Homework 3 - Solutions

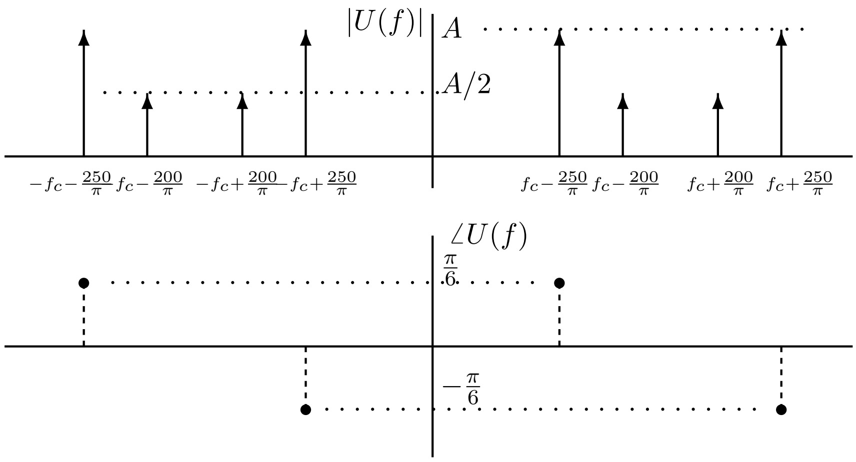
Problem 1) The modulated signal (see top of page 72) is

$$
\begin{aligned}
u_{DSB-SC}(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t)\\
&\quad \text{We'll just write things in the same format of } \cos(2\pi Dt+E)\\
u_{DSB-SC}(t) &= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t+\frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t)\\
&\quad \text{Now, use} \quad \cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}\\
&\quad \text{as well as} \quad \sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}\\
u_{DSB-SC}(t) &= A\cos(2\pi(4\times 10^3+\frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3-\frac{200}{\pi})t)\\
&\quad +2A\sin(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})
\end{aligned}
$$

You can take the Fourier transform of any of the previous expressions. It seems easier to take the Fourier transform of the second expression as use the "multiplication in time domain equals convolution is frequency domain" property:

$$
\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi}) + \delta(f+\frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right]\\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3)+\delta(f+4\times 10^3)]\\
&= \frac{A}{2}\Big[\delta(f-4\times 10^3-\frac{200}{\pi}) + \delta(f-4\times 10^3+\frac{200}{\pi})\\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3+\frac{250}{\pi})\\
&\quad +\delta(f+4\times 10^3-\frac{200}{\pi}) + \delta(f+4\times 10^3+\frac{200}{\pi})\\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3+\frac{250}{\pi})\Big]
\end{aligned}
$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



To find the power content of the modulated signal we write $u^2(t)$ as

$$
u^2_{DSB-SC}(t) = A^2\cos^2(2\pi(4\times 10^3+\frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times 10^3-\frac{200}{\pi})t)
$$

$$+4A^2\sin^2(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3})+4A^2\sin^2(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})$$

+terms of cosine and sine functions in the first power

which integrate to zero. (1)

Hence,

$$P=\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)dt=\frac{A^2}{2}+\frac{A^2}{2}+\frac{4A^2}{2}+\frac{4A^2}{2}=5A^2$$

Problem 2) FOR DSB-SC, the information carrying signal multiplies the cosine carrier wave. So you get a cosine whose envelope looks like the P-3.3 figures.

Figure 1 shows the modulated signals for $A = 1$ and $f_0 = 10$ for $m_1(t)$, $Am_1(t)\cos(2\pi f_0 t)$. When $m_2(t)$ is used, $Am_2(t)\cos(2\pi f_0 t)$, you get the signal seen in Figure 2.

As observed, both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal with respect to the first signal. This discontinuity is shown clearly in Figure 3 where we plot $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.

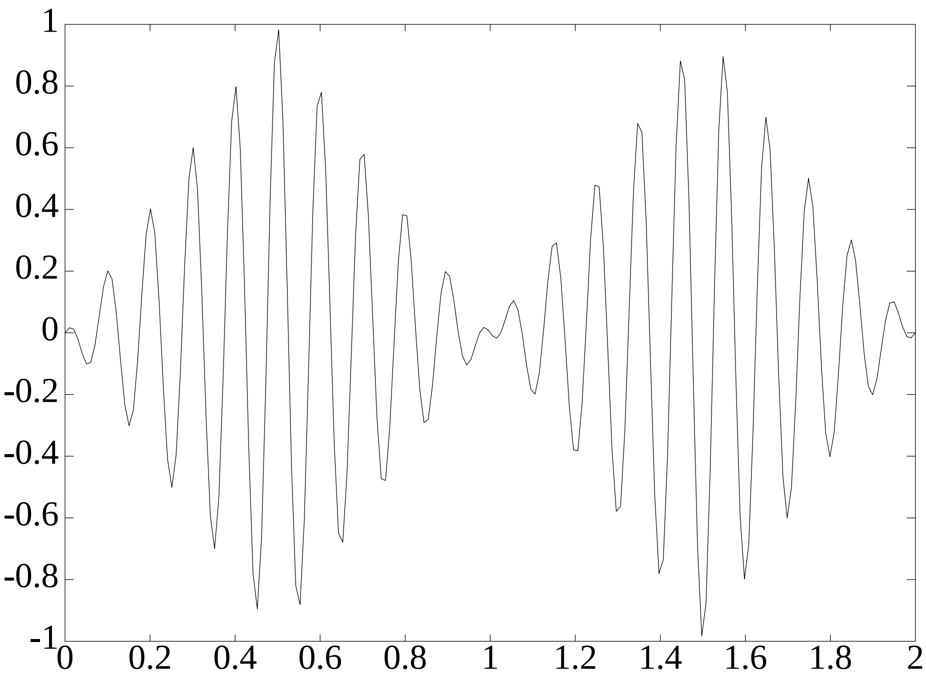


Figure 1: Problem 2

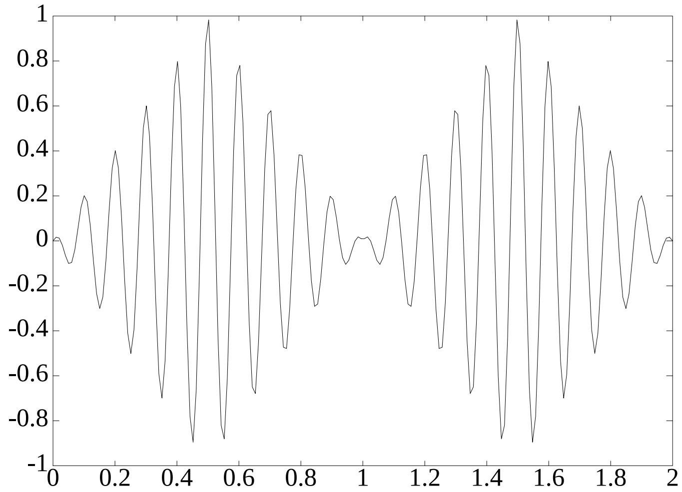


Figure 2: Problem 2

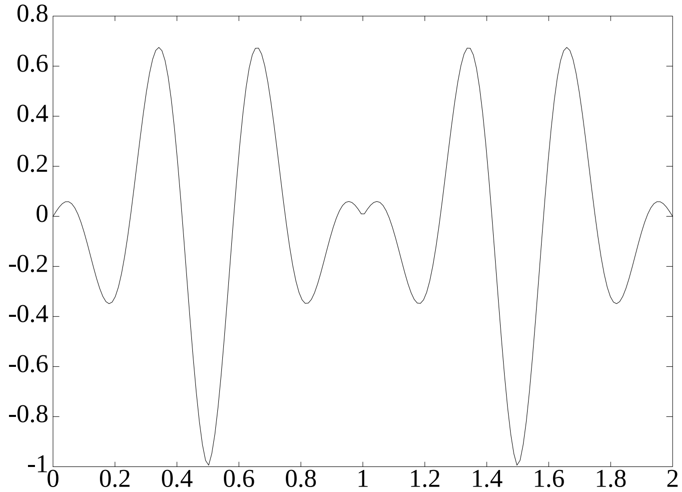


Figure 3: Problem 2.

Problem 3) We know that the USB could theoretically be obtained by producing a DSB-SC signal, and then passing that through a bandpass signal (*Recall, this is good for analysis, but we saw in class that it's a poor way to implement it!!*). See Figure 3.9 of the text, where

$$H_{bp}(f) = \begin{cases} 1 & |f| > f_c \\ 0 & \text{otherwise} \end{cases}$$

1) We simply rewrite the modulated signal in a way that shows the frequency content clearly... easy! They're just cosines!

$$\begin{aligned} u(t) &= m(t)c(t) \\ &= 100(\cos(2\pi 1000t) + 2\cos(2\pi 2000t))\cos(2\pi f_c t) \\ &= 100\cos(2\pi 1000t)\cos(2\pi f_c t) + 200\cos(2\pi 2000t)\cos(2\pi f_c t) \\ &\quad \text{See trigonometric identities from problem 1.} \\ &= \frac{100}{2}\left[\cos(2\pi(f_c + 1000)t) + \cos(2\pi(f_c - 1000)t)\right] \end{aligned}$$

$$\frac{200}{2}\left[\cos(2\pi(f_c+2000)t)+\cos(2\pi(f_c-2000)t)\right]$$

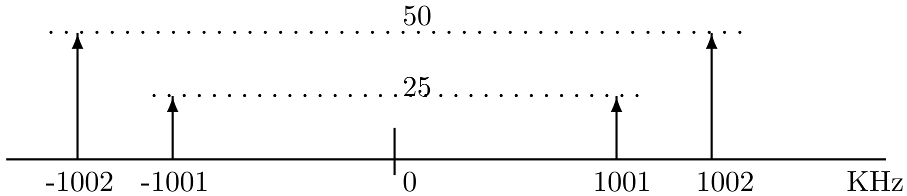
So, by passing this signal through the filter $H_{bp}(f)$, we're discarding all parts with frequencies below f_c. Thus, the upper sideband (USB) signal is whatever is left over:

$$u_{USB}(t) = 50\cos(2\pi(f_c+1000)t) + 100\cos(2\pi(f_c+2000)t)$$

2) Taking the Fourier transform of $u_{USB}(t)$, we obtain

$$\begin{aligned} U_{USB}(f) &= 25\left(\delta(f-(f_c+1000))+\delta(f+(f_c+1000))\right) \\ &\quad +50\left(\delta(f-(f_c+2000))+\delta(f+(f_c+2000))\right) \end{aligned}$$

A plot of $U_{USB}(f)$ is given in the next figure.



Problem 4)

1) The modulated signal is

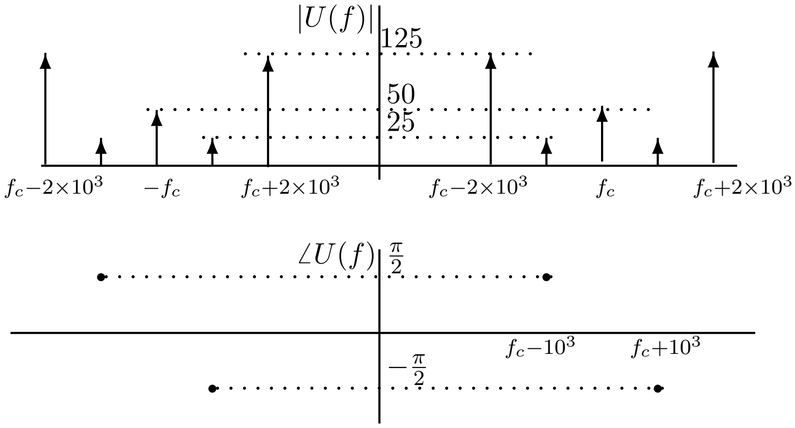
$$\begin{aligned} u(t) &= 100[1+m(t)]\cos(2\pi 8\times 10^5 t) \\ &\quad \text{Now, we just write it out:} \\ &= 100\cos(2\pi\ 8\times 10^5\ t) + 100\sin(2\pi\ 10^3\ t)\cos(2\pi\ 8\times 10^5\ t) \\ &\quad +500\cos(2\pi\ 2\times 10^3\ t)\cos(2\pi\ 8\times 10^5\ t) \\ &\quad \text{Use the same trigonometric identities used in problem 1} \\ &= 100\cos(2\pi\ 8\times 10^5\ t) + 50[\sin(2\pi\ (10^3+8\times 10^5)\ t) - \sin(2\pi\ (8\times 10^5-10^3)\ t)] \\ &\quad +250[\cos(2\pi\ (2\times 10^3+8\times 10^5)\ t)+\cos(2\pi\ (8\times 10^5-2\times 10^3)\ t)] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50[\delta(f-8\times 10^5)+\delta(f+8\times 10^5)] \\ &\quad +25\left[\frac{1}{j}\delta(f-8\times 10^5-10^3)-\frac{1}{j}\delta(f+8\times 10^5+10^3)\right] \\ &\quad -25\left[\frac{1}{j}\delta(f-8\times 10^5+10^3)-\frac{1}{j}\delta(f+8\times 10^5-10^3)\right] \\ &\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \\ &\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \end{aligned} \tag{2}$$

The $\frac{1}{j}$ terms affect the phase, so we must not ignore them. We can rewrite them using a format that leaves the phase readily readable:

$$
\begin{aligned}
U(f) &= 50[\delta(f-8\times 10^5)+\delta(f+8\times 10^5)] \\
&+25\left[\delta(f-8\times 10^5-10^3)e^{-j\frac{\pi}{2}}+\delta(f+8\times 10^5+10^3)e^{j\frac{\pi}{2}}\right] \\
&+25\left[\delta(f-8\times 10^5+10^3)e^{j\frac{\pi}{2}}+\delta(f+8\times 10^5-10^3)e^{-j\frac{\pi}{2}}\right] \\
&+125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \\
&+125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right]
\end{aligned}
$$



2) The average power in the carrier is

$$P_{\text{carrier}} = \frac{A_c^2}{2} = \frac{100^2}{2} = 5000$$

The power in the sidebands is (see page 80, or class notes)

$$P_{\text{sidebands}} = \frac{50^2}{2} + \frac{50^2}{2} + \frac{250^2}{2} + \frac{250^2}{2} = 65000$$

3) The modulation index is described in pages 78 and 79. We write $m(t)$ in a way that we can easily check what its minimum value is, and then we see what factor we must multiply it so we get -1. The message signal can be written as

$$
\begin{aligned}
m(t) &= \sin(2\pi\ 10^3\ t) + 5\cos(2\pi\ 2\times 10^3\ t) \\
&\quad \text{using the identity} \cos(2a) = -2\sin^2(a)+1 \\
&= \sin(2\pi\ 10^3\ t) - 10\sin^2(2\pi\ 10^3\ t) + 5
\end{aligned}
$$

The minimum value of $m(t)$ is -6 (you can plot the expression to obtain it, if you like). Hence, the modulation index is $\alpha = 6$.

4) The power delivered to the load is $p = v\cdot i = v\cdot v/r = v^2/r$.

$$P_{\text{load}} = \frac{|u(t)|^2}{50} = \frac{100^2(1+m(t))^2\cos^2(2\pi f_c t)}{50}$$

The maximum absolute value of is about

$$\max(P_{\text{load}}) \approx 7256.$$

Problem 5)

1) So we want to write it in the format given by equations (3.3.2) and (3.3.6) of the book. So, we have f_c (well, assume it's given), and we have $m(t)$, but we need k_f. We can use the general definition for k_f given in equation (3.3.17) of the text. For this, we'll need the bandwidth of the message, W. Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_0^t \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is (see equation (3.3.9) of the text and replace k_f by using equation (3.3.17))

$$\Delta f_{\max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule (equation (3.3.45) of the text), the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6+1)200 = 2800 \text{ Hz}$$