Homework 2 - Solutions

Problem 1) We start by writing the formal transform formula:

$$\mathcal{F}[x(t)y(t)] = \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt$$

Now, we want $X(f)$ and $Y(f)$ in there, so we replace $x(t)$ with the inverse transform of $X(\theta)$ (*Note, we use "θ" instead of "f" as the dummy variable so that we don't mix the variables up*). That means $x(t) = \mathcal{F}^{-1}\{X(\theta)\}$:

$$\begin{aligned}\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right) y(t)e^{-j2\pi ft}dt\end{aligned}$$

O.K., so now we have $X(\theta)$, which is good. We now want $X(\theta)$ convolved with $Y(\theta)$ (Please, don't get hung up on the dummy variables. If it's easier for you, just swap f with θ. It doesn't matter as long as you're consistent.) Since we don't want to mess with our $X(\theta)$, let's switch the order of the integrals and bring the $X(\theta)$ out:

$$\mathcal{F}[x(t)y(t)] = \int_{-\infty}^{\infty} X(\theta)\left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right)d\theta$$

Now the thing inside the parenthesis is just the transform of $y(t)$ with a frequency shift:

$$\mathcal{F}[x(t)y(t)] = \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta$$

But, this is exactly convolution:

$$\begin{aligned}\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta \\ &= X(f) \star Y(f)\end{aligned}$$

Problem 2)

1) $x(t) = e^{-\alpha t}\cos(\beta t)u_{-1}(t)$ (here $u_{-1}(\cdot)$ is the unit step function).

$$\begin{aligned}E_x &= \int_{-\infty}^{\infty} x^2(t)\,dt \qquad \text{using equation (2.3.2), and since } x(\cdot) \text{ is real,} \\ &= \int_{-\infty}^{\infty} e^{-2\alpha t}\cos^2(\beta t)u_{-1}^2(t)\,dt \qquad \alpha,\beta > 0 \\ &= \int_{0}^{\infty} e^{-2\alpha t}\cos^2(\beta t)\,dt \qquad \alpha,\beta > 0 \\ &\leq \int_{0}^{\infty} e^{-2\alpha t}\,dt = \frac{1}{2\alpha}\end{aligned}$$

where we have used $0 \leq \cos^2(\beta t) \leq 1 \quad \forall t$. Therefore, $x(\cdot)$ is energy type.

To find the energy we have from above:

$$\begin{aligned}\int_0^\infty e^{-2\alpha t}\cos^2(\beta t)\,dt &= \frac{1}{2}\int_0^\infty e^{-2\alpha t}\,dt+\frac{1}{2}\int_0^\infty e^{-2\alpha t}\cos(2\beta t)\,dt\\ &\quad \text{here we used the identity } \cos^2(r)=(\cos(2r)+1)/2\\ &= \frac{1}{4\alpha}+\frac{1}{2}\,\frac{e^{-2\alpha t}}{4\alpha^2+4\beta^2}\left[2\beta\sin(2\beta t)-2\alpha\cos(2\beta t)\right]\quad \Big|_0^\infty\\ &\quad \text{I got this integral from Schaum's "Calculus", but you could also}\\ &\quad \text{replace the cosine with exponentials}\\ &= \frac{1}{4\alpha}+\frac{1}{2}\left(0\cdot[\text{something finite}]-\frac{1}{4\alpha^2+4\beta^2}(0-2\alpha\cdot 1)\right)\\ &= \frac{1}{4\alpha}+\frac{\alpha}{4\alpha^2+4\beta^2}\end{aligned}$$

The energy spectral density can be calculated by $\mathcal{G}_x(f)=|X(f)|^2$ (see top of page 42 of text). Therefore, we take the time-domain product of lines 5 and 11 of Table 2.1 of the text and realize that it's the convolution:

$$\begin{aligned}X(f) &= [\tfrac{1}{2}\delta(f-\beta/2\pi)+\tfrac{1}{2}\delta(f+\beta/2\pi)]\star\tfrac{1}{\alpha+j2\pi f}\\ &= \tfrac{1}{2}\tfrac{1}{\alpha+j2\pi\frac{\beta}{2\pi}}+\tfrac{1}{2}\tfrac{1}{\alpha-j2\pi\frac{\beta}{2\pi}}\\ &= \tfrac{1}{2}\cdot\tfrac{1}{\alpha+j\beta}+\tfrac{1}{\alpha-j\beta}\\ &= \tfrac{\alpha}{\alpha^2+\beta^2}\end{aligned}$$

$$\Rightarrow \mathcal{G}_x(f)=\left|\frac{\alpha}{\alpha^2+\beta^2}\right|^2=\frac{\alpha^2}{(\alpha^2+\beta^2)^2}$$

2) $x(t)=\operatorname{sinc}(t)$. We'll use Rayleigh's Property of the Fourier Transform, since it's easier to calculate the frequency domain integral for the energy (seen in Equation (2.3.3)). From Table 2.1, $X(f)=\Pi(f)$.
The energy content of the signal is

$$E_X=\int_{-\infty}^{\infty}\Pi^2(f)df=\int_{-\frac{1}{2}}^{\frac{1}{2}}\Pi(f)df=1$$

so it's energy type. Again, we get the energy spectral density by $\mathcal{G}_X(f)=|X(f)|^2=\Pi^2(f)=\Pi(f)$.

3) $x(t)=\sum_{n=-\infty}^{\infty}\Lambda(t-2n)$. The signal is periodic and thus it is not of the energy type, but power-type (see the paragraph right after equation (2.3.9)). Since this signal is periodic, we're free to use equation (2.3.10) by setting $\tau=0$. So, the power content of the signal is

$$\begin{aligned}P_x &= \frac{1}{2}\int_{-1}^{1}|x(t)|^2dt=\frac{1}{2}\left(\int_{-1}^{0}(t+1)^2dt+\int_0^1(-t+1)^2dt\right)\\ &= \frac{1}{2}\left(\frac{1}{3}t^3+t^2+t\right)\Big|_{-1}^{0}+\frac{1}{2}\left(\frac{1}{3}t^3-t^2+t\right)\Big|_0^1\\ &= \frac{1}{3}\end{aligned}$$

For the power spectral density, we can use equation (2.3.13), so if the Fourier series coefficients of $x(\cdot)$ are $x_k = \frac{1}{(k\pi)^2}\cos(k\pi)$, then

$$S_x(f) = \sum_{k=-\infty}^{\infty} |x_k|^2 \delta(f - \frac{k}{2}) = \sum_{k=-\infty}^{\infty} \left| \frac{1}{(k\pi)^2}\cos(k\pi) \right|^2 \delta(f - \frac{k}{2}) = \sum_{k=-\infty}^{\infty} \frac{1}{(k\pi)^4}\delta(f - \frac{k}{2})$$

4) From the top of page 41, we use

$$E_x = \int_{-\infty}^{\infty} |u_{-1}(t)|^2 dt = \int_{-\infty}^{\infty} u_{-1}^2(t) dt = \int_0^{\infty} 1 \; dt = t \; |_0^{\infty} \to \infty$$

Thus, the signal is not of the energy type. Now trying equation (2.3.5)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \quad t \; |_0^{T/2} = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Hence, the signal is of the power type and its power content is 1/2. To find the power spectral density we find first the autocorrelation $R_{X,\tau}(\tau)$ from equation (2.3.4)

$$\begin{aligned} R_{X,\tau}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}(t) u_{-1}(t-\tau) dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{\tau}^{\frac{T}{2}} 1 \; dt \\ &= \lim_{T\to\infty} \frac{1}{T}(\frac{T}{2} - \tau) = \frac{1}{2} \end{aligned}$$

Thus, $\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)] = \frac{1}{2}\delta(f)$.

5) Again, we'll use Rayleigh's Property of the Fourier Transform, since it's easier to calculate the frequency domain integral for the energy (seen in Equation (2.3.3)).
From Table 2.1 we have that $|X(f)|^2 = \pi^2 \mathrm{sgn}^2(f) = \pi^2$ and $E_X = \int_{-\infty}^{\infty} \pi^2 dt \to \infty$. The signal is not of the energy type for the energy content is not bounded.
Now, we see if it's power type

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\frac{1}{t}|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} t^{-2} dt = \lim_{T\to\infty} \frac{1}{T} (-1/t) \; |_{-\frac{T}{2}}^{\frac{T}{2}} = 0$$

The signal is not of the power type either.

Problem 3)

1) The energy spectral density is (see top of page 42) is just

$$\begin{aligned} G_x(f) = |Y(f)|^2 &= |X(f) \; H(f)|^2 \quad \text{just because it's an LTI system.} \\ &= |X(f)|^2 \; |H(f)|^2 \\ &= |\frac{1}{\alpha + j2\pi f} \star \frac{1}{2}(\delta(f - \beta/2\pi) - \delta(f + \beta/2\pi))|^2 \; |\frac{1}{\gamma + j2\pi f}|^2 \\ &= |\frac{1}{2}\frac{1}{\alpha + j2\pi(f - \beta/2\pi)} + \frac{1}{2}\frac{1}{\alpha + j2\pi(f + \beta/2\pi)}|^2 \; |\frac{1}{\gamma + j2\pi f}|^2 \end{aligned}$$

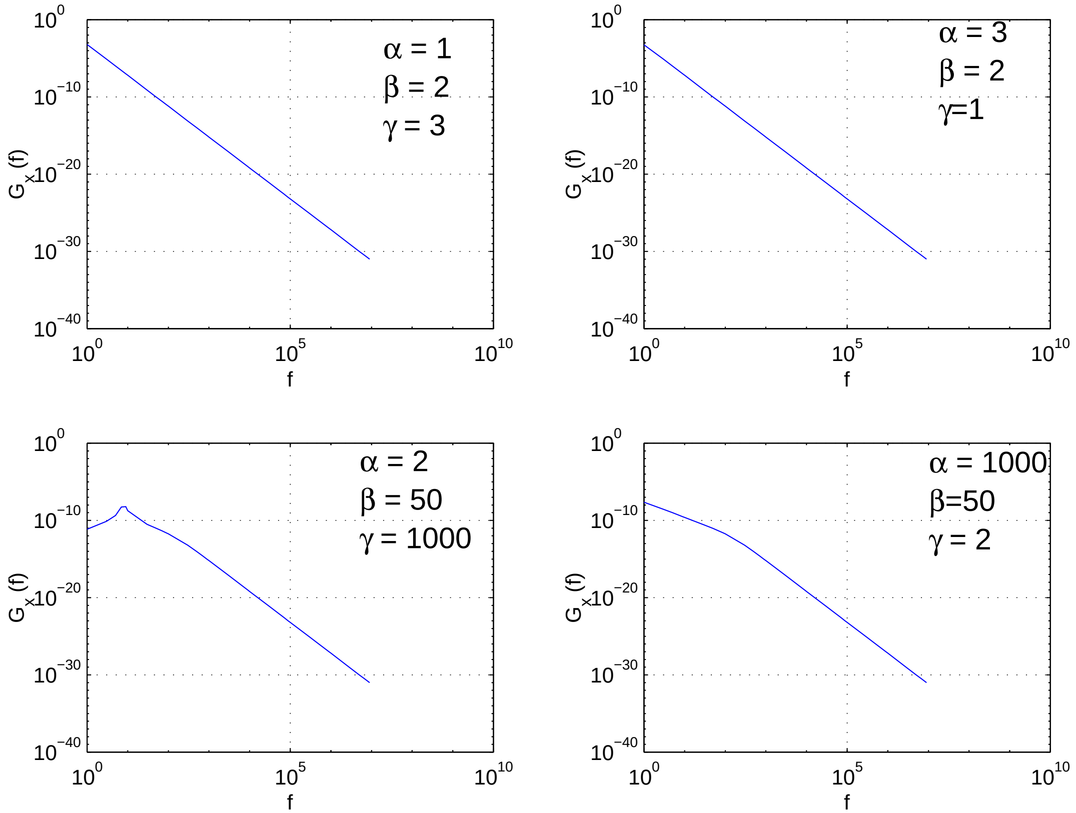

Figure 1: Problem 2 a

The expression becomes difficult to handle, so using a computer aid, such as Matlab (copyright Mathworks), you obtain an output as shown in Figure 1.

2) $H(f) = \frac{1}{\gamma + j2\pi f} \Longrightarrow |H(f)|^2 = \frac{1}{\gamma^2 + 4\pi^2 f^2}$. The energy spectral density of the output is

$$\mathcal{G}_Y(f) = |X(f)|^2 \; |H(f)|^2 = \frac{1}{\gamma^2 + 4\pi^2 f^2} \Pi(f)$$

The energy content of the signal is (see equation (2.3.3))

$$\begin{aligned} E_Y &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\gamma^2 + 4\pi^2 f^2} df = \frac{1}{2\pi\gamma} \arctan \frac{f\gamma}{2\pi} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{1}{\pi\gamma} \arctan \frac{\gamma}{4\pi} \end{aligned}$$

3) The power spectral density of the output is (see equation (2.3.15))

$$\begin{aligned} \mathcal{S}_Y(f) &= \sum_{n=-\infty}^{\infty} |x_n|^2 \left| H(\frac{n}{2}) \right|^2 \delta(f - \frac{n}{2}) \\ &= \frac{1}{4\gamma^2} \delta(f) + 2 \sum_{l=0}^{\infty} \frac{|x_{2l+1}|^2}{\gamma^2 + \pi^2 (2l+1)^2} \delta(f - \frac{2l+1}{2}) \\ &= \frac{1}{4\gamma^2} \delta(f) + \frac{8}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4 (\gamma^2 + \pi^2 (2l+1)^2)} \delta(f - \frac{2l+1}{2}) \end{aligned}$$

The power content of the output signal is (see the equation at the bottom of page 44)

$$
\begin{aligned}
P_Y &= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{2})|^2 \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2} \sum_{l=0}^{\infty} \left[\frac{1}{\gamma^2(2l+1)^4} + \frac{\pi^4}{\gamma^4(\gamma^2+\pi^2(2l+1)^2)} - \frac{\pi^2}{\gamma^4(2l+1)^2} \right] \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2} \left(\frac{\pi^2}{\gamma^2 96} - \frac{\pi^4}{8\gamma^4} + \frac{\pi^2}{\gamma^4} \sum_{l=0}^{\infty} \frac{1}{\frac{\gamma^2}{\pi^2} + (2l+1)^2} \right) \\
&= \frac{1}{3\gamma^2} - \frac{\pi^2}{\gamma^4} + \frac{2\pi^2}{\gamma^5} \tanh(\frac{\gamma}{2})
\end{aligned}
$$

where we have used the fact

$$
\tanh(\frac{\pi x}{2}) = \frac{4x}{\pi} \sum_{l=0}^{\infty} \frac{1}{x^2+(2l+1)^2}, \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}
$$

4) The power spectral density of the output signal is (see equation (2.3.15))

$$
\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{1}{2}\frac{1}{\gamma^2+4\pi^2 f^2}\delta(f) = \frac{1}{2\gamma^2}\delta(f)
$$

The power content of the signal is (see equation (2.3.7))

$$
P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f) df = \frac{1}{2\gamma^2}
$$

5) The signal here was not energy-type or power-type. We simply use the fact that it's an LTI system. $X(f) = -j\pi \text{sgn}(f)$ so that $|X(f)|^2 = \pi^2$ for all f except $f = 0$ for which $|X(f)|^2 = 0$. Thus, the energy spectral density of the output is

$$
\mathcal{G}_Y(f) = |X(f)|^2 |H(f)|^2 = \frac{\pi^2}{\gamma^2 + 4\pi^2 f^2}
$$

and the energy content of the signal

$$
E_Y = \pi^2 \int_{-\infty}^{\infty} \frac{1}{\gamma^2+4\pi^2 f^2} df = \pi^2 \frac{1}{2\pi\gamma} \arctan(\frac{f 2\pi}{\gamma}) \Big|_{-\infty}^{\infty} = \frac{\pi^2}{2\gamma}
$$

Problem 4) Let $y(t)$ be the output signal, which is the convolution of $x(t)$, and $h(t)$, $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. Using the Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\
&\leq \left[\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau \right]^{\frac{1}{2}} \\
&\leq E_h^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau \right]^{\frac{1}{2}} \qquad \text{where we've defined } E_h \text{ to be that integral above it}
\end{aligned}
$$

Squaring the previous inequality and integrating from $-\infty$ to ∞ we obtain

$$\int_{-\infty}^{\infty} |y(t)|^2 dt \le E_h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau dt$$

But by assumption $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau dt$, E_h are finite, so that the energy of the output signal is finite. ♯

Consider the LTI system with impulse response $h(t) = \sum_{n=-\infty}^{\infty} \Pi(t-2n)$. The signal is periodic with period $T=2$, and the power content of the signal is $P_H = \frac{1}{2}$. If the input to this system is the energy type signal $x(t) = \Pi(t)$, then

$$y(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$$

which is a power type signal (hence, not energy type) with power content $P_Y = \frac{1}{2}$.

Problem 5) The more savvy student will have noticed that the signal $x(t)$ is, in fact, not a power-type signal, but an energy-type signal. This brings the question: why are the instructions to produce the *power content* and *power spectral density*?

Consider in a communications environment that just about every signal that we wish to transmit will be time-limited (has a beginning and an end and is zero beyond those borders). That would make all of them energy-type, at best. Yet, as a designer, we're interested not in the energy, but in the power content, and not of the entire signal (who cares about the $x(t) = 0$ for $t \notin [0, 10]$?) but only where the signal is interesting ($t \in [0, 10]$).

Therefore, we limit our power calculation to the area of interest. say that area has a length T. Then

$$P_x = \frac{1}{T}\int_0^T x^2(t)dt \quad ,$$

but in Matlab, we have to approximate this integral with discrete samples of the signal. Say we sample the signal every T_s over the portion of interest of T seconds. Then, we'll end up with some number of samples, N, where $T = N \cdot T_s$.

$$\begin{aligned} P_x &= \underbrace{\frac{1}{T}}\ \underbrace{\int_0^T x^2(t)dt} && \text{power content for the area of interest of } x(t) \\ &= \tfrac{1}{N\cdot T_s}\ T_s \sum_{n=0}^{N-1} x^2[n] && \text{think of the definition of the Riemann integral } \lim_{\Delta t \to 0} \sum_k f(t_k)\Delta t \\ &= \tfrac{1}{N} \sum_{n=0}^{N-1} x^2[n] \end{aligned}$$

Therefore, the Matlab procedure may look like this:

```
>> ts=0.001;                % This is the sampling interval (1000 samples per second).

>> t=[0:ts:10];              % This is the time line.

>> x=cos(2*pi*47*t) + cos(2*pi*219*t);              % This is the function.
```

```
>> power=norm(x)^2/length(x)            % This is the power content.

power =

                  1.0003
```

For the Power Spectral density, you may refer to homework 1 for getting the Fourier Transform of a signal, or you may use Matlab's in-built functions: spectrum.m and specplot.m (both are described by Matlab's help).

```
>> powerspectraldens = spectrum(x);            % This estimates the PSD.

>> specplot(powerspectraldens, 1/ts);          % This plots it. Here 1/ts is the sampling frequency.
```

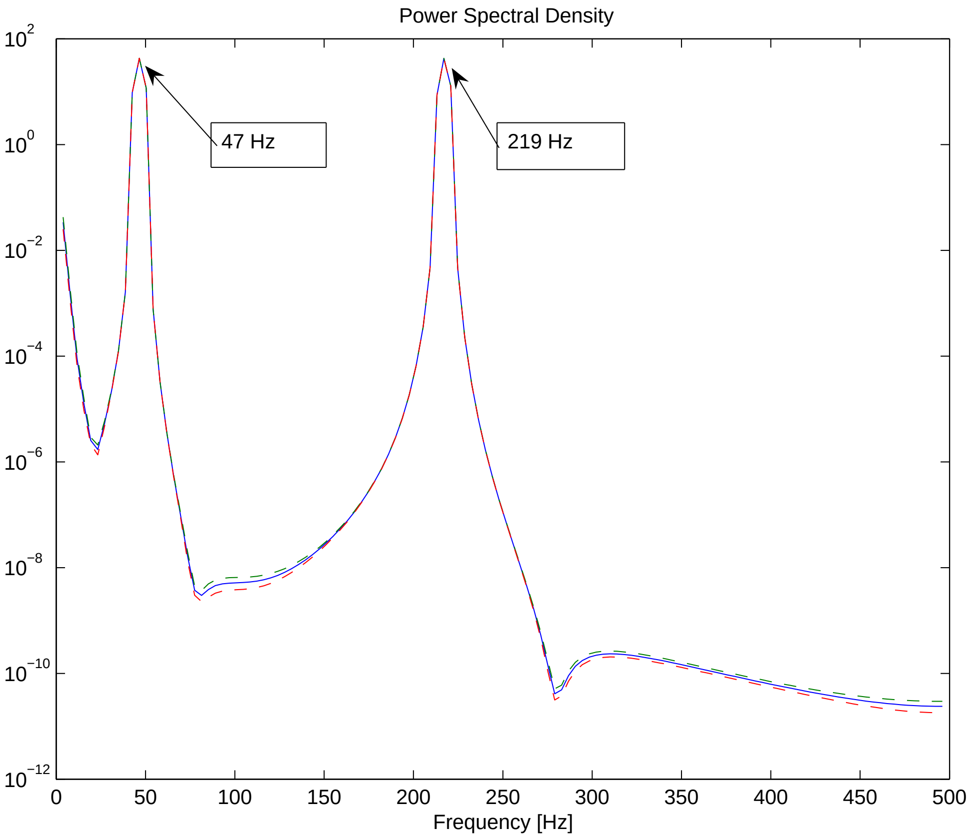

Figure 2: Plot of the Power Spectral Density of $x(t)$ for the area of interest.