of variable for the inner integral, $u = t-\tau$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi f(t-\tau)}dt &= \int_{-\infty}^{\infty}y(u)e^{-j2\pi fu}du \\
&= \mathcal{F}[y(t)] \\
&= Y(f)
\end{aligned}$$

Note: we got away with that change of variable because the inner integral was going from $-\infty$ to ∞ and the τ was just a shift. So, things simplify to

$$\begin{aligned}
\mathcal{F}[x(t)\star y(t)] &= \int_{-\infty}^{\infty}x(\tau)Y(f)e^{-j2\pi f\tau}d\tau \\
&= X(f)\cdot Y(f)
\end{aligned}$$

Problem 3) We start with the Fourier transform of $x(t-t_0)$,

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty}x(t-t_0)e^{-j2\pi ft}dt$$

With a change of variable of $u = t-t_0$, we obtain

$$\begin{aligned}
\mathcal{F}[x(t-t_0)] &= \int_{-\infty}^{\infty}x(u)e^{-j2\pi ft_0}e^{-j2\pi fu}du \\
&= e^{-j2\pi ft_0}\int_{-\infty}^{\infty}x(u)e^{-j2\pi fu}du \\
&= e^{-j2\pi ft_0}\mathcal{F}[x(t)]
\end{aligned}$$

Problem 4) The "Convolution theorem" is:

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

So, we apply it to sinc(·):

$$\mathcal{F}[\mathrm{sinc}(t) \star \mathrm{sinc}(t)] = \Pi(f) \cdot \Pi(f)$$

But we don't want $\mathcal{F}[\mathrm{sinc}(t) \star \mathrm{sinc}(t)]$, we just want $\mathrm{sinc}(t) \star \mathrm{sinc}(t)$, so we take the inverse transform:

$$\mathrm{sinc}(t) \star \mathrm{sinc}(t) = \mathcal{F}^{-1}[\Pi(f)\Pi(f)]$$

Note that $\Pi(f) \cdot \Pi(f)$ is the same as $\Pi(f)$, except for at the points $f = \pm 1/2$. For the inverse transform integral, these two solitary points don't make any difference:

$$\begin{aligned} \mathrm{sinc}(t) \star \mathrm{sinc}(t) &= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] \\ &= \mathcal{F}^{-1}[\Pi(f)] \\ &= \mathrm{sinc}(t) \end{aligned}$$

where we used line 8 of Table 2.1 to get the last line.

Problem 5) If you looked at Example 15 (part 2) of the Matlab notes handed out in class, you could have seen the problem solved for you. Pages 17-23 of the notes would clarify how to solve this problem.

First, how do we get the sampling interval? We need the signal's bandwidth (which we don't have and wish to learn). Since the signal is rather smooth, we can approximate the bandwidth by saying it's proportional to the inverse of the time duration of the signal. Since this is just an approximation, and to be on the safe side, let's say the bandwidth is *ten times bigger* than that amount:

$$BW = 10 \cdot \frac{1}{total \;\; time} = 10 \cdot \frac{1}{2} = 5 \quad .$$

So if we accept the bandwidth to be 5, then the Nyquist frequency is twice that amount: choose the sampling frequency $f_s = 2 \cdot 5 = 10$. That means that the sampling interval is $T_s = 1/f_s = 1/10 = 0.1$.

Now, we must generate a signal in Matlab that has the values described by $x(t)$. The "t" should increment in steps of T_s. I would give $x(t)$ some zero padding before and after the signal just to drive in the fact that that's the whole signal and there's nothing else beyond it.

Then we take the *Fast Fourier Transform* of the signal (don't forget to normalize by f_s as seen in the Matlab notes).

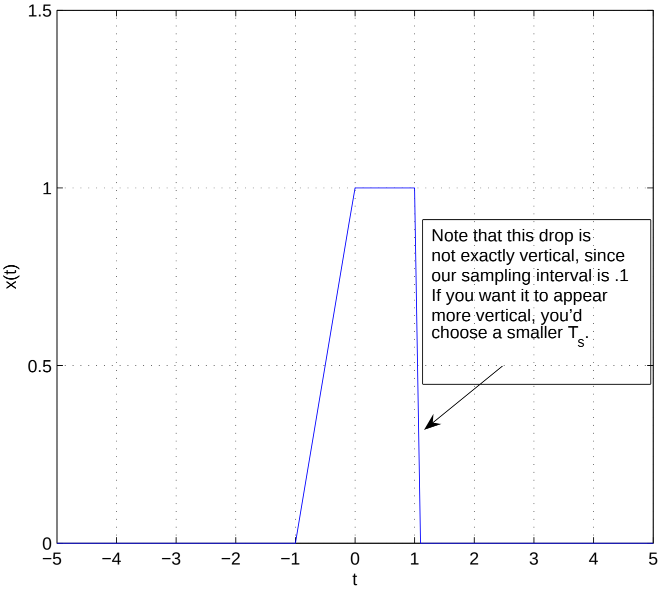
Well, the big important stuff is done by now. We just need to plot things. We need to generate the frequencies for the plot's abscissa. How big is the sampling frequency? How many points do I have to divide it by? Then I have to make sure that instead of going from points $(0, f_s)$ the plot goes from $(-f_s/2, f_s/2)$. That means centering the frequencies as well as the transform values.

Finally, we get down to plotting the results and tweaking the plots to look nice.

```
>> ts=0.1;              % This is the sampling interval we decided on.
>> t=[-5:ts:5];         % This is the vector with the times for sampling (note it goes from
                        % (−5, 5) though the function only goes from (−1, 1)).

                        % I labelled it “t”, but you may argue that we’re in discrete times
                        % now, so it should be “n”.
>> x=zeros(size(t));         % This is the basis for, x(t) (or x[n], for the purists). We put all
                             % zeros for now and then add the values.
>> x(41:51)=t(41:51)+1;           % This is the ramp part (note that Matlab understands
                                  % that I add 1 to each value of the vector).
>> x(52:61)=ones(size(x(52:61)));        % This is the flat part of x(t) (of x[n], for the purists).
>> plot(t,x);                            % Sanity check.
>> X=fft(x)*ts;                   % Using the FFT of x[n] times Ts, we approximate X(jΩ).
>> Xcentered=fftshift(X);         % Rearrange it so the center frequency is zero.
>> fs=1/ts;                  % This is the sampling frequency.
>> l=length(X) - 1;          % This is the number of points of the transform in the frequency
                             % domain minus 1.
>> df=fs/l;                  % frequency resolution (so we can use it for the plot’s abscissa).
>> f=[0:df:fs]-fs/2;              % Frequencies for the abscissa of the plot.
>> subplot(1,2,1),plot(f,abs(Xcentered))       % Make first graph.
>> xlabel=('frequency')        % Labels the abscissa axis.
>> ylabel=('abs(X(j omega))')       % Labels the ordinate axis.

>> subplot(1,2,2),plot(f,angle(Xcentered))       % Make second graph.
>> xlabel=('frequency')        % Labels the abscissa axis.
>> ylabel=('angle of X(j omega)')       % Labels the ordinate axis.
```



Note that this drop is not exactly vertical, since our sampling interval is .1 If you want it to appear more vertical, you'd choose a smaller T_s.

Figure 1: Plot of $x(t)$. I used the menu item "Tools" to "enable plot editing". When I double clicked on the axis, it gave me a window that allowed me to choose the tic marks and limits of the axis. That window also allowed me to label the abscissa and the ordinate. Under the "insert" menu item, there is one item called "text box"; I used that to add the comment about the slope.

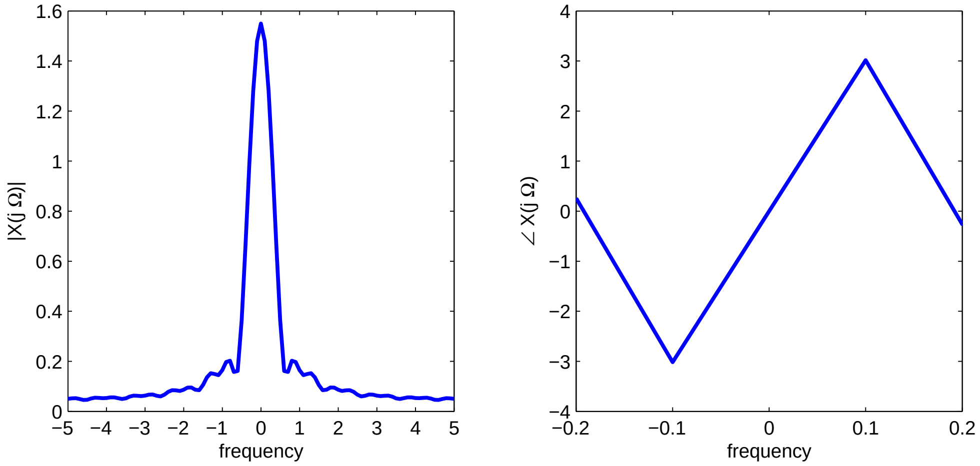


Figure 2: Plot of $X(j\Omega)$.