COHERENCE IN SIGNAL PROCESSING: A FUNDAMENTAL RE-DEFINITION

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ABSTRACT

The term "coherence" in the signal processing community has been defined for nearly 50 years in terms of a narrowband statistical power spectral density ratio. This definition is very limiting with modern digital electronics and sensor array technology that is very broadband and which uses sophisticated wideband modulations. In this paper, we provide a fundamental overhaul of the definition of coherence, with many specialized variations that revert to the old coherence definition in one case and that also handle the wideband cases as well.

1. INTRODUCTION

The concept of **coherence** originated in the electromagnetic [EM] spectrum of light literature around the turn of the century. The terminology **coherent signal** and **coherent signal processing** is now found in the modern signal processing literature, yet the signal processing community has not formally defined a concept of coherence applicable to generic signals from any source, particularly those sources capable of generating sophisticated or broadband (wideband) waveforms. The community has instead continued to rely on outdated concepts of coherence originally appropriate for unsophisticated narrowband EM waves. A two-signal metric, composed of the cross spectral density between the signals normalized by the product of each signal's auto spectral density, is currently the only measure used by the community that carries the name **coherence**, yet it is relevant only for narrowband stochastic signals. The author has been hampered in his work with digital beamforming systems in describing precisely the sense of coherence being conveyed using the existing inadequate coherence concepts, and this has motivated this article. It is the objective of this article to formally establish the concept of coherence as used in the signal processing community and to provide a number of coherence definitions, metrics, and estimation approaches appropriate for a variety of signal conditions.

The most common attribute ascribed to a coherent signal is its phase, often expressed relative to some reference signal (most often a sinusoidal waveform). This is a legacy of the original use of the term **coherent** in optics to describe interference effects of monochromatic (single frequency) light in which a reference light source is combined with another light source to produce a fringe pattern. The earliest reference found by the author that defines coherent appeared in a German text by Drude [1900] that was translated [1902] and later republished as a Dover paperback [1959]. On page 134 of the Dover publication, it states: "For if two sources are to produce interference, their phases must always be either exactly the same or else must have a constant difference. Such sources are called **coherent**." The IEEE Standard Dictionary [1984] similarly defines **coherent** as "characterized by a fixed phase relationship between points on an electromagnetic wave." The dictionary further defines **coherent signal processing** in the radar context as "echo integration, filtering or detection using amplitude and phase of the signal referred to a coherent oscillator, an oscillator that provides a reference phase by which changes in the radio-frequency [RF] phase of successively received pulses may be recognized."

There are those who believe that coherency is an inherent attribute of individual signals that have preserved their "amplitude and phase". If this were so, then white noise that preserves amplitude and phase would be coherent with itself by correlating the noise signal by selecting any two points along the time waveform. This is self-evidently an erroneous concept. One of the key points of this article is that coherence is actually relative; that is, the degree of coherence is established with respect to a reference signal, often in conjunction with other reference conditions. This is reminescent of the decibel scale for signal power. Stating that the signal level is so many dB is meaningless unless the reference power (for example, 1 milliwatt=0 dB) is also stated. This article seeks to debunk the frequently made statement that preservation of a signal's amplitude and phase is a sufficient condition to establish coherency. The source of the error is again the failure to place the preservation of signal information in the context of (relative to) a reference. In fact, we shall show that a signal can still be coherent even if the phase information is deleted/filtered out.

2. REAL SIGNALS AND COMPLEX SIGNALS

If a signal $x(t)$ is complex-valued, there are two common complex representations for describing such a signal. The **real–imaginary representation** describes the signal as

$$x(t) = \mathrm{Re}\{x(t)\} + j\Im\{x(t)\} = x_r(t) + jx_i(t) \tag{1}$$

in which $\mathrm{Re}\{x(t)\} = x_r(t)$ is the real component, $\Im\{x(t)\} = x_i(t)$ is the imaginary component, and $j = (-1)^{1/2}$. Note that both $x_r(t)$ and $x_i(t)$ are real-valued functions, despite the adjective imaginary used to describe $x_i(t)$. The polar coordinate transformation

$$a(t) = \left[x_r^2(t) + x_i^2(t)\right]^{1/2} \tag{2}$$

$$\phi(t) = \arctan\left\{x_i(t)/x_r(t)\right\}, \tag{3}$$

yields an equivalent complex representation known as the **amplitude–phase representation**

$$x(t) = a(t)\exp\{j\phi(t)\}, \tag{4}$$

in which $a(t)$ is the time-varying amplitude component and $\phi(t)$ is the time-varying phase component. Note by definition that the amplitude component $a(t)$ must be a postive real value. The inverse transformation

$$x_r(t) = a(t)\cos\{\phi(t)\} \tag{5}$$

$$x_i(t) = a(t)\sin\{\phi(t)\} \tag{6}$$

Continuous-time Fourier transform for finite-energy deterministic signal (real and complex cases)

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \tag{7}$$

Continuous-time energy theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df \tag{8}$$

Energy is preserved under time shift, frequency shift, and time/frequency scaling ($a > 0$):

$$E = \int_{-\infty}^{\infty} |x(t-\tau)|^2\,dt \tag{9}$$

$$= \int_{-\infty}^{\infty} |x(t)\exp(+j2\pi\mathrm{f}t)|^2\,dt \tag{10}$$

$$= \int_{-\infty}^{\infty} |\sqrt{a}x(at)|^2\,dt \tag{11}$$

$$= \int_{-\infty}^{\infty} |X(f)\exp(-j2\pi f\tau)|^2\,df \tag{12}$$

$$= \int_{-\infty}^{\infty} |X(f-\mathrm{f})|^2\,df \tag{13}$$

$$= \int_{-\infty}^{\infty} |X(f/a)/\sqrt{a}|^2\,df \tag{14}$$

Continuous-time Parseval theorem

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)\,df \tag{15}$$

Sampled bandlimited signal $x[n] = x(nT)$ to produce discrete-time Fourier transform

$$X(f) = T\sum_{n=-\infty}^{\infty} x[n]\exp(-j2\pi fnT) \tag{16}$$

for $|f| \le 1/2T$ periodicity. Discrete-time energy theorem

$$E = T\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-1/2T}^{1/2T} |X(f)|^2\,df \tag{17}$$

Discrete-time Parseval theorem

$$T\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \int_{-1/2T}^{1/2T} X_1(f)X_2^*(f)\,df \tag{18}$$

3. COHERENCE DEFINITIONS AND TWO-SIGNAL COHERENCE METRICS

IEEE Standard Dictionary [1984] defines **coherence** as "the correlation between electromagnetic fields at points which are separated in space or in time, or both."

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt \tag{19}$$

is a matched filter process. Broadband continuous-time **temporal correlation coherence** (TCC) (complex-valued in general) normalizes the correlation value

$$c_t = \frac{\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt}{\left(\int_{-\infty}^{\infty} |x_1(t)|^2\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} |x_2(t)|^2\,dt\right)^{1/2}} \tag{20}$$

$$= \frac{\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt}{E_1^{1/2}E_2^{1/2}} \tag{21}$$

to produce, due to Schwarz inequality, unity magnitude bounds $0 \leq |c_t|^2 \leq 1$ for complex-valued $x(t)$ or $-1 \leq c_t \leq 1$ for real-valued $x(t)$. Here $|c_t|$ is the **magnitude TCC** or simply the **magnitude coherence**, $|c_t|^2$ is the **magnitude-squared TCC** or simply **magnitude-squared coherence** (MSC), and $\theta = \arctan(\mathrm{Im}\{c_t\}/\mathrm{Re}\{c_t\})$ is the **TCC phase** or simply **coherence phase**. The integration range can be set to the range t_1 to t_2 if the signal energy is essentially within this finite time interval; it is also the practical situation for implementation with actual signals as infinite integration intervals are not possible and only analysis windows can be used. Whenever one of the denominator terms is zero, then there is no energy from that signal with which to be correlated and, therefore, the coherence metric is defined to be zero. This will be the case with any of the coherence metrics in this text: a zero-value denominator term yields a zero-value coherence. The broadband continuous-time **spectral (frequency) correlation coherence** (SCC) (complex-valued in general) normalizes the spectral correlation value

$$c_f = \frac{\int_{-\infty}^{\infty} X_1(f)X_2^*(f)\,df}{\left(\int_{-\infty}^{\infty} |X_1(f)|^2\,df\right)^{1/2}\left(\int_{-\infty}^{\infty} |X_2(f)|^2\,df\right)^{1/2}} \tag{22}$$

$$= \frac{\int_{-\infty}^{\infty} X_1(f)X_2^*(f)\,df}{E_1^{1/2}E_2^{1/2}} \tag{23}$$

If a real-valued signal is essentially bandlimited to a range between f_1 to f_2, then the infinite integration range above should be limited to $-f_2 \leq f \leq -f_1$ and $f_1 \geq f \geq f_2$. In the case of a complex-valued signal, the inegration range is simply limited to f_1 to f_2 where f_1 and f_2 can be either positive or negative frequencies. Due to continuous-time energy theorem and Parseval's theorem, $c_t = c_f = c$. In practical situations, one cannot use infinite integration intervals, and finite time or frequence intervals will yield estimates $\hat{c}_t$ and $\hat{c}_f$ that are not identical (the symbol ˆ denotes estimate). Special case for the traditional single sinusoid at frequency f_0 is

$$c = \frac{X_1(f_0)X_2^*(f_0)}{|X_1(f_0)||X_2(f_0)|} = \exp(j\theta_1 - j\theta_2) \tag{24}$$

which has unit magnitude and the coherence isa function only of the phase difference.

Broadband discrete-time **temporal correlation coherence** (TCC) (complex-valued in general) normalizes the correlation value

$$c_t = \frac{T\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]}{\left(T\sum_{n=-\infty}^{\infty} |x_1[n]|^2\right)^{1/2}\left(T\sum_{n=-\infty}^{\infty} |x_2[n]|^2\right)^{1/2}} \tag{25}$$

$$= \frac{T\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]}{E_1^{1/2}E_2^{1/2}} \tag{26}$$

to produce, due to Cauchy inequality, unity magnitude bounds $0 \leq |c_t|^2 \leq 1$ for complex-valued $x[n]$ or $-1 \leq c_t \leq 1$ for real-valued $x[n]$. Here $|c_t|$ is the **magnitude TCC** or simply the **magnitude coherence**, $|c_t|^2$ is the **magnitude-squared TCC** or simply **magnitude-squared coherence** (MSC), and $\theta = \arctan(\mathrm{Im}\{c_t\}/\mathrm{Re}\{c_t\})$ is the **TCC phase** or simply **coherence phase**. Note that the factor T can be cancelled from both the numerator and denominator. The broadband discrete-time **spectral (frequency) correlation coherence** (SCC)

$$c_f = \frac{\int_{-1/2T}^{1/2T} X_1(f)X_2^*(f)\,df}{\left(\int_{-1/2T}^{1/2T} |X_1(f)|^2\,df\right)^{1/2}\left(\int_{-1/2T}^{1/2T} |X_2(f)|^2\,df\right)^{1/2}} \tag{27}$$

$$= \frac{\int_{-1/2T}^{1/2T} X_1(f)X_2^*(f)\,df}{E_1^{1/2}E_2^{1/2}} \tag{28}$$

Due to discrete-time energy theorem and Parseval's theorem, $c_t = c_f = c$.

4. ENERGY COHERENCE METRICS

Broadband continuous-time **temporal energy coherence** (TEC) (real-valued in general) normalizes the summed signals energy to produce the semblance value

$$e_t = \frac{\int_{-\infty}^{\infty} |x_1(t) + x_2(t)|^2\,dt}{2\left[\int_{-\infty}^{\infty} |x_1(t)|^2\,dt + \int_{-\infty}^{\infty} |x_2(t)|^2\,dt\right]} \tag{29}$$

$$= \frac{\int_{-\infty}^{\infty} |x_1(t) + x_2(t)|^2\,dt}{2\left[E_1 + E_2\right]} \tag{30}$$

to produce, due to inequality, unity magnitude bounds $0 \leq e_t \leq 1$ for real-valued or complex-valued $x(t)$. The broadband continuous-time **spectral (frequency) energy coherence** (SEC), aka **semblance**

$$e_f = \frac{\int_{-\infty}^{\infty} |X_1(f) + X_2(f)|^2\,df}{2\left[\int_{-\infty}^{\infty} |X_1(f)|^2\,df + \int_{-\infty}^{\infty} |X_2(f)|^2\,df\right]} \tag{31}$$

$$= \frac{\int_{-\infty}^{\infty} |X_1(f) + X_2(f)|^2\,df}{2\left[E_1 + E_2\right]} \tag{32}$$

Due to continuous-time energy theorem, $e_t = e_f = e$. Special case for the traditional single sinusoid at frequency f_0 is

$$e = \frac{|X_1(f_0) + X_2(f_0)|^2}{2\left[|X_1(f_0)|^2 + |X_2(f_0)|^2\right]} \tag{33}$$

which has unit magnitude if both sinusoids have the same amplitude and phase. Analysis when either are different...

Broadband discrete-time **temporal energy coherence** (TEC) (real-valued in general) normalizes the summed signals value

$$e_t = \frac{T\sum_{n=-\infty}^{\infty} |x_1[n] + x_2[n]|^2}{2\left[T\sum_{n=-\infty}^{\infty} |x_1[n]|^2 + T\sum_{n=-\infty}^{\infty} |x_2[n]|^2\right]} \tag{34}$$

$$= \frac{T\sum_{n=-\infty}^{\infty} |x_1[n] + x_2[n]|^2}{2\left[E_1 + E_2\right]} \tag{35}$$

to produce, due to inequality, unity magnitude $0 \leq e_t \leq 1$ for real-valued or complex-valued $x[n]$. The broadband discrete-time **spectral (frequency) energy coherence** (SEC)

$$e_f = \frac{\int_{-1/2T}^{1/2T} |X_1(f) + X_2(f)|^2\, df}{2\left[\int_{-1/2T}^{1/2T} |X_1(f)|^2\, df + \int_{-1/2T}^{1/2T} |X_2(f)|^2\, df\right]} \tag{36}$$

$$= \frac{\int_{-1/2T}^{1/2T} |X_1(f) + X_2(f)|^2\, df}{2\left[E_1 + E_2\right]} \tag{37}$$

Due to discrete-time energy theorem, $e_t = e_f = e$.

5. COHERENCE FUNCTIONS

The coherence metrics all presumed that the signals were temporally-aligned, phase-aligned, frequency-aligned, and scale-aligned in order to compute a single coherence value. However, in practice one must adjust one or the other of the two signals to produce such alignment. Since one oftens searches with respect to time shift, phase shift, frequency shift, or scale, the various coherence metrics will become functions of the variable over which the search is performed. We illustrate one case below. Others will be shown at the conference.

6. TIME-SHIFT AND SPECTRAL-PHASE-SHIFT COHERENCE

For signals which are **coherent with respect to time shift (delay) operations**, use the time-shift (delay) correlation coherence $c(\tau)$ that is a function of temporal shift τ is defined by making the substitutions $x_1(t) = x(t)$ and $x_2(t) = y(t-\tau)$ yielding

$$c(\tau) = \frac{\int_{-\infty}^{\infty} x(t) y^*(t-\tau)\, dt}{\left(\int_{-\infty}^{\infty} |x(t)|^2\, dt\right)^{1/2} \left(\int_{-\infty}^{\infty} |y(t-\tau)|^2\, dt\right)^{1/2}} \tag{38}$$

$$= \frac{\int_{-\infty}^{\infty} X(f) Y^*(f) \exp(-j2\pi f\tau)\, df}{\left(\int_{-\infty}^{\infty} |X(f)|^2\, df\right)^{1/2} \left(\int_{-\infty}^{\infty} |Y(f)\exp(-j2\pi f\tau)|^2\, df\right)^{1/2}} \tag{39}$$

in which the denominators of $c(\tau)$ can be replaced by $E_x^{1/2} E_y^{1/2}$. Also applies to spatial-shift coherence. A signal can be self-coherent. The discrete-time version, which only permits integer delay intervals, may require interpolation in order to obtain a satisfactory estimate of delay. See section 8 for a method with the FFT for performing trigonmetric interpolation.

The time-shift (delay) energy coherence $e(\tau)$

$$e(\tau) = \frac{\int_{-\infty}^{\infty} |x(t) + y(t-\tau)|^2\, dt}{2\left[\int_{-\infty}^{\infty} |x(t)|^2\, dt + \int_{-\infty}^{\infty} |y(t-\tau)|^2\, dt\right]} \tag{40}$$

$$= \frac{\int_{-\infty}^{\infty} |X(f) + Y(f)\exp(-j2\pi f\tau)|^2\, df}{2\left[\int_{-\infty}^{\infty} |X(f)|^2\, df + \int_{-\infty}^{\infty} |Y(f)\exp(-j2\pi f\tau)|^2\, df\right]} \tag{41}$$

in which the denominators of $e(\tau)$ can be replaced by $2[E_x + E_y]$.

7. REFERENCES

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