7

The Analytical Design of Discrete Systems

7.1 Introduction

The analytical design of discrete systems presents a most interesting challenge. A digital processor, which may be a general-purpose computer or a special-purpose switching circuit, may be effectively employed to generate a control input to a system. The use of a digital computer as a controller of a system is particularly attractive, since the implementation of a control law requires only the preparation of a computer program. A program permits almost unlimited flexibility.

In this chapter we shall investigate a variety of design techniques, all of which are aimed at defining a control algorithm implementable as a computer program. As with all design techniques, it will be necessary to employ a design objective or performance criterion. Usually a design objective implies the optimization or minimization of a certain factor influencing system performance. In so doing, we are applying the concepts of optimal control theory, which has been developed in the last decade. Most of the optimization problems postulated in this chapter have solutions that are obtained by solving a

set of linear equations. Such is not the case in the continuous counterpart of optimization theory.

7.2 Time-domain Synthesis with Minimum Settling Time

In this section we will consider procedures that utilize state transition matrices to generate computer algorithms. The design objective will be to achieve minimum settling time in system response. In order to make the presentation of the required calculations possible, it is necessary to restrict the discussion to simple examples. However, in each example given, an attempt will be made to outline a more general case.

A number of different design approaches may be taken to insure that the system's output $c(t)$ will equal its input $r(t)$ in the minimum number of sample times (N). For example, it is possible to synthesize a controller such that $c(t) = r(t)$ at the sampling times but not necessarily in between. Alternatively, it may be desired to have $c(t) = r(t)$ for all $t \geq NT$ for the smallest value of N. A design based on the first approach will be less complex to implement than the latter because of its simpler control task. The complexity of each will depend on the nature of the input signal $r(t)$ and the order of the system being controlled.

Consider the system shown in Figure 7.2-1. It will be our objective to determine the digital transfer function $D(z)$ so that the system may respond with minimum settling time on a closed-loop basis. First we will consider the solution of this problem in the time domain. A later section will present a parallel solution in the z domain.

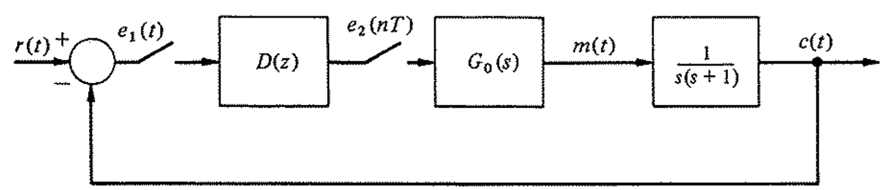

Figure 7.2-1. Digital control system.

7.2-1 Response to a Step Input

For the continuous plant of the system shown in Figure 7.2-1,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} m(t) \tag{7.2-1}$$

$$c(t) = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For $T = 1.0$ second, the discrete state equations are

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(k) \qquad (7.2\text{-}2)$$

$$k = 0, 1, 2, \ldots$$

and

$$c(k) = x_2(k)$$

The sequence $e_2(k)$ represents the input to the zero-order hold.

The objective of minimum settling time requires us to determine the control inputs $e_2(0), e_2(1), \ldots$, which drive the plant from an arbitrary initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

to a state such that the output $c(t)$ is equal to the input $r(t)$ for $t = t_1 > 0$. Furthermore, the output is to remain equal to the input from that time on. For the case at hand, the input is a step function; i.e.,

$$r(t) = R, \qquad t \geq 0 \qquad (7.2\text{-}3)$$

Therefore, we require that

$$\left.\begin{array}{l} c(t) = R \\ \text{and} \\ \dot{c}(t) = 0 \end{array}\right\} \quad t \geq t_1 > 0 \qquad (7.2\text{-}4)$$

The derivative of the output must be set equal to zero to guarantee that $c(t)$ will not change after it has reached the magnitude of the input. Conditions (7.2-4) may be related to the discrete state variable by use of the state equations (7.2-1); that is,

$$\left.\begin{array}{l} x_2(t) = R \\ x_1(t) = 0 \end{array}\right\} \quad t \geq t_1 > 0 \qquad (7.2\text{-}5)$$

The time t_1 at which conditions (7.2-5) are satisfied is unknown at this time. Since the input to the zero-order hold circuit occurs only at the discrete times $0, T, 2T, \ldots$, it is necessary to modify equations (7.2-5) to

$$\left.\begin{array}{l} x_2(NT) = R \\ x_1(NT) = 0 \end{array}\right\} \quad \text{for some integer } N > 0 \qquad (7.2\text{-}6)$$

Then the desired system state is synchronized timewise with the input. The integer N for which (7.2-6) holds is unknown. We must, therefore, investigate for what value of N it is possible to satisfy (7.2-6). We will investigate three cases; that is, $N = 1$, 2, and 3. For $N = 1$, it will be possible for the output to be equal to the input at the sampling instants; however, the system's response shows an undesirable ripple. This type of response is called the *minimal prototype response*. It is generally possible for any order system to exhibit a minimal prototype response ($N = 1$) to a step input. For $N = 2$, we will see that the system responds in a *deadbeat* manner; that is, the output equals the input at all times, not just at the sampling instants. In general, an nth order system requires $N = n$ sampling periods for deadbeat response. For $N = 3$, the system will be seen to respond in a deadbeat manner; in addition, it will be possible to impose some constraints on the magnitude of the control variable. These three cases will be investigated in detail for the second-order system which we are presently considering.

Case a. $N = 1$ *(Minimal Prototype Design)*

The transition equations for the first period are obtained from (7.2-2) by setting $k = 0$. This yields

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(0) \qquad (7.2\text{-}7)$$

If initial conditions are assumed to be zero,* equations (7.2-7) yield

$$x_1(1) = 0 = .632e_2(0) \qquad (7.2\text{-}8a)$$

and

$$x_2(1) = R = .368e_2(0) \qquad (7.2\text{-}8b)$$

which have to be satisfied simultaneously for a single $e_2(0)$. This is not possible. We conclude that a single period is not sufficient to accomplish the desired objective. Despite this negative result, it is interesting to pursue this case further to determine what can be accomplished in one period.

Solving (7.2-8b) for $e_2(0)$ yields

$$e_2(0) = \frac{R}{.368} = 2.72R$$

Setting $e_2(0)$ to this value will assure that $x_2(1) = R$, or $c(1) = R$, but will not produce the desired condition $\dot{c}(1) = 0$. Thus, the system output will

*No loss of generality arises by assuming the initial state to be zero.

be equal to the system input at the sampling instant, but it will not stay there. Indeed, for $e_2(0) = 2.72R$

$$x_1(1) = (.632)2.72R = 1.74R$$

Proceeding to the second interval, we have, from the transition equations,

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} 1.74R \\ R \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(1)$$

Again we seek to satisfy the position equation $x_2(2) = R$ and disregard the velocity condition $x_1(2)$. This yields

$$e_2(1) = -2.95R$$

and

$$x_1(2) = -1.225R$$

For the next period a similar calculation yields

$$e_2(2) = 2.12R$$

The remaining terms of the sequence $e_2(k)$ may be calculated accordingly. Since the z-transform of $e_2(k)$ is given by

$$E_2(z) = \sum_{k=0}^{\infty} e_2(k)z^{-k}$$

we can write

$$E_2(z) = 2.72R - 2.95Rz^{-1} + 2.12Rz^{-2} + \cdots \tag{7.2-9}$$

From the diagram of Figure 7.2-1 it is clear that

$$E_2(z) = D(z) \cdot E_1(z) \tag{7.2-10}$$

Therefore, if $E_1(z)$ is known, $D(z)$ may be specified. Now

$$e_1(t) = r(t) - c(t) \qquad t \geq 0 \tag{7.2-11}$$

For $t = kT$

$$e_1(k) = r(k) - c(k) \qquad k = 0, 1, \ldots$$

Thus, for $k = 0$

$$e_1(0) = R - 0 = R$$

and for $k > 0$

$$e_1(k) = 0$$

Thus

$$E_1(z) = R \tag{7.2-12}$$

Now, dividing (7.2-9) by (7.2-12) yields

$$D(z) = \frac{2.72 - 2.95z^{-1} + 2.12z^{-2} + \cdots}{1} \tag{7.2-13}$$

From developments to be introduced later, this transfer function can be shown to be expressible as

$$\begin{aligned} D(z) &= (2.72 - z^{-1})[1 - .717z^{-1} + (.717)^2z^{-2} - (.717)^3z^{-3} + \cdots] \\ &= \frac{2.72 - z^{-1}}{1 + .717z^{-1}} \end{aligned} \tag{7.2-14}$$

Equation (7.2-14) prescribes the program for the digital processor as the familiar ratio of two polynomials in z^{-1}. It will drive the system from the zero state to that for which the output of the system matches up with the input at the sampling instants but *not* in between, since the derivative of the output is not zero. The response of the system to a unit step input is shown in Figure 7.2-2. A system that has a step response as shown by this plot is called a minimal prototype system. It shows a pronounced ripple, which is sustained over a large number of periods. Because of this, its practical application is severely limited, and it is primarily of academic interest.

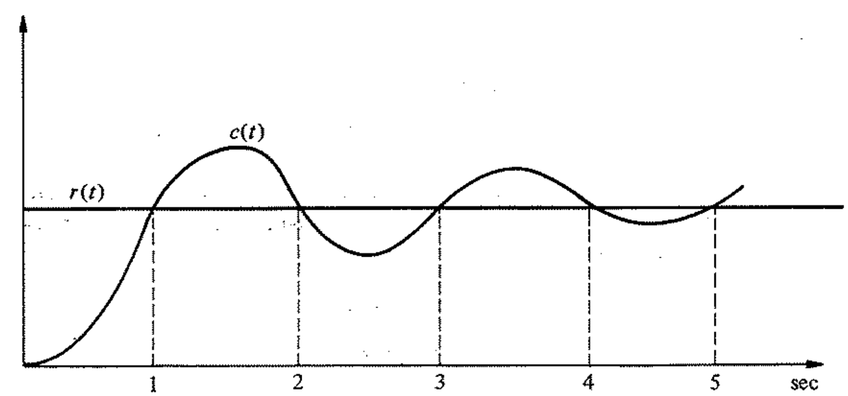

Figure 7.2-2. Response of system.

Case b. *$N = 2$ (Ripple-free System)*

If two sampling periods are allotted to reach the desired state as given by (7.2-6) it will be possible to insure that the system output will be ripple-free. To show this, we consider the state transition equation over two periods:

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}^2 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(0) + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_1(1) \tag{7.2-15}$$

Since

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we may simplify (7.2-15) to

$$\begin{bmatrix} 0 \\ R \end{bmatrix} = \begin{bmatrix} .233 & .632 \\ .768 & .368 \end{bmatrix} \begin{bmatrix} e_2(0) \\ e_2(1) \end{bmatrix}$$

which may easily be solved to yield

$$\begin{bmatrix} e_2(0) \\ e_2(1) \end{bmatrix} = \begin{bmatrix} 1.58 \\ -.58 \end{bmatrix} R$$

The numerical values of $e_2(0)$ and $e_2(1)$ uniquely define the first two members of the control sequence $e_2(k)$ required to drive the system from the state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

to the state

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

Since the desired state has been reached at the end of the second period, the remaining members of the control sequence will be zero. To verify this fact, apply (7.2-2) with

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

and $e_2(k) = 0$ for $k \geq 2$. This yields

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix} \quad \text{for } k \geq 2$$

The z-transform of the entire sequence is then

$$E_2(z) = R(1.58 - .58z^{-1}) \tag{7.2-16}$$

Equation (7.2-16) establishes the numerator for the digital computer transfer function.

To determine $E_1(z)$, we again make use of (7.2-11). Thus

$$e_1(0) = R$$

and

$$\begin{aligned} e_1(1) &= R - c(1) \\ &= R - x_2(1) \end{aligned}$$

But from (7.2-7) we have

$$\begin{aligned} x_2(1) &= (.368)e_2(0) \\ &= (.368)1.58R \\ &= .582R \end{aligned}$$

Thus

$$e_1(1) = R(1 - .582) = .418R$$

Furthermore,

$$e_1(k) = 0, \qquad k \geq 2$$

since the error is zero.

Collecting terms, we have

$$E_1(z) = (1 + .418z^{-1})R \tag{7.2-17}$$

The digital transfer function is, therefore,

$$D(z) = \frac{1.58 - .58z^{-1}}{1 + .418z^{-1}} \tag{7.2-18}$$

This transfer function will permit a ripple-free response of the system to a step input of any magnitude within two sampling periods of one second each, provided the system is initially at rest. A typical response to a sequence

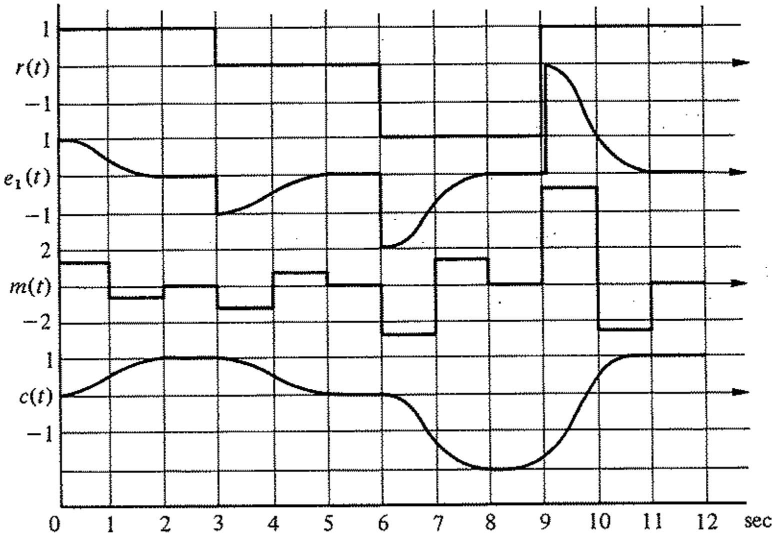

Figure 7.2-3. System response to a sequence of step inputs.

of unit step functions is shown in the response traces of Figure 7.2-3, consisting of input, error, plant input, and output.

In the previous two cases it was shown that for the example presently being discussed each sampling period that is allotted for the transition provides for one degree of freedom. Thus for $N = 1$, one degree of freedom existed; this was used to specify the conditions on one state variable, i.e., $x_2(k)$. For $N = 2$, two degrees of freedom existed; they were used to implement the constraints on two state variables $x_1(k)$ and $x_2(k)$. This resulted in ripple-free response. If more than two periods are available to complete the response to a step input, additional degrees of freedoms are provided which may be used to advantage in incorporating additional constraints. Of interest in this respect are amplitude constraints on the plant input, or on the maximum velocity or acceleration that may be tolerated during a typical transition. In what follows we shall give an illustration of this idea in the form of an input amplitude-constrained system.

Case c. $N \geq 3$ *(Input Amplitude-constrained System)*

The specific case to be considered here is

i. The plant is to respond ripple-free to a unit step input in the smallest number of sampling periods.
ii. The plant input $m(t)$ must satisfy the condition

$$|m(t)| \leq M = 1$$

In general, this problem is extremely difficult to solve. An approximation

to the desired control will now be obtained. The digital computer output $e_2(k)$ is calculated for two consecutive periods starting with $k = 0$. If the desired state can be reached without violating the plant input constraint, the problem is completed. If the constraints are not satisfied, we set $e_2(k)$ equal to ± 1, depending on its polarity, and add one more period to the total response time.

For the first two periods we consider (7.2-15) for $R = 1$.

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}\begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(0) + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(1) \qquad (7.2\text{-}19)$$

Solving for $e_2(0)$ and $e_2(1)$, we obtain

$$\begin{bmatrix} e_2(0) \\ e_2(1) \end{bmatrix} = \begin{bmatrix} 1.58 \\ -.58 \end{bmatrix}$$

It is seen that $e_2(0) > 1$. This exceeds the allowable limit. Consequently, we set $e_2(0)$ equal to the closest admissible value $+1$, and calculate the state of the plant at the end of the first period in response to this input.

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} .632 \\ .368 \end{bmatrix}$$

Proceeding now to the next two sampling periods, we have

$$\begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}^2 \begin{bmatrix} .632 \\ .368 \end{bmatrix} + \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}\begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(1) + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(2)$$

Solving for $e_2(1)$ and $e_2(2)$ yields

$$\begin{bmatrix} e_2(1) \\ e_2(2) \end{bmatrix} = \begin{bmatrix} .215 \\ -.215 \end{bmatrix}$$

These values satisfy the constraints. Thus the complete control sequence is

$$e_2(k) = [1, .215, -.215, 0, 0, \ldots]$$

or

$$E_2(z) = 1 + .215z^{-1} - .215z^{-2} \qquad (7.2\text{-}20)$$

Having determined the plant input sequence, we can now calculate the error sequence $e_1(k)$. Using (7.2-11), we have, for $k = 0, 1, 2,$

$$k = 0 \qquad e_1(0) = r(0) - c(0) = 1 - 0 = 1$$

$$k = 1 \qquad e_1(1) = r(1) - c(1) = r(1) - x_2(1) = 1 - .368 = .632$$

$$k = 2 \qquad e_1(2) = r(2) - x_2(2)$$

but $x_2(2)$ is calculated from the transition equations for the second interval

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(1)$$

or

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix} \begin{bmatrix} .632 \\ .368 \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} .215 = \begin{bmatrix} .368 \\ .847 \end{bmatrix}$$

so that

$$e_1(2) = 1 - .847 = .153$$

$$k \geq 3 \qquad e_1(k) = 0$$

Consequently,

$$E_1(z) = \mathscr{Z}\{1, .632, .153, 0, \ldots\} = 1 + .632z^{-1} + .153z^{-2} \tag{7.2-21}$$

With $E_1(z)$ and $E_2(z)$ determined, we can now specify $D(z)$.

$$D(z) = \frac{E_2(z)}{E_1(z)} = \frac{1 + .215z^{-1} - .215z^{-2}}{1 + .632z^{-1} + .153z^{-2}} \tag{7.2-22}$$

Equation (7.2-22) represents the digital transfer function that will guarantee a ripple-free response of the system shown in Figure 7.2-4 when subjected to a unit step input. The design of this system is limited to a unit step input and a unit input amplitude constraint. Should either of these two conditions be changed, a new digital transfer function will have to be determined. It seems plausible, for instance, that if the magnitude of the step input is increased it would take more than three sampling periods to complete the desired transition, since the plant input would be saturated for more than one period. It will be left as an exercise to explore this point.

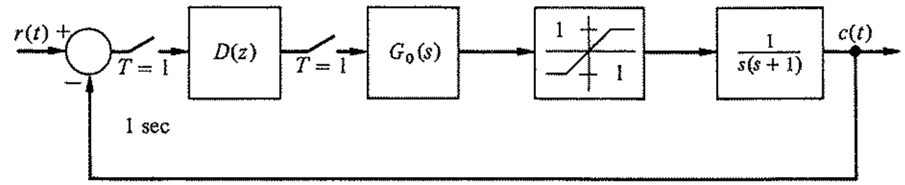

Figure 7.2-4. Block diagram of ripple-free sampled system with input amplitude constraints.

7.2-2 Response to a Ramp Input

An approach similar to the one presented in the previous section may be followed when the input is a ramp function. Let us consider the design of a digital transfer function for the system shown in Figure 7.2-5. The input is assumed to be

$$r(t) = Vt \quad \text{for } t \geq 0$$

and the initial state is taken to be zero.

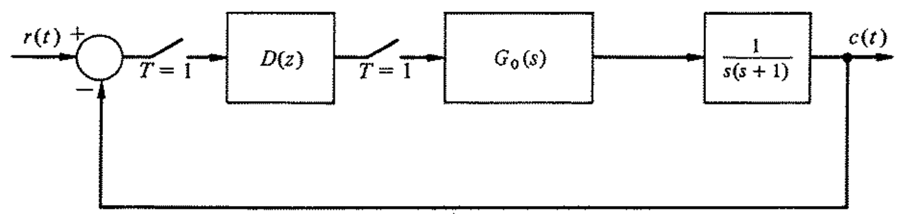

Figure 7.2-5. Sampled system with digital compensation.

The objective, as before, is to determine a digital transfer function $D(z)$ that will provide a ripple-free system response in a minimum number of sampling periods. To realize this response the following terminal conditions must be specified:

$$\begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} = \begin{bmatrix} V \\ VN \end{bmatrix} \quad \text{for some integer } N > 0 \tag{7.2-23}$$

These conditions derive from the requirement that

$$\dot{c}(t) = V$$
$$c(t) = Vt$$

and from the relationships between state variables and output variables as given by (7.2-4) through (7.2-6).

The number of sampling periods required to reach steady state, which is indicated by the integer N, is to be minimized. We consider the following cases.

Case a. $N = 1$

This case is meaningless, since no input can be generated to the digital processor during the first period. This input is generated from equation (7.2-11).

$$e_1(t) = r(t) - c(t) \quad \text{for } t = 0, e_1(0) = 0$$

Thus, the first period cannot be used for generating an output from the digital computer. Hence, this problem does not become meaningful until $N \geq 2$.

Case b. $N = 2$ *(Minimal Prototype)*

It is possible to design a digital processor that is functional within two periods. However, it can be shown to suffer from the same shortcoming as demonstrated for Case a of the step input design: only one of the conditions of (7.2-23) can be satisfied. If the condition selected is

$$x_2(N) = VN, \qquad N \geq 2$$

then the design will contain a ripple between sampling periods. Such a design represents the minimal prototype for a ramp input. It is left as an exercise to the student to verify this.

Case c. $N = 3$ *(Ripple-free Response)*

When N is selected as 3, it is possible to design a system that will satisfy both conditions of (7.2-23). From the state transition equations (7.2-2) we have

$$\begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} V \\ 3V \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}\begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(1) + \begin{bmatrix} .632 \\ .368 \end{bmatrix} e_2(2) \qquad (7.2\text{-}24)$$

Use has been made of the fact that the initial conditions are zero and that $e_2(0)$ is taken as zero as per discussion of Case a. Solving for $e_2(1)$ and $e_2(2)$, we obtain

$$\begin{bmatrix} e_2(1) \\ e_2(2) \end{bmatrix} = \begin{bmatrix} 3.810 \\ .173 \end{bmatrix} V$$

In determining the remainder of the computer output sequence $e_2(k)$,

it is readily seen that

$$e_2(k) = V, \qquad k \geq 3$$

For instance, for $k = 3$ we have from the state transition equations,

$$\begin{bmatrix} x_1(4) \\ x_2(4) \end{bmatrix} = \begin{bmatrix} .368 & 0 \\ .632 & 1 \end{bmatrix}\begin{bmatrix} V \\ 3V \end{bmatrix} + \begin{bmatrix} .632 \\ .368 \end{bmatrix} V = \begin{bmatrix} V \\ 4V \end{bmatrix}$$

In summary, then,

$$E_2(z) = (0 + 3.81z^{-1} + .173z^{-2} + z^{-3} + z^{-4} + \cdots)V \qquad (7.2\text{-}25)$$

The iterative application of the error equation permits the computation of the input sequence to the digital computer. This yields

$$E_1(z) = (0 + z^{-1} + .6z^{-2})V \qquad (7.2\text{-}26)$$

To determine the digital transfer function, $E_2(z)$ is divided by $E_1(z)$.

$$D(z) = \frac{3.81 + .173z^{-1} + z^{-2} + z^{-4} + \cdots}{1 + .6z^{-1}}$$

The numerator may be expressed in closed form by use of the geometric series identity.

$$\begin{aligned} D(z) &= \frac{3.81 + .173z^{-1}}{1 + .6z^{-1}} + \frac{z^{-2}}{(1 - z^{-1})(1 + .6z^{-1})} \\ &= \frac{3.81 + 2.637z^{-1} + .827z^{-2}}{1 - .4z^{-1} - .6z^{-2}} \end{aligned} \qquad (7.2\text{-}27)$$

7.2-3 The General Case (for Step Inputs)

A digital control system with minimum settling time as the performance objective may be designed in a very general sense. The examples presented so far were selected subject to two important restrictions. First, the plant was a single input–single output plant; second, the plant contained at least one free integrator. The latter condition provides a simple way of guaranteeing that the output equal the input with all derivatives of the output equal to zero, such as is, for instance, specified by equations (7.2-4). The presence of a free integrator in the plant lets the output of the digital computer in the steady state reach zero. It will now be shown that identical control characteristics may be obtained for plants without a free integrator.

Consider the system shown in the vector block diagram of Figure 7.2-6.

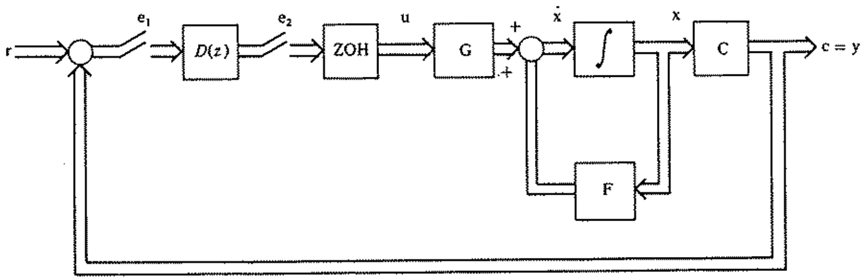

Figure 7.2-6. Multiple input–multiple output computer control system.

It is a multiple input–multiple output system. The plant is described by the linear vector differential equations

$$\frac{d}{dt}\mathbf{x} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \tag{7.2-28}$$

$$\mathbf{c} = \mathbf{y} = \mathbf{C}\mathbf{x} \tag{7.2-29}$$

where $\mathbf{x}$ is an $n \times 1$ state vector
$\mathbf{u}$ is the $m \times 1$ control vector
$\mathbf{y}$ is the $p \times 1$ output vector.

Furthermore, let N be the number of sampling intervals until deadbeat response is achieved.

The objective of this problem is to design $\mathbf{D}(z)$ to cause $\mathbf{c}(t)$ to respond in a deadbeat manner to a step input $\mathbf{r}(t)$.

The computation of $\mathbf{D}(z)$ proceeds in two parts. First, the sequence of vectors for $\mathbf{e}_2(k)$ is computed to obtain the deadbeat response of $\mathbf{y}(t)$ in a minimum number of sampling periods. From the $\mathbf{y}(t)$ response, the sequence of vector $\mathbf{e}_1(k)$ may be computed. $\mathbf{D}(z)$ is to be computed such that $\mathbf{e}_2(k)$ is generated from $\mathbf{e}_1(k)$.

The output at the kth sampling period is related to the $\mathbf{e}_2$ sequence and the disturbance in the following way.

Because of the zero-order hold, $\mathbf{x}(k)$ may be computed as follows:

$$\mathbf{x}(k) = [e^{\mathbf{F}T}]^k\mathbf{x}(0) + \sum_{l=0}^{k-1}[e^{\mathbf{F}T}]^{k-l-1}\int_0^T e^{\mathbf{F}T}dt\mathbf{G}\mathbf{e}_2(l) \tag{7.2-30}$$

The initial condition $\mathbf{x}(0)$ is taken as zero for this problem.

Let

$$\mathbf{A} = e^{\mathbf{F}T}$$

and

$$\mathbf{B} = \int_0^T e^{\mathbf{F}T}dt\mathbf{G}$$

Since $\mathbf{x}(0) = \mathbf{0}$, $\mathbf{x}(k)$ may be written as

$$\mathbf{x}(k) = \sum_{l=0}^{k-1}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{e}_2(l)$$

Thus, $\mathbf{y}(k)$ becomes

$$\mathbf{y}(k) = \sum_{l=0}^{k-1}\mathbf{C}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{e}_2(l) \tag{7.2-31}$$

Following a vector step input of arbitrary size, denoted $\mathbf{r}_0$, we want $\mathbf{e}_1$ to go to zero in the minimum number of sampling periods. Thus,

$$\mathbf{r}_0 = \sum_{l=0}^{N-1}\mathbf{C}\mathbf{A}^{N-l-1}\mathbf{B}\mathbf{e}_2(l) \tag{7.2-32}$$

or, in matrix form,

$$[\mathbf{C}\mathbf{A}^{N-1}\mathbf{B} \quad \mathbf{C}\mathbf{A}^{N-2}\mathbf{B} \quad \ldots \quad \mathbf{C}\mathbf{B}]\begin{bmatrix}\mathbf{e}_2(0)\\ \mathbf{e}_2(1)\\ \cdot\\ \cdot\\ \cdot\\ \mathbf{e}_2(N-1)\end{bmatrix} = \mathbf{r}_0 \tag{7.2-33}$$

This expression does not guarantee that the response will be deadbeat as required; it only forces the output to $\mathbf{r}_0$. To guarantee a deadbeat response, $\dot{\mathbf{x}}(NT)$ must be zero. Since $\mathbf{u}(t)$ is constant in the interval $NT \le t < (N+1)T$ and $\dot{\mathbf{x}}(NT) = \mathbf{0}$, then $\mathbf{x}(t)$ cannot change from NT to $(N+1)T$. Thus, $\mathbf{u}(NT)$ will be the control required to effect the step change in the output response.

$$\mathbf{u}(t) = \mathbf{u}(NT) = \mathbf{e}_2(NT) \quad \text{for } t \ge NT \tag{7.2-34}$$

The expression for $\dot{\mathbf{x}}(NT)$ may be computed as follows. From the state equation

$$\dot{\mathbf{x}}(NT) = \mathbf{F}\mathbf{x}(NT) + \mathbf{G}\mathbf{e}_2(NT)$$

Equating $\dot{\mathbf{x}}(NT) = \mathbf{0}$ and substituting the expression for $\mathbf{x}(NT)$, we can write that

$$\mathbf{0} = \mathbf{F}\left[\sum_{l=0}^{N-1}\mathbf{A}^{N-l-1}\mathbf{B}\mathbf{e}_2(l)\right] + \mathbf{G}\mathbf{e}_2(N)$$

or

$$\mathbf{0} = \sum_{l=0}^{N-1}\mathbf{F}\mathbf{A}^{N-l-1}\mathbf{B}\mathbf{e}_2(l) + \mathbf{G}\mathbf{e}_2(N)$$

In matrix form, this equation may be written

$$[\mathbf{FA}^{N-1}\mathbf{B} \quad \mathbf{FA}^{N-2}\mathbf{B} \quad \ldots \quad \mathbf{FB} \quad \mathbf{G}]\begin{bmatrix} \mathbf{e}_2(0) \\ \mathbf{e}_2(1) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_2(N-1) \\ \mathbf{e}_2(N) \end{bmatrix} = [\mathbf{0}] \qquad (7.2\text{-}35)$$

Equations (7.2-33) and (7.2-35) may be combined to form the system of equations that must be solved to determine the $\mathbf{e}_2$ sequence.

$$\begin{bmatrix} \mathbf{CA}^{N-1}\mathbf{B} & \mathbf{CA}^{N-2}\mathbf{B} & \ldots & \mathbf{CB} & \mathbf{0} \\ \mathbf{FA}^{N-1}\mathbf{B} & \mathbf{FA}^{N-2}\mathbf{B} & \ldots & \mathbf{FB} & \mathbf{G} \end{bmatrix}\begin{bmatrix} \mathbf{e}_2(0) \\ \mathbf{e}_2(1) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_2(N-1) \\ \mathbf{e}_2(N) \end{bmatrix} = \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{0} \end{bmatrix} \qquad (7.2\text{-}36)$$

A discussion of the solution to equation (7.2-36) will be given later, but first let us consider computation of the $\mathbf{e}_1$ sequence and of $\mathbf{D}(z)$. From the error equation we have that

$$\mathbf{e}_1(k) = \mathbf{r}_0 - \mathbf{y}(k)$$

But $\mathbf{y}(k)$ may be computed from equation (7.2-31).

$$\mathbf{e}_1(k) = \mathbf{r}_0 - \sum_{l=0}^{k-1} \mathbf{C}\{\mathbf{A}^{k-l-1}\mathbf{Be}_2(l)\} \qquad (7.2\text{-}37)$$

On the other hand, $\mathbf{e}_2(l)$ is obtained from the solution of equation (7.2-36). Let us denote that solution in the following way:

$$\mathbf{e}_2(l) = \mathbf{P}(l)\mathbf{r}_0, \qquad l = 0, 1, \ldots, N$$

Thus, equation (7.2-37) may be written

$$\mathbf{e}_1(k) = \left[\mathbf{I} - \sum_{l=0}^{k-1} \{\mathbf{CA}^{k-l-1}\mathbf{BP}(l)\}\right]\mathbf{r}_0 \qquad (7.2\text{-}38)$$

Finally, $\mathbf{D}(z)$ may be computed as follows. Taking the z-transform of the $\mathbf{e}_1(k)$ and $\mathbf{e}_2(k)$ sequences, we get

$$\mathbf{E}_1(z) = \sum_{k=0}^{\infty} \mathbf{e}_1(k)z^{-k}$$
$$= \sum_{k=0}^{N-1} z^{-k}\left[\mathbf{I} - \sum_{l=0}^{k-1} \{\mathbf{CA}^{k-l-1}\mathbf{BP}(l)\}\right]\mathbf{r}_0 \qquad (7.2\text{-}39)$$

The infinite series due to the z-transform of $\mathbf{e}_1(k)$ is terminated at $N - 1$, since the coefficients $\mathbf{e}_1(k)$ for $k > N - 1$ are identically zero.

$$\mathbf{E}_2(z) = \sum_{k=0}^{\infty} \mathbf{P}(k)\mathbf{r}_0 z^{-k}$$

Since the input to the plant is constant after $N - 1$ sampling periods, we have $\mathbf{e}_2(k) = \mathbf{P}(N)$ for $k \geq N$, so that

$$\mathbf{E}_2(z) = \left[\sum_{k=0}^{N-1} z^{-k}\mathbf{P}(k) + \mathbf{P}(N)\sum_{k=N}^{\infty} z^{-k}\right]\mathbf{r}_0$$

or

$$\mathbf{E}_2(z) = \left[\sum_{k=0}^{N-1} z^{-k}\mathbf{P}(k) + \mathbf{P}(N)\frac{z^{-N}}{1 - z^{-1}}\right]\mathbf{r}_0 \qquad (7.2\text{-}40)$$

Since

$$\mathbf{E}_2(z) = \mathbf{D}(z)\,\mathbf{E}_1(z)$$

we may combine equations (7.2-39) and (7.2-40) to obtain

$$\left[\sum_{k=0}^{N-1} z^{-k}\mathbf{P}(k) + \mathbf{P}(N)\frac{z^{-N}}{1 - z^{-1}}\right]\mathbf{r}_0$$
$$= \mathbf{D}(z)\left[\sum_{k=0}^{N-1} z^{-k}\left(\mathbf{I} - \sum_{l=0}^{k-1} \{\mathbf{CA}^{k-l-1}\mathbf{BP}(l)\}\right)\right]\mathbf{r}_0 \qquad (7.2\text{-}41)$$

Since equation (7.2-41) must hold for arbitrary $\mathbf{r}_0$, the coefficient matrices premultiplying $\mathbf{r}_0$ must be equal. Thus,

$$\left[\sum_{k=0}^{N-1} z^{-k}\mathbf{P}(k) + \mathbf{P}(N)\frac{z^{-N}}{1 - z^{-1}}\right]$$
$$= \mathbf{D}(z)\left[\sum_{k=0}^{N-1} z^{-k}\left(\mathbf{I} - \sum_{l=0}^{k-1} \{\mathbf{CA}^{k-l-1}\mathbf{BP}(l)\}\right)\right] \qquad (7.2\text{-}42)$$

Solving equation (7.2-42) for $\mathbf{D}(z)$, we obtain the result

$$\mathbf{D}(z) =$$
$$\left[\sum_{k=0}^{N-1} z^{-k}\mathbf{P}(k) + \mathbf{P}(N)\frac{z^{-N}}{1 - z^{-1}}\right]\left[\sum_{k=0}^{N-1} z^{-k}\left(\mathbf{I} - \sum_{l=0}^{N-1} \{\mathbf{CA}^{k-l-1}\mathbf{BP}(l)\}\right)\right]^{-1} \qquad (7.2\text{-}43)$$

The computation of $\mathbf{P}(k)$ may be demonstrated in the course of the development of four cases.

7.2-4 Scalar Case: $n = m = p = 1$

State equation

$$\dot{x} = ax + bu$$

Output equation

$$y = cx$$

$$\mathbf{A} = e^{aT}, \qquad \mathbf{B} = \int_0^T e^{at} dt b = a^{-1}[e^{aT} - 1]b$$

In this case equation (7.2-36) has two rows (i.e., two equations), so at the end of the first sampling period the output will be zero.

$$\begin{bmatrix} ca^{-1}[e^{aT} - 1]b & 0 \\ [e^{aT} - 1]b & b \end{bmatrix} \begin{bmatrix} e_2(0) \\ e_2(1) \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \end{bmatrix}$$

Solving for $e_2(0)$ and $e_2(1)$, we determine the inverse of the coefficient matrix, obtaining

$$\begin{bmatrix} e_2(0) \\ e_2(1) \end{bmatrix} = \begin{bmatrix} \dfrac{a/bc}{e^{aT} - 1} & 0 \\ -\dfrac{a}{bc} & \dfrac{1}{b} \end{bmatrix} \begin{bmatrix} r_0 \\ 0 \end{bmatrix}$$

Thus

$$e_2(0) = \frac{a}{b(e^{aT} - 1)c} r_0 \quad \text{and} \quad e_2(1) = -\frac{a}{bc} r_0$$

and

$$\mathbf{P}(0) = \frac{a}{b(e^{aT} - 1)c} \quad \text{and} \quad \mathbf{P}(1) = -\frac{a}{bc}$$

It is seen that $N = 1$ in this case. The digital transfer function is, therefore,

$$D(z) = [z^{-0}(1 - 0)]^{-1}\left[z^{-0}\mathbf{P}(0) + \mathbf{P}(1)\frac{z^{-1}}{1 - z^{-1}}\right]$$

Upon substituting for $\mathbf{P}(0)$ and $\mathbf{P}(1)$, we obtain the following:

$$D(z) = \frac{a(1 - e^{aT} z^{-1})}{bc(1 - e^{aT})(1 - z^{-1})}$$

7.2-5 Single Input–Second-order System: $n = 2$, $m = p = 1$

Let the system be represented by the state equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

with the output given by

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

An inspection of equation (7.2-36) reveals that $N = 2$.

The state transition matrix for the case $T = 1$ is

$$\mathbf{A} = \begin{bmatrix} e^{-1} & 0 \\ 1 - e^{-1} & 1 \end{bmatrix}$$

The input transition matrix is

$$\mathbf{B} = \begin{bmatrix} 1 - e^{-1} \\ e^{-1} \end{bmatrix}$$

Other expressions needed are

$$\mathbf{CAB} = [0 \quad 1] \begin{bmatrix} e^{-1} & 0 \\ 1 - e^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 - e^{-1} \\ e^{-1} \end{bmatrix} = (1 - e^{-1})^2 + e^{-1}$$

$$\mathbf{CB} = e^{-1}$$

$$\mathbf{FAB} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 1 - e^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 - e^{-1} \\ e^{-1} \end{bmatrix} = \begin{bmatrix} -e^{-1}(1 - e^{-1}) \\ e^{-1}(1 - e^{-1}) \end{bmatrix}$$

$$\mathbf{FB} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - e^{-1} \\ e^{-1} \end{bmatrix} = \begin{bmatrix} e^{-1} - 1 \\ 1 - e^{-1} \end{bmatrix}$$

Thus, equation (7.2-36) becomes

$$\begin{bmatrix} (1 - e^{-1})^2 + e^{-1} & e^{-1} & 0 \\ -e^{-1}(1 - e^{-1}) & e^{-1} - 1 & 1 \\ e^{-1}(1 - e^{-1}) & 1 - e^{-1} & 0 \end{bmatrix} \begin{bmatrix} e_2(0) \\ e_2(1) \\ e_2(2) \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \\ 0 \end{bmatrix}$$

With the exponentials evaluated, this becomes

$$\begin{bmatrix} .768 & .368 & 0 \\ -.232 & -.632 & 1 \\ .232 & .632 & 0 \end{bmatrix} \begin{bmatrix} e_2(0) \\ e_2(1) \\ e_2(2) \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \\ 0 \end{bmatrix}$$

When the inverse of the coefficient matrix is computed, the vector $\mathbf{e}_2$ can be calculated.

$$\begin{bmatrix} e_2(0) \\ e_2(1) \\ e_2(2) \end{bmatrix} = \begin{bmatrix} 1.58 \\ -.58 \\ 0 \end{bmatrix} r_0$$

Thus

$$\mathbf{P}(0) = 1.58, \qquad \mathbf{P}(1) = 0.58, \qquad \mathbf{P}(2) = 0.0$$

$\mathbf{P}(2)$ in this case is zero, because the system contains a free integrator.

The digital transfer function becomes, by use of (7.2-43),

$$D(z) = \left[\mathbf{P}(0) + \mathbf{P}(1)z^{-1} + \mathbf{P}(2)\frac{z^{-2}}{1 - z^{-1}}\right]\{z^0(1) + z^{-1}[1 - \mathbf{CBP}(0)]\}^{-1}$$

With the numbers substituted, this is

$$D(z) = \frac{1.58 - .58z^{-1}}{1 + .418z^{-1}}$$

It is seen that this is identical to the expression derived earlier by equation (7.2-18).

7.2-6 *Multiple Input–Multiple Output Fourth-order System:* $n = 4, m = p = 2$

The state equations of this system are given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -5 & -1 \\ 0 & -2 & 0 & 0 \\ 2 & 1 & -6 & -1 \\ -2 & -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The output equations of this system are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -3 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

For $T = .1$ the state transition matrix is computed as

$$\mathbf{A} = \begin{bmatrix} 1.0 & .0779 & -.398 & -.0705 \\ .0 & .819 & .0 & .0 \\ .164 & .0779 & .506 & -.0705 \\ -.164 & -.0779 & .164 & .741 \end{bmatrix}$$

Similarly, the input transition matrix is

$$\mathbf{B} = \begin{bmatrix} .104 & .0734 \\ .0 & .1813 \\ .0088 & .169 \\ -.0088 & -.0861 \end{bmatrix}$$

Because the system has two inputs and two outputs, it follows that $N = 2$. Hence, equations (7.2-36) take on the form

$$\begin{bmatrix} .214 & -.086 & .268 & -.095 & .0 & .0 \\ .064 & .259 & .086 & .346 & .0 & .0 \\ .019 & -.346 & .068 & -.501 & 1.0 & 1.0 \\ .0 & -.297 & .0 & -.362 & .0 & 2.0 \\ .106 & -.432 & .164 & -.597 & .0 & 2.0 \\ -.106 & .211 & -.164 & -.267 & .0 & -1.0 \end{bmatrix} \begin{bmatrix} e_2^1(0) \\ e_2^1(1) \\ e_2^1(2) \\ e_2^2(0) \\ e_2^2(1) \\ e_2^2(2) \end{bmatrix} = \begin{bmatrix} r_0^1 \\ r_0^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the corresponding entries are computed by using the matrices **F**, **A**, **B**, and **C**.

The matrices $\mathbf{P}(l)$ may be obtained upon inverting the coefficient matrix of the last equation. Thus,

$$\begin{bmatrix} \mathbf{P}(0) \\ \\ \mathbf{P}(1) \\ \\ \mathbf{P}(2) \\ \end{bmatrix} = \begin{bmatrix} 24.01 & 1.575 \\ -1.969 & 9.843 \\ -15.750 & .195 \\ .962 & -4.814 \\ .529 & .353 \\ -.118 & .588 \end{bmatrix}$$

Now,

$$\mathbf{P}(0) = \begin{bmatrix} 24.01 & 1.575 \\ -1.969 & 9.843 \end{bmatrix}, \qquad \mathbf{P}(1) = \begin{bmatrix} -15.750 & .195 \\ .962 & -4.814 \end{bmatrix},$$

$$\mathbf{P}(2) = \begin{bmatrix} .529 & .353 \\ -.118 & .588 \end{bmatrix}$$

Finally, the digital transfer function may be computed by use of equation (7.2-43) for $k = 2$.

$$\mathbf{D}(z) = \left[\mathbf{P}(0) + z^{-1}\mathbf{P}(1) + \mathbf{P}(2)\frac{z^{-2}}{1 - z^{-1}}\right]\{\mathbf{I} + z^{-1}[\mathbf{I} - \mathbf{CBP}(0)]\}^{-1}$$

Upon substitution of the numerical values of the respective matrices, this becomes a matrix of four transfer functions; that is

$$\mathbf{D}(z) = \begin{bmatrix} \dfrac{24.01 - 39.76z^{-1} + 16.339z^{-2}}{1 - 6.621z^{-1} - 1.08z^{-2}} & \dfrac{1.575 - 1.38z^{-1} + .258z^{-2}}{.515z^{-1} - .515z^{-2}} \\ \dfrac{1.969 + 2.931z^{-1} + 1.08z^{-2}}{1.393z^{-1} - 1.393z^{-2}} & \dfrac{9.843 - 3.542z^{-1} + 5.402z^{-2}}{1 - 3.542z^{-1} + 2.542z^{-2}} \end{bmatrix}$$

7.3 Minimal Prototype Design Using z-Transform Method

We now consider an alternative to the time-domain approach of designing a system with minimum settling time by using methods available to us from the study of sampled-data systems using the z-transform. Namely, design a controlled system such that for a given continuous plant:

1. The overall response and the response of all elements of the system must be nonanticipative.
2. The steady-state error for all polynomial inputs of degree equal to or less than q is zero.
3. The transient response should be as fast as possible, and the settling time should be equal to a finite number of sampling intervals.

Consider the system shown in Figure 7.3-1 for a typical example. From (6.5-13), the relationship

$$\frac{C(z)}{R(z)} = K(z) = \frac{D(z)\mathscr{Z}[G_m(s)G(s)]}{1 + D(z)\mathscr{Z}[G_m(s)G(s)]} \tag{7.3-1}$$

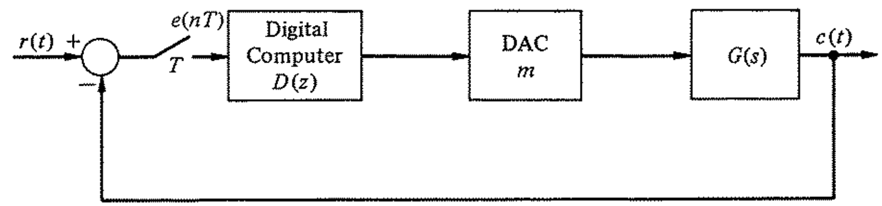

Figure 7.3-1. Sampled-data feedback system.

was established. For rational $G(s)$, we have

$$\begin{aligned} \mathscr{Z}[G_m(s)G(s)] &= \frac{p_r z^{-l} + \cdots + p_s z^{-t}}{q_0 + qz^{-1} + \cdots + q_b z^{-b}} \\ &= c_1 z^{-l} + c_2 z^{-l-1} + \cdots \end{aligned} \tag{7.3-2}$$

In order that $\mathscr{Z}[G_m(s)G(s)]$ be nonanticipative, the integer l must be equal to or greater than zero. If this is not true, then the output precedes the input, which corresponds to an anticipative system. This is predicated on the presence of q_0 in (7.3-2), which is a necessary condition for nonanticipativeness.

The transfer function $D(z)$ is of the form

$$D(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_u z^{-u}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_j z^{-j}} \tag{7.3-3}$$

Inserting (7.3-2) and (7.3-3) into (7.3-1) and simplifying, we obtain

$$K(z) = \frac{k_l z^{-l} + \cdots + k_p z^{-p}}{l_0 + l_1 z^{-1} + \cdots + l_q z^{-q}} \tag{7.3-4}$$

In order that $K(z)$ be nonanticipative, its numerator must contain z^{-1} to a power equal to or greater than the lowest power of l appearing in $\mathscr{Z}[G_m(s)G(s)]$ in (7.3-2). Again, l_0 must appear for reasons similar to the appearance of q_0 in (7.3-2). The requirement of nonanticipativeness is met if the above properties are satisfied.

It is further desired that $D(z)$ be selected so that the steady-state error at the sampling times is zero. If the desired form of $K(z)$ can be found, then $D(z)$ may be determined by using the identity (7.3-1); that is,

$$D(z) = \frac{K(z)}{\mathscr{Z}[G_m(s)G(s)][1 - K(z)]} \tag{7.3-5}$$

From Figure 7.3-1, the relationship

$$E(z) = R(z) - C(z)$$

follows. Since $C(z) = K(z)R(z)$, we have

$$E(z) = [1 - K(z)]R(z) \tag{7.3-6}$$

Since the steady-state error at the sampling time is required to be zero, this implies $e(\infty) = 0$. Using the final value theorem, we have

$$e(\infty) = \lim_{z \to 1} \{(1 - z^{-1})[1 - K(z)]R(z)\}$$

For polynomial inputs of order less than or equal to q, $R(z)$ is of the form

$$R(z) = \frac{A(z)}{(1 - z^{-1})^{q+1}}$$

with $A(z)$ being a polynomial in z^{-1}. Therefore,

$$e(\infty) = \lim_{z \to 1} (1 - z^{-1})[1 - K(z)]\frac{A(z)}{(1 - z^{-1})^{q+1}} \tag{7.3-7}$$

To guarantee that the steady-state error is zero for all such polynomials, the term $1 - K(z)$ must be selected so that

$$1 - K(z) = (1 - z^{-1})^{q+1}F(z) \tag{7.3-8}$$

where $F(z)$ is a ratio of polynomials in z^{-1} that is analytic at $z = 1$. Substituting (7.3-8) into (7.3-6) gives

$$\begin{aligned} E(z) &= (1 - z^{-1})^{q+1}F(z)R(z) \\ &= F(z)A(z) = \frac{N(z)}{D(z)}A(z) \end{aligned} \tag{7.3-9}$$

By dividing $D(z)$ into $N(z)A(z)$, the time history of the error will evolve; that is,

$$E(z) = e_0 + e_1 z + \cdots + e_N z^N + \cdots$$

Requirement 3 is satisfied if this expression is terminated with a finite number of terms and the highest power of z^{-1} is the minimum possible. An investigation of (7.3-9) reveals that these properties are satisfied if $F(z)$ is set equal to a constant; for convenience we choose $F(z) = 1$. From (7.3-8), we have

$$1 - K(z) = (1 - z^{-1})^{q+1} \tag{7.3-10}$$

so that the expression for the desired transfer of the digital computer compensator is, by (7.3-5),

$$D(z) = \frac{1 - (1 - z^{-1})^{q+1}}{(1 - z^{-1})^{q+1}\mathscr{Z}[G_m(s)G(s)]} \tag{7.3-11}$$

EXAMPLE 7.3-1

Determine $K(z)$ and the settling times for a step, ramp, and acceleration input.

(i) *Step Input*

For a step input $q = 0$; therefore, (7.3-10) gives us

$$K(z) = 1 - (1 - z^{-1})^1 = z^{-1}$$

(ii) *Ramp Input*

For ramp inputs $q = 1$; therefore,

$$K(z) = 1 - (1 - z^{-1})^2 = 2z^{-1} - z^{-2}$$

(iii) For acceleration inputs $q = 2$; therefore,

$$K(z) = 1 - (1 - z^{-1})^3 = 3z^{-1} - 3z^{-2} + z^{-3}$$

The settling times for the various inputs can be determined by noting that for $F(z) = 1$, $E(z)$ is given by

$$E(z) = [1 - K(z)]R(z)$$

For different values of q, we have

Step input of magnitude R:

$$E(z) = [1 - z^{-1}]\frac{R}{1 - z^{-1}} = R$$

Therefore,

$$\begin{aligned} e(nT) &= 0 \quad \text{for } n = 1, 2, \ldots \\ e(0) &= R \end{aligned}$$

Ramp input of slope V:

$$E(z) = [1 - z^{-1}]^2\frac{VTz^{-1}}{(1 - z^{-1})^2} = VTz^{-1}$$

Therefore,

$$\begin{aligned} e(T) &= VT \\ e(nT) &= 0 \quad \text{for } n = 2, 3, 4, \ldots \end{aligned}$$

Acceleration input of magnitude A:

$$E(z) = [(1 - z^{-1}]^2 \frac{ATz^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$
$$= ATz^{-1} + ATz^{-2}$$

Therefore,

$$e(T) = AT$$
$$e(2T) = AT$$
$$e(nT) = 0 \quad \text{for } n = 3, 4, 5, \ldots$$

Table 7.3-1 summarizes these results.

Table 7.3-1 Minimal Response Characteristics

Input	$R(t)$	$R(z)$	$K(z) = \frac{C(z)}{R(z)}$	Settling Time in Sampling Periods
Step	$\alpha u(t)$	$\frac{\alpha}{1 - z^{-1}}$	z^{-1}	T
Ramp	$\alpha tu(t)$	$\frac{\alpha T z^{-1}}{(1 - z^{-1})^2}$	$2z^{-1} - z^{-2}$	$2T$
Acceleration	$\alpha t^2 u(t)$	$\frac{\alpha T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$	$3z^{-1} - 3z^{-2} + z^{-3}$	$3T$

In the minimal response design, the value of the error signal is driven to zero at the sampling instants. However, between sampling periods, the error signal need not be zero. If this is the case, the actual output signal will tend to oscillate about the desired output, which for unity feedback is the applied input.

EXAMPLE 7.3-2

Design a minimal response digital computer compensator for the system shown in Figure 7.3-1 when a unit step is applied. For this problem

$$G(s) = \frac{1}{s(s + 1)}$$

This is the same problem treated in Section 7.2.

From Table 7.3-1, we have $K(z) = z^{-1}$ for the step input. Now

$$\mathscr{Z}\left[G_0(s)\frac{1}{s(s + 1)}\right] = (1 - z^{-1})\mathscr{Z}\left[\frac{1}{s^2(s + 1)}\right] = \frac{z^{-1}[1 + (e - 2)z^{-1}]}{(1 - z^{-1})(e - z^{-1})}$$

Inserting this result along with $K(z) = z^{-1}$ into (7.3-5) gives us the desired digital computer compensator transfer function. Namely,

$$D(z) = \frac{e - z^{-1}}{1 + (e - 2)z^{-1}}$$

which is in agreement with the results (7.2-13) found in Section 7.2.

EXAMPLE 7.3-3

Design a minimal-response digital computer compensator for the system shown in Figure 7.3-2 when ramp inputs are applied.

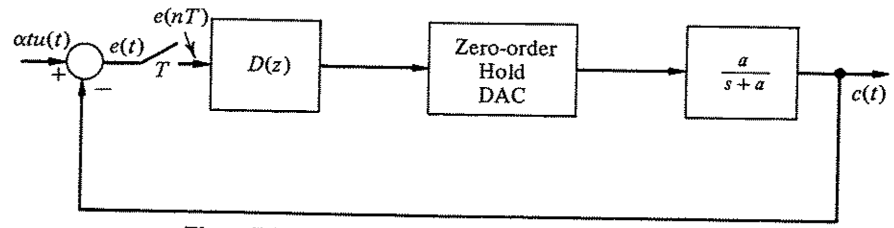

Figure 7.3-2. Minimal response system for step inputs.

From Table 7.3-1, we can find that the desired transfer function $K(z)$ is

$$K(z) = \frac{C(z)}{R(z)} = 2z^{-1} - z^{-2} \tag{7.3-12}$$

It can be shown that

$$\mathscr{Z}\left[G_0(s)\frac{a}{s + a}\right] = (1 - e^{-aT})\left[\frac{z^{-1}}{1 - z^{-1}e^{-aT}}\right] \tag{7.3-13}$$

Inserting (7.3-12) and (7.3-13) into (7.3-5), we find the desired transfer function of the digital computer compensator to be

$$D(z) = \frac{(2 - z^{-1})(1 - z^{-1}e^{-aT})}{(1 - e^{-aT})(1 - z^{-1})^2}$$

A typical response of this system is shown in Figure 7.3-3.

It is possible to design systems that have both finite settling times and zero error for all time after the system has settled. The design of such systems was treated in Section 7.2. We have treated unity feedback systems exclusively for minimal-response design. However, similar methods may be used if the output signal $C(s)$ is fed back through a system with transfer function $H(s)$.

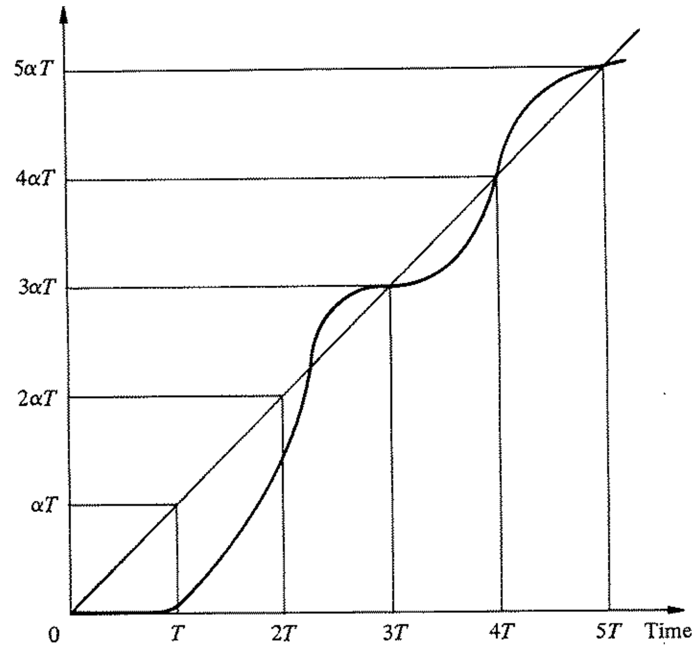

Figure 7.3-3. Typical minimal ramp response.

7.4 Controllability and Observability

Before an in-depth discussion of the design of controllers for linear discrete systems may be made, it is first necessary to introduce the concept of controllability. Consider a linear discrete system characterized by the vector difference equation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \tag{7.4-1}$$

where $\mathbf{x}(k)$ is the $n \times 1$ state vector at the kth iteration
$\mathbf{A}$ is the $n \times n$ transition matrix that has an inverse
$\mathbf{B}$ is the $n \times 1$ control vector
$u(k)$ is the scalar input at the kth iteration
k is the discrete iteration time.

7.4-1 Controllability

System (7.4-1) is said to be completely controllable if it is possible to force the system from any arbitrary initial state $\mathbf{x}(0)$ to any arbitrary desired state $\mathbf{x}_D$ in a finite number of iteration times.

Essentially, controllability indicates whether a system may be properly controlled. In order to determine what properties a linear discrete system must

satisfy so as to be controllable, we repeatedly apply (7.4-1), obtaining

$$\begin{aligned}
\mathbf{x}(1) &= \mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0)\\
\mathbf{x}(2) &= \mathbf{A}\mathbf{x}(1) + \mathbf{B}u(1)\\
&= \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}u(0) + \mathbf{B}u(1)\\
&\cdots\\
\mathbf{x}(N) &= \mathbf{A}^N\mathbf{x}(0) + \mathbf{A}^{N-1}\mathbf{B}u(0) + \cdots + \mathbf{A}\mathbf{B}u(N-2) + \mathbf{B}u(N-1)
\end{aligned} \tag{7.4-2}$$

According to the definition of controllability, system (7.4-1) is controllable if it is possible to select a value of N and $u(0), u(1), \ldots, u(N-1)$ such that $\mathbf{x}(N) = \mathbf{x}_D$ with both $\mathbf{x}(0)$ and $\mathbf{x}_D$ being fixed but arbitrary. Assume that such a selection has been made; that is,

$$\mathbf{x}_D - \mathbf{A}^N\mathbf{x}(0) = \mathbf{h}_{N-1}u(0) + \mathbf{h}_{N-2}u(1) + \cdots + \mathbf{h}_0 u(N-1) \tag{7.4-3}$$

where

$$\mathbf{h}_k = \mathbf{A}^k\mathbf{B} \quad \text{for } k = 0, 1, 2, \ldots$$

is an $n \times 1$ vector.

For (7.4-3) to be satisfied for arbitrary $\mathbf{x}(0)$ and $\mathbf{x}_D$, it is necessary that there be n linearly independent vectors in the set $\{\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{N-1}\}$. This immediately implies that $N \geq n$. In fact, it may be shown that for the nth-order linear discrete system of (7.4-1) to be controllable, it is necessary and sufficient that the n vectors $\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$ form a set of linearly independent vectors. To help comprehend this fact, rewrite (7.4-3) for $N = n$ as

$$\mathbf{x}_D - \mathbf{A}^n\mathbf{x}(0) = [\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1}]\begin{bmatrix} u(0)\\ u(1)\\ \vdots\\ u(n-1)\end{bmatrix} = \mathbf{H}_n\mathbf{u}_n \tag{7.4-4}$$

where $\mathbf{H}_n$ is an $n \times n$ matrix whose jth column is $\mathbf{h}_{j-1}$
$\mathbf{u}_n$ is an $n \times 1$ vector whose jth element is $u(j-1)$.

Equation (7.4-4) will have a solution $\mathbf{u}_n$ for arbitrary vectors $\mathbf{x}(0)$ and $\mathbf{x}_D$ if and only if $\mathbf{H}_n$ is nonsingular. This immediately implies that the columns $\mathbf{h}_{j-1}$ of $\mathbf{H}_n$ are linearly independent.

Conventionally, it is said that system (7.4-1) is controllable if the vectors $(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n)$ are linearly independent, where

$$\mathbf{f}_k = -\mathbf{A}^{-k}\mathbf{B} \tag{7.4-5}$$

is an $n \times 1$ vector.

It is an easy exercise to show that if $\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$ are linearly independent then so are $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ and vice versa.

Fortunately, most practical linear discrete systems are controllable.

EXAMPLE 7.4-1

To illustrate how one determines the controllability characteristics of a specific system, consider a system governed by the second-order differential equation

$$\frac{d^2c}{dt^2} + \frac{dc}{dt} = u \tag{7.4-6}$$

as shown in Figure 7.4-1

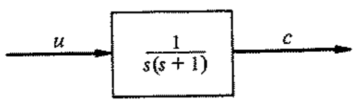

Figure 7.4-1. Second-order system.

The system (7.4-6) may be characterized by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

and

$$c(t) = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x_1(t) = c$ and $x_2(t) = dc/dt$.

Taking into account the fact that the input $u(t)$ is constrained to be constant over one-second intervals, we find

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & .632 \\ 0 & .368 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} .368 \\ .632 \end{bmatrix} u(k)$$

from which we have

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1.718 \\ 0 & 2.718 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} .368 \\ .632 \end{bmatrix}$$

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ as given by (7.4-5) are

$$\mathbf{f}_1 = \begin{bmatrix} .718 \\ -1.718 \end{bmatrix}, \qquad \mathbf{f}_2 = \begin{bmatrix} 3.671 \\ -4.671 \end{bmatrix}$$

which we may verify to be linearly independent. This system is therefore controllable.

Some simplification, in the thought process at least, is obtained if the desired state $\mathbf{x}_D$ is set equal to the zero state (origin in state space). No loss of generality results from this assumption, since it is possible to define a new state vector $\mathbf{x}_{new}$ related to the former state vector $\mathbf{x}_{old}$ by

$$\mathbf{x}_{new} = \mathbf{x}_{old} - \mathbf{x}_D$$

for which when $\mathbf{x}_{old} = \mathbf{x}_D$ it follows that $\mathbf{x}_{new} = \mathbf{0}$ (the origin in the new state space). This is merely a shifting of the origin of the original state space by the amount $\mathbf{x}_D$.

Unless otherwise noted, the desired state vector $\mathbf{x}_D$ will be taken to be equal to the zero vector in all discussions following.

Let us now determine those initial states $\mathbf{x}(0)$ that may be forced to the zero state ($\mathbf{x}_D = 0$) as a function of the number of iteration times N. Premultiplying both sides of (7.4-2) by $\mathbf{A}^{-N}$ and setting $\mathbf{x}(N) = 0$, we obtain

$$\mathbf{x}(0) = \mathbf{f}_1 u(0) + \mathbf{f}_2 u(1) + \cdots + \mathbf{f}_N u(N-1) \tag{7.4-7}$$

where $\mathbf{f}_k = -\mathbf{A}^{-k}\mathbf{B}$. If (7.4-7) is satisfied, then it is guaranteed that $\mathbf{x}(N) = 0$.

Specifically, let $N = 1$. Therefore, any initial state that may be forced to the zero state must be representable as

$$\mathbf{x}(0) = \mathbf{f}_1 u(0) \tag{7.4-8}$$

where $\mathbf{f}_1$ is an $n \times 1$ vector and $u(0)$ is a scalar. Any initial state that lies on the line passing through the tip of vector $\mathbf{f}_1$ and the origin in state space may be forced to the zero state (the origin) in one iteration time. All other initial states not positioned on this line cannot be forced to the zero state in one iteration time.

For example, consider the system given in Example 7.4-1. For this system the vector $\mathbf{f}_1$ was given by

$$\mathbf{f}_1 = \begin{bmatrix} .718 \\ -1.718 \end{bmatrix}$$

Figure (7.4-2) shows all initial states that may be forced to the origin in one iteration time.

For $N = 2$, equation (7.4-7) becomes

$$\mathbf{x}(0) = \mathbf{f}_1 u(0) + \mathbf{f}_2 u(1) \tag{7.4-9}$$

Expression (7.4-9) is simply a linear combination of the two $n \times 1$ vectors $\mathbf{f}_1$ and $\mathbf{f}_2$. The set of initial states that may be forced to the zero state has been expanded (if it is assumed that the order of the system is greater than one)

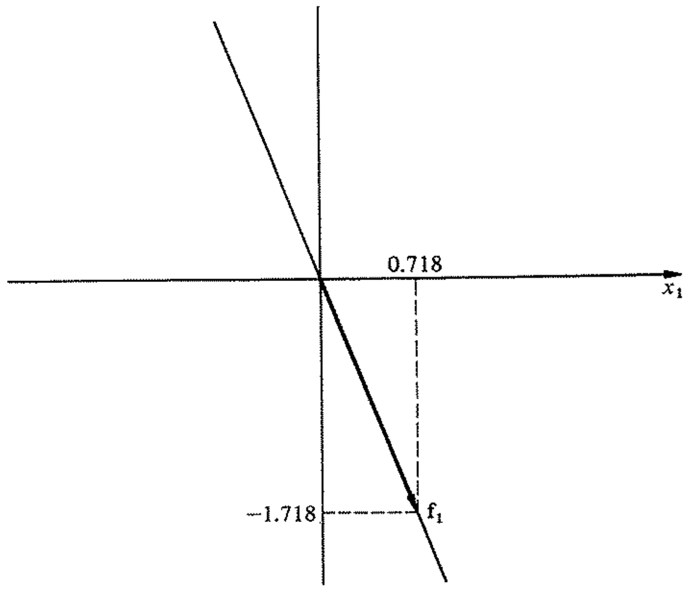

Figure 7.4-2. Initial states lying on line through tip of $\mathbf{f}_1$ and origin may be forced to the zero state in one iteration for the system of Example 7.4-1.

over the case $n = 1$ under the assumption that the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent. However, if $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly dependent, that is, if

$$\mathbf{f}_2 = \alpha \mathbf{f}_1 \quad \text{for some scalar}$$

then (7.4-9) becomes

$$\mathbf{x}(0) = \mathbf{f}_1\{u(0) + \alpha u(1)\} = \mathbf{f}_1 \hat{\mathbf{u}}(0)$$

and the same set of initial states may be forced to the origin as was the case for $N = 1$. Therefore, it is desirable, from a control viewpoint, to have $\mathbf{f}_1$ and $\mathbf{f}_2$ linearly independent.

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ for Example 7.4-1 are shown in Figure 7.4-3 for the system given in Example 7.4-1. We see readily from Figure 7.4-3 that the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent. Since, in this example, state space is two-dimensional, it follows that *any* initial state $\mathbf{x}(0)$ may be written as a linear combination of the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$. To demonstrate this fact algebraically, rewrite (7.4-9) as

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} .718 \\ -1.718 \end{bmatrix} u(0) + \begin{bmatrix} 3.671 \\ -4.671 \end{bmatrix} u(1) = \begin{bmatrix} .718 & 3.671 \\ -1.718 & -4.671 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \tag{7.4-10}$$

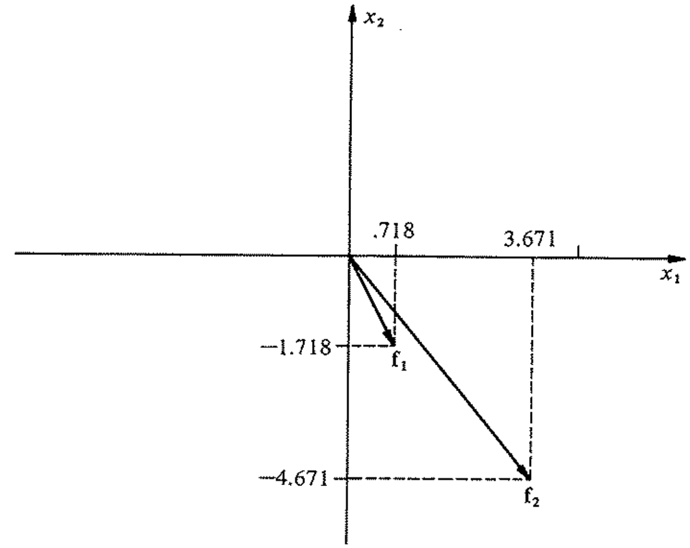

Figure 7.4-3. Vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ for the system of Example 7.4-1.

Now the matrix multiplying the vector

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$$

is invertible, so (7.4-10) has a unique set of scalars $\{u(0), u(1)\}$, which will satisfy this equation for arbitrary $x_1(0)$, $x_2(0)$.

For systems that are of order exceeding 1 or 2, it follows that not all initial states may be forced to the zero state in one or two iteration times. For example, if (7.4-9) were viewed as a set of n equations in two unknowns [i.e., $u(0)$, $u(1)$], only under very specific choices of $x(0)$ will (7.4-9) have a solution.

We might think of the set of initial states as written in (7.4-8) as forming a one-dimensional subspace of the n-dimensional state space. Similarly, if $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent, the set of vectors written in the form given by (7.4-9) can be thought of as forming a two-dimensional subspace of the n-dimensional state space.

This notion may be extended to $N = 3, 4, 5, \ldots$. The set of initial states that may be forced to the origin in N iteration times is given by

$$\mathbf{R}_N = \left\{\mathbf{x}(0): \quad \mathbf{x}(0) = \sum_{i=0}^{N-1} \mathbf{f}_{i+1} u(i)\right\} \tag{7.4-11}$$

For $N \leq n$, $\mathbf{R}_N$ is a subspace of state space. If the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$ are linearly independent, subspace $\mathbf{R}_N$ has dimension N.

7.4-2 Control Amplitude Limitations

All practical control systems have some limitation on the amplitude of the control that may be applied. In such cases, the set of initial states that may be forced to the zero state in N iteration times changes drastically. For example, let the control inputs be constrained by

$$-1 \leq u(i) \leq 1 \quad \text{for } i = 0, 1, 2, \ldots \tag{7.4-12}$$

The set of initial states that may be forced to the origin in N iteration times and this constraint is given by

$$\mathbf{R}_N^s = \left\{\mathbf{x}(0): \quad \mathbf{x}(0) = \sum_{i=0}^{N-1} \mathbf{f}_{i+1} u(i) \quad \text{with } |u(i)| \leq 1\right\} \tag{7.4-13}$$

For the system considered in Example 7.4-1, $\mathbf{R}_1^s$ and $\mathbf{R}_2^s$ are shown in Figures 7.4-4 and 7.4-5, respectively.

An investigation of Figures 7.4-1 through 7.4-4 reveals the control loss one suffers when control amplitude constraints are imposed.

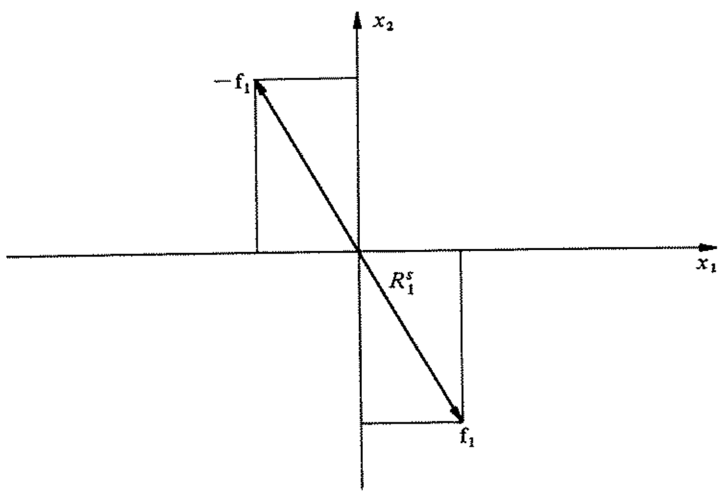

Figure 7.4-4. Region $\mathbf{R}_1^s$.

7.4-3 Observability

To develop some of the basic properties of system observability, we shall study the discrete system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{7.4-14}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \tag{7.4-15}$$

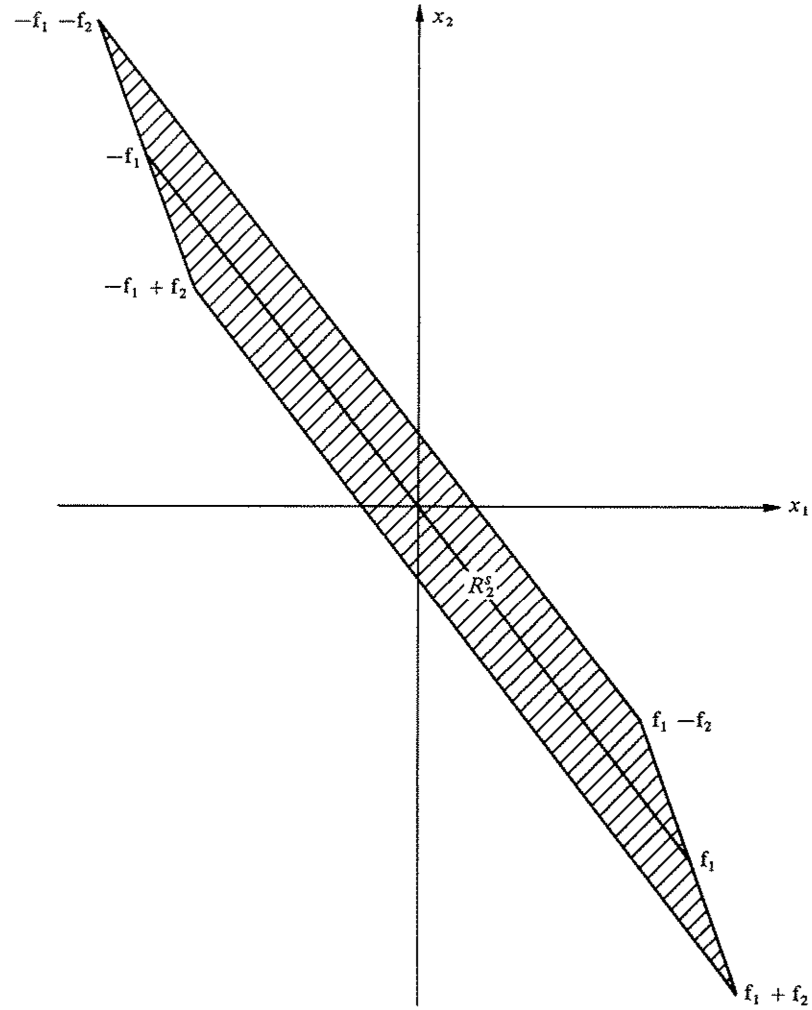

Figure 7.4-5. Region $\mathbf{R}_2^s$.

where $\mathbf{x}(k)$ is a $n \times 1$ state vector, $\mathbf{u}(k)$ is the $m \times 1$ input vector, $y(k)$ is the measurable output signal, while $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $n \times n$, $n \times m$, and $1 \times n$ matrices, respectively. The lack of a $\mathbf{Du}(k)$ term in (7.4-15) is not very restrictive, since any system whose transfer function has more poles than zeros may always be put into this form.

It is frequently desirable to be able to determine the state of the system $\mathbf{x}(k)$ from a knowledge of the measurable output signal $y(k)$. Since the state vector $\mathbf{x}(k)$ has n components while the output signal is only a scalar, it is apparent that we need many values of $y(k)$, both present and past, in order to determine $\mathbf{x}(k)$. In essence, this is basically the meaning of observability. A system such as (7.4-14) and (7.4-15) is said to be "completely observable" if it is possible to determine the state $\mathbf{x}(k)$ from present and past values of the measurable output signal.

Since the control input vector $\mathbf{u}(k)$ and input matrix $\mathbf{B}$ are known quantities, we may investigate the zero input case [i.e., $\mathbf{u}(k) = 0$] to develop the necessary and sufficient condition for system observability. Namely, we shall be concerned with the system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) \tag{7.4-16}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \tag{7.4-17}$$

Iteratively applying (7.4-16), we have

$$\mathbf{x}(k-m) = \mathbf{A}^{-m}\mathbf{x}(k) \quad \text{for } m = 0, 1, 2, \ldots$$

so that

$$y(k-m) = \mathbf{C}\mathbf{A}^{-m}\mathbf{x}(k) = \mathbf{g}_m\mathbf{x}(k)$$

where $\mathbf{g}_m$ is a $1 \times n$ vector. Therefore,

$$\begin{aligned} y(k) &= \mathbf{g}_0\mathbf{x}(k) \\ y(k-1) &= \mathbf{g}_1\mathbf{x}(k) \\ y(k-n+1) &= \mathbf{g}_{n-1}\mathbf{x}(k) \end{aligned}$$

or, equivalently,

$$\begin{bmatrix} y(k) \\ y(k-1) \\ \cdot \\ \cdot \\ \cdot \\ y(k-n+1) \end{bmatrix} = \mathbf{G}\mathbf{x}(k) \tag{7.4-18}$$

where $\mathbf{G}$ is an $n \times n$ matrix whose first row is $\mathbf{g}_0$, whose second row is $\mathbf{g}_1$, etc. A sufficient condition that (7.4-18) have a unique solution is that the matrix $\mathbf{G}$ be invertible. If this be the case, then

$$\mathbf{x}(k) = \mathbf{G}^{-1}\begin{bmatrix} y(k) \\ y(k-1) \\ \cdot \\ \cdot \\ \cdot \\ y(k-n+1) \end{bmatrix} \tag{7.4-19}$$

Expression (7.4-19) indicates that if the value of the output signal at times $k, k-1, \ldots, k-n+1$ is retained, then the state $\mathbf{x}(k)$ may be directly determined. Therefore, the invertibility of $\mathbf{G}$ is a sufficient condition for "complete observability," and it turns out to be a necessary condition also. $\mathbf{G}$ is invertible if and only if it has independent rows, which implies that an alternate necessary and sufficient condition for "complete observability" is that the set of $1 \times n$ vectors

$$\{\mathbf{A}, \mathbf{C}\mathbf{A}, \mathbf{C}\mathbf{A}^2, \ldots, \mathbf{C}\mathbf{A}^{n-1}\}$$

must be linearly independent. This gives us a very straightforward method for determining the "complete observability" of a system.

With the concepts of controllability and observability established, we shall now treat some of the more important optimal control problems.

7.5 Regulator Problem

The most basic control problem is one of regulation. Suppose that we wish to control a system governed by the vector difference equation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \tag{7.5-1}$$

in such a manner as to transfer the system from some arbitrary initial state $\mathbf{x}(0)$ to a desired state $\mathbf{x}_D$. What is the form of the required control sequence and the control law that generates this sequence?

It will be assumed that this nth-order discrete system with one control input is completely controllable; that is, the set of vectors $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$, where

$$\mathbf{f}_k = -\mathbf{A}^k\mathbf{B} \quad (\text{an } n \times 1 \text{ vector for } k \geq 1)$$

forms a set of linearly independent vectors.

Repeatedly applying equation (7.5-1) gives

$$\begin{aligned} \mathbf{x}(N) = \mathbf{A}^N\mathbf{x}(0) &+ \mathbf{A}^{N-1}\mathbf{B}u(0) + \mathbf{A}^{N-2}\mathbf{B}u(1) \\ &+ \cdots + \mathbf{A}\mathbf{B}u(N-2) + \mathbf{B}u(N-1) \end{aligned}$$

If it is possible to select a control sequence $u(0), u(1), \ldots, u(N-1)$ so that the state $\mathbf{x}(N) = \mathbf{x}_D$, this implies that the control sequence must satisfy the relationship

$$\mathbf{x}(0) - \mathbf{A}^{-N}\mathbf{x}_D = \mathbf{f}_1u(0) + \mathbf{f}_2u(1) + \cdots + \mathbf{f}_Nu(N-1)$$

or in matrix form

$$\mathbf{x}(0) - \mathbf{A}^{-N}\mathbf{x}_D = \mathbf{F}_N\mathbf{u}_N \tag{7.5-2}$$

where $\mathbf{F}$ is an $n \times N$ matrix with columns $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$ while $\mathbf{u}_N$ is an $N \times 1$ vector whose first element is $u(0)$, whose second element is $u(1)$, etc. Equation (7.5-2) is, in fact, a system of n equations in N unknowns [the $u(i)$'s]. If N is less than n, that is, if the number of control sequences is less than the order of the system, it may not be possible to find a control vector $\mathbf{u}_N$ to satisfy (7.5-2). With this in mind, let $N = n$; now the $n \times n$ matrix $\mathbf{F}_n$ has an inverse, since it has n linearly independent columns (the $\mathbf{f}_k$'s) because of the assumption of system controllability. Premultiplying both sides of (7.5-2) by $\mathbf{F}_n^{-1}$ gives the required control vector to effect the desired state transformation; that is,

$$\mathbf{u}_n = \mathbf{F}_n^{-1}[\mathbf{x}(0) - \mathbf{A}^{-N}\mathbf{x}_D] \tag{7.5-3}$$

The control law as given by (7.5-3) is an open-loop control law, since the required control vector $\mathbf{u}_n$ depends only on the initial state $\mathbf{x}(0)$. Therefore, once the initial state $\mathbf{x}(0)$ is monitored and the desired state $\mathbf{x}_D$ given, the control sequence $u(0), u(1), \ldots, u(n-1)$ is immediately calculated from (7.5-3). This sequence is unique because for $N = n$, equation (7.5-2) is a set of n equations in n unknowns and, since $\mathbf{F}_n$ is a nonsingular matrix, there exists one solution, which is given by (7.5-3).

For controllable systems, expression (7.5-3) gives the control vector that will drive the system from any initial state to any desired state in n iteration times. If $N < n$, it is not always possible to select the control sequence $\{u(k)\}$ so that $\mathbf{x}(N) = \mathbf{x}_D$. When $N > n$, the control engineer has some design freedom. In this latter case (i.e., $N > n$), there exists an infinite number of different control sequences to effect the desired control, and the designer may select from this multitude of choices one that may satisfy secondary constraints such as minimum energy regulation, minimum amplitude regulation, etc.

EXAMPLE 7.5-1

Consider the system studied in Example 7.4-1. Suppose that it is desired to drive this system from any arbitrary initial state to the zero state in two iteration (sample times) times. From (7.5-2) with $\mathbf{x}_D = \mathbf{0}$ and $N = 2$, we have

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} .7183 & 3.6708 \\ -1.7183 & -4.6708 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \tag{7.5-4}$$

Premultiplying both sides of (7.5-4) by $\mathbf{F}_2^{-1}$ yields

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} -1.5820 & -1.2433 \\ .5820 & .2433 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

which, when we set $x_1(0) = c(0)$ and $x_2(0) = dc(0)/dt$, results in

$$\begin{aligned} u(0) &= -1.5820c(0) - 1.2433\frac{dc(0)}{dt} \\ u(1) &= .5820c(0) + .2433\frac{dc(0)}{dt} \end{aligned} \tag{7.5-5}$$

The control law as given by (7.5-5) reveals the open-loop nature of this type of control.

In the regulation problem, it is not necessary to equate the number of control iterations (N) to the order of the linear discrete system under control (n). Let us now investigate the case when $N \geq n$ for controllable systems. Equation (7.5-2) is a set of n equations in N [the $u(i)$'s] unknowns, and when $N > n$, we have more unknowns than equations. When these equations are consistent (have at least one solution), there exists an infinite number of different solutions. Because the system under control is assumed controllable, this set of n equations in N unknowns is *always* consistent for $N \geq n$. This fact is best illustrated by means of an example.

EXAMPLE 7.5-2

For the system investigated in Example 7.4-1, determine the properties of the control sequence needed to force $\mathbf{x}(3) = \mathbf{0}$.

In this case, for $N = 3$, we find that

$$\mathbf{f}_1 = \begin{bmatrix} .7183 \\ -1.7183 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 3.6708 \\ -4.6708 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} 11.6965 \\ -12.6965 \end{bmatrix}$$

so that (7.5-3) becomes

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} .7183 & 3.6708 & 11.6965 \\ -1.7183 & -4.6708 & -12.6965 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

or, in equation form,

$$\begin{aligned} x_1(0) &= .7183u(0) + 3.6708u(1) + 11.6965u(2) \\ x_2(0) &= -1.7183u(0) - 4.6708u(1) - 12.6965u(2) \end{aligned}$$

that is, $2(n = 2)$ equations in $3(N = 3)$ unknowns. We may easily verify that this set of equations has at least one solution [e.g., let $u(2) = 0$ and see Example 7.5-1].

7.6 Minimum Energy Control

In many applications, we desire to accomplish a given control task, using the minimum amount of control energy necessary. Such a problem will now be formulated. (See also Ref. 5.)

The system under study is governed by the vector difference equation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \tag{7.6-1}$$

A linear, time-invariant vector difference equation of the form given by (7.6-1) will occur whenever a continuous system that is characterized by a linear, time-invariant differential equation is driven by a single input that is constant over fixed intervals of time (sampled-data systems). This was demonstrated in Chapters 2 and 3.

We desire to drive such a system from any arbitrary initial state $\mathbf{x}(0)$ to a desired state $\mathbf{x}_D$ in N iteration times, using a minimum of control energy. Control energy will be measured by the quantity

$$E_N = \sum_{k=0}^{N-1} u^2(k) \tag{7.6-2}$$

If the alotted number of control iterations N is smaller than the order of the system n, it will not always be possible to accomplish the desired control action. In order to take this factor into account, a slightly different control problem is postulated.

Given a discrete system governed by (7.6-1), design a controller that generates the control input $u(0), u(1), \ldots, u(N-1)$ that

1. Takes the system from any initial state $\mathbf{x}(0)$ to a desired state $\mathbf{x}_D$ in N iterations while minimizing control energy [as measured by (7.6-2)] and, if this is not possible,
2. Minimizes the Euclidean distance of the state of the discrete system from the desired state at the end of N iteration times; that is,

$$(\mathbf{x}_D - \mathbf{x}(N))^T(\mathbf{x}_D - \mathbf{x}(N)) \tag{7.6-3}$$

Expression (7.6-3) gives a measure of the distance that the error vector $\mathbf{x}_D - \mathbf{x}(N)$ is from the zero vector. For the remainder of this section, the desired state is taken to be the zero state. No loss in generality is incurred under this assumption, since if the zero state can be reached in N iteration times, then so can any other state.

It is desired to force $\mathbf{x}(N) = \mathbf{0}$ by a proper selection of control inputs $u(0), u(1), \ldots, u(N-1)$. If this is to be possible for any initial state $\mathbf{x}(0)$, then it must be possible to select the parameters (control inputs) $u(0), u(1), \ldots, u(N-1)$ so that

$$\mathbf{x}(N) = \mathbf{0} = \mathbf{A}^N\mathbf{x}(0) + \mathbf{A}^{N-1}\mathbf{B}u(0) + \cdots + \mathbf{A}\mathbf{B}u(N-2) + \mathbf{B}u(N-1)$$

or, if we premultiply both sides by $\mathbf{A}^{-N}$ and use the identity $\mathbf{f}_k = -\mathbf{A}^{-k}\mathbf{B}$,

$$\mathbf{x}(0) = u(0)\mathbf{f}_1 + u(1)\mathbf{f}_2 + \cdots + u(N-1)\mathbf{f}_N \tag{7.6-4}$$

Equation (7.6-4) indicates that if $\mathbf{x}(N) = \mathbf{0}$, then the initial state $\mathbf{x}(0)$ must be expressible as a linear combination of the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$. For controllable systems, the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ form a set of linearly independent vectors, so that any $n \times 1$ vector may be expressed as a linear combination of such a set. This suggests that for controllable systems it is always possible to force $\mathbf{x}(N) = \mathbf{0}$ for $N \geq n$ for arbitrary initial states $\mathbf{x}(0)$. When $N < n$, it is not possible to force all initial states to the zero state. Two cases will then be considered: (i) $N \geq n$, and (ii) $N < n$.

Case (i) $N \geq n$

For $N \geq n$, the state vector after N iteration times may always be forced to the zero state.

Thus, expressing 7.6-4 in matrix form,

$$\mathbf{x}(0) = \mathbf{F}\mathbf{u} \tag{7.6-5}$$

where $\mathbf{F}$ = the $n \times N$ matrix with columns $\mathbf{f}_i$
$\mathbf{u}$ = the $N \times 1$ control sequence vector with elements u_i

The optimal control sequence must satisfy the matrix equation (7.6-5) and also must be a minimum energy solution. It should be noted that if the control sequence satisfies equation (7.6-5), then $\mathbf{x}(N) = \mathbf{0}$ is guaranteed.

Case (ii) $N < n$

The case when the plant is controllable and $N < n$ is treated analogously to the case when $N \geq n$, but with one basic difference. The assumption that any initial state can be forced to the zero state after N sampling periods is not valid. If $\mathbf{x}(N)$ can be forced to the zero state, the control sequence required for $N < n$ is unique, and if it is not possible to force $\mathbf{x}(N) = \mathbf{0}$ then the Euclidean distance

$$\|\mathbf{x}(N)\|^{1/2} = \left(\sum_{i=1}^{n} x_i^2(N)\right)^{1/2} \tag{7.6-6}$$

must be minimized. It will be observed that no mention of minimum energy is made for $N < n$. Thus the minimum energy problem in this case is equivalent to selecting a control sequence that minimizes $\|\mathbf{x}(N)\|$. The solution to the minimum energy problem for controllable systems is illustrated in Table 7.6-1.

Table 7.6-1 Solution of Minimum Energy Problem for Controllable Plants

Case	Solution
$N \geq n$	Minimum energy solution of $\mathbf{x}(0) = \mathbf{Fu}$
$N < n$	Minimize $\|\mathbf{x}(N)\|$

Before we consider the techniques of obtaining the solution to this problem, a few developments in inverse matrix theory will be presented.

7.6-1 Right and Left Inverse Matrix Theory

To demonstrate the use of inverse matrix theory in the solving of a system of n linear equations in m unknowns, the following set of equations will be considered.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

or in its equivalent matrix form

$$\mathbf{Ax} = \mathbf{b} \tag{7.6-7}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

Consider the special case when $m = n$; it can be easily shown that if the rank of matrix $\mathbf{A}$ is n, then equation (7.6-7) has a unique solution given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

where the $n \times n$ matrix $\mathbf{A}^{-1}$ is that unique matrix that satisfies the two properties

$$\mathbf{AA}^{-1} = \mathbf{I}_n$$
$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

$\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$.

An alternate necessary and sufficient condition for a square matrix $\mathbf{A}$ to have an inverse matrix $\mathbf{A}^{-1}$ is for det $(\mathbf{A}) \neq 0$. If the determinant of $\mathbf{A}$ is equal to zero, then equation (7.6-7) may have infinitely many solutions or in fact may have no solutions at all.

When $m \neq n$, matrix theory can still play an important role in the seeking of solutions to a system of linear equations.

Definition. An $n \times m$ matrix $\mathbf{A}$ is said to have a right inverse $\mathbf{A}^R$ if $\mathbf{AA}^R = \mathbf{I}_n$. Similarly, the $n \times m$ matrix $\mathbf{A}$ is said to have a left inverse $\mathbf{A}^L$ if $\mathbf{A}^L\mathbf{A} = \mathbf{I}_m$.

Theorem 7.6-1. If the $n \times m$ matrix $\mathbf{A}$ has a right inverse matrix $\mathbf{A}^R$, then

$$\mathbf{x} = \mathbf{A}^R\mathbf{b}$$

is a solution to the consistent matrix equation $\mathbf{Ax} = \mathbf{b}$. The term *consistent* as used above indicates that the matrix equation has at least one solution.

Proof. Substitute the assumed solution into the matrix equation; if it is indeed a solution it will satisfy the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

Let

$$\mathbf{x} = \mathbf{A}^R\mathbf{b}$$
$$\mathbf{Ax} = \mathbf{AA}^R\mathbf{b} = \mathbf{b}$$

Theorem 7.6-2. If the $n \times m$ matrix $\mathbf{A}$ has a right inverse matrix $\mathbf{A}^R$, then

$$\mathbf{x} = (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}$$

is a solution to the homogeneous matrix equation $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{y}$ is "any" $m \times 1$ vector.

Proof. $\mathbf{Ax} = \mathbf{0}$

Let

$$\mathbf{x} = (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}$$
$$\mathbf{Ax} = \mathbf{A}(\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y} = (\mathbf{A} - \mathbf{AA}^R\mathbf{A})\mathbf{y} = \mathbf{0}$$

Corollary. If the $n \times m$ matrix $\mathbf{A}$ has a right inverse matrix $\mathbf{A}^R$, then

$$\mathbf{x} = \mathbf{A}^R\mathbf{b} + (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y} \tag{7.6-8}$$

is a general solution to the consistent matrix equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{y}$ is any $m \times 1$ vector.

It will be noted that the solution space of the matrix equation $\mathbf{Ax} = \mathbf{b}$ is of dimension equal to the rank of $\mathbf{I}_m - \mathbf{A}^R\mathbf{A}$. So that, in general, if the matrix of $\mathbf{A}$ has a right inverse, there is an infinite number of solutions all contained in the subset specified by

$$\mathbf{x} = \mathbf{A}^R\mathbf{b} + (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}$$

where the vector $\mathbf{A}^R\mathbf{b}$ is fixed (for a given $\mathbf{A}^R$) and the vector $\mathbf{y}$ is allowed to span the m-dimensional space.

It should be pointed out that a right inverse matrix $\mathbf{A}^R$, if it exists, is in general not unique, as will be demonstrated in the next example.

EXAMPLE 7.6-1

Consider the set of equations

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_2 + 2x_3 &= 3 \end{aligned} \tag{7.6-9}$$

Therefore,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

and, as is easily verified, one right inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^R = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \tag{7.6-10}$$

The general solution using equation (7.6-8) becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 & -4 & -4 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The rank of $\mathbf{I}_m - \mathbf{A}^R\mathbf{A}$ in this example is one so that in general any solution to equations (7.6-9) is located on a line in three-dimensional real space which passes through the point (9, −3, 3). The equation of this line is specified by $(\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}$ and is obtained in the following manner:

$$\begin{bmatrix} 0 & -4 & -4 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4(y_2 + y_3) \\ 2(y_2 + y_3) \\ -(y_2 + y_3) \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} (y_2 + y_3)$$

The range of $y_2 + y_3$ is $(-\infty, \infty)$. The solutions to equations (7.6-9) become

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 - 4a \\ -3 + 2a \\ 3 - a \end{bmatrix}, \qquad -\infty < a < \infty$$

To demonstrate that the right inverse of matrix $\mathbf{A}$ as given by equation (7.6-10) is not unique, we may easily verify that the following matrix is also a right inverse of $\mathbf{A}$.

$$\mathbf{A}^R = \begin{bmatrix} 3 & 2 \\ -1 & -1 \\ \frac{1}{2} & 1 \end{bmatrix}$$

It can be shown that an $n \times m$ matrix $\mathbf{A}$ has a right inverse if and only if $\mathbf{A}$ is of rank n and has a left inverse if and only if the rank of $\mathbf{A}$ is m.

7.6-2 Minimal Right and Left Inverse Matrices

If the $n \times m$ matrix $\mathbf{A}$ has a right inverse, then the solutions to the matrix equation $\mathbf{Ax} = \mathbf{b}$ are given by

$$\mathbf{x} = \mathbf{A}^R\mathbf{b} + (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}$$

Of all the solutions existent, we are particularly interested in the solution $\mathbf{x}^0$, which is smallest in the Euclidean norm sense; i.e.,

$$\sum_{i=1}^{n} (x_i^0)^2 = \|\mathbf{x}^0\| = \min_{\mathbf{y}} [\|\mathbf{A}^R\mathbf{b} + (\mathbf{I}_m - \mathbf{A}^R\mathbf{A})\mathbf{y}\|] \tag{7.6-11}$$

The problem to be considered here is the determination of that right inverse of $\mathbf{A}$ (which will be denoted by $\mathbf{A}^{RM}$) that will yield the solution $\mathbf{x}^0$ given by equation (7.6-11); i.e.,

$$\mathbf{x}^0 = \mathbf{A}^{RM}\mathbf{b}$$

Definition. $\mathbf{x}^0$ is the "minimal Euclidean" solution to the consistent matrix equation $\mathbf{Ax} = \mathbf{b}$ if

(i) $\mathbf{Ax}^0 = \mathbf{b}$
(ii) $\|\mathbf{x}^0\| \leq \|\mathbf{x}\|$ for all $\mathbf{x}$ that satisfy $\mathbf{Ax} = \mathbf{b}$

Theorem 7.6-3. If the $n \times m$ matrix $\mathbf{A}$ is of rank n, then the "minimal Euclidean" solution to the consistent matrix equation $\mathbf{Ax} = \mathbf{b}$ is given by

$$\mathbf{x}^0 = \mathbf{A}^{RM}\mathbf{b} \tag{7.6-12}$$

where $\mathbf{A}^{RM} = \mathbf{A}^T(\mathbf{AA}^T)^{-1}$ and $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. $\mathbf{A}^{RM}$ will be called the minimal right inverse of $\mathbf{A}$.

EXAMPLE 7.6-2

The "minimal Euclidean" solution to Example 7.6-1 will now be determined.

We recall that

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Therefore,

$$\mathbf{AA}^T = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\mathbf{A}^{RM} = \mathbf{A}^T(\mathbf{AA}^T)^{-1} = \frac{1}{21}\begin{bmatrix} 5 & -2 \\ 8 & 1 \\ -4 & 10 \end{bmatrix}$$

and

$$\mathbf{x}^0 = \mathbf{A}^{RM}\mathbf{b} = \begin{bmatrix} \frac{3}{7} \\ \frac{9}{7} \\ \frac{6}{7} \end{bmatrix}$$

Definition. $\mathbf{x}^0$ is the "minimal Euclidean" approximation solution to the matrix equation $\mathbf{Ax} = \mathbf{b}$ if

$$\|\mathbf{Ax} - \mathbf{b}\| > \|\mathbf{Ax}^0 - \mathbf{b}\| \quad \text{for all } \mathbf{x} \neq \mathbf{x}^0$$

Theorem 7.6-4. If the $n \times m$ matrix $\mathbf{A}$ is of rank m, then the "minimal Euclidean" approximation solution to the matrix equation $\mathbf{Ax} = \mathbf{b}$ is given by

$$\mathbf{x}^0 = \mathbf{A}^{LM}\mathbf{b} \tag{7.6-13}$$

where $\mathbf{A}^{LM} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ and will be called the minimal left inverse of $\mathbf{A}$.

7.6-3 Application of the Minimal Right Inverse Matrix to the Minimum Energy Problem for Controllable Plants

In the case where the plant is controllable and $N \geq n$, it was shown that the minimum energy problem was obtained by satisfying the matrix equation

$$\mathbf{x}(0) = \mathbf{Fu} \tag{7.6-14}$$

and the control sequence vector $\mathbf{u}$ which satisfies this equation must be a minimal energy solution. As $N \geq n$, this control sequence will guarantee that the state of the plant after N sampling periods will be the zero state vector.

Since the plant is assumed controllable, the $n \times N$ matrix has rank n; thus it has a minimal right inverse. Using Theorem 7.6-3, we find that the matrix $\mathbf{F}$ takes the place of matrix $\mathbf{A}$ and the "minimal Euclidean" solution to equation (7.6-14) becomes

$$\mathbf{u}^0 = \mathbf{F}^T(\mathbf{FF}^T)^{-1}\mathbf{x}(0)$$

That $\mathbf{u}^0$ is the minimal energy solution follows from the fact that $\mathbf{u}^0$ is the "minimal Euclidean" solution to equation (7.6-14); i.e.,

$$\|\mathbf{u}^0\| = \left(\sum_{i=0}^{N-1} (\mathbf{u}_i^0)^2\right)$$

which is a measure of the energy. Since $\mathbf{u}^0$ satisfies equation (7.6-14), it follows, by the remarks leading to Theorem 7.6-3, that $\mathbf{u}^0$ is the minimal energy control sequence that forces $\mathbf{x}(N) = \mathbf{0}$.

The matrix $\mathbf{F}^T(\mathbf{FF}^T)^{-1}$ is an $N \times n$ matrix and right-multiplying it by $\mathbf{x}(0)$ requires Nn multiplications to generate the minimal energy control sequence vector $\mathbf{u}^0$. When the optimal controller first senses the initial disturbance $\mathbf{x}(0)$, it performs the n multiplications needed to generate the control function for the first sampling period; i.e.,

$$u(0) = \sum_{j=1}^{n} \alpha_{ij} x_j \tag{7.6-15}$$

where α_{ij} = the (i, j) element of $\mathbf{F}^T(\mathbf{FF}^T)^{-1}$
x_j = the jth component of $\mathbf{x}(0)$.

The optimal controller then applies $u(0)$ during the first sampling period and simultaneously determines the remaining components $u(1)$, $u(2), \ldots,$ $u(N-1)$. The time between when the initial disturbance is detected and when $u(0)$ is applied to the plant is essentially the time required to carry out the n multiplications as given by equation (7.6-15). If this time is small in comparison to the sampling period and the plant time constants, then a real-time minimum energy control strategy is feasible.

It is felt that the case when the plant is controllable and $N \geq n$ is the most practical situation a control engineer will meet. This is because the great majority of plants encountered in practice are controllable and by selecting $N \geq n$ it is guaranteed that any initial disturbance will always be completely extinguished in N sampling periods.

The optimal controller for this case is illustrated in block diagram form in Figure 7.6-1.

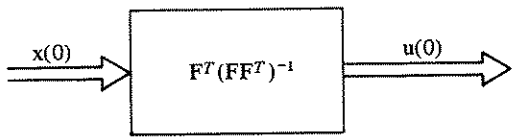

Figure 7.6-1. Optimal controller for a controllable plant, $N \geq n$.

7.6-4 Numerical Example

The plant under study is characterized by the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

For this plant we have

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The fundamental matrix $\mathbf{A}$ is evaluated by standard techniques, and for a sampling period of one second it is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 - e^{-1} \\ 0 & e^{-1} \end{bmatrix} \qquad (7.6\text{-}16)$$

$\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} e^{-1} \\ 1 - e^{-1} \end{bmatrix} \qquad (7.6\text{-}17)$$

The vectors $\mathbf{f}_k$ which are the columns of $\mathbf{F}$, are given by

$$\mathbf{f}_k = -\mathbf{A}^{-k}\mathbf{B} \qquad (7.6\text{-}18)$$

It can be shown that

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 - e \\ 0 & e \end{bmatrix}$$

Evaluating $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, and $\mathbf{f}_4$ as given by equation (7.6-18) we obtain

$$\mathbf{f}_1 = \begin{bmatrix} .7183 \\ -1.7183 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 3.6708 \\ -4.6708 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} 11.6965 \\ -12.6965 \end{bmatrix},$$

$$\mathbf{f}_4 = \begin{bmatrix} 33.5126 \\ -34.5126 \end{bmatrix}$$

Since $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent, the plant is controllable.

Problem. Find the control sequence vector for the minimal energy problem for $N = 2, 3, 4$.

Since the order of the system under study is $n = 2$, the minimal energy solution for $N = 2, 3, 4$ will force any initial disturbance to zero. The results of Section 7.6 with $N \geq n$ are used.

Solution. The problem will be solved in detail for $N = 4$. The results for $N = 2$ and $N = 3$ will be given without detail.

$$\mathbf{F}\mathbf{u} = \mathbf{x}(0)$$

for $N = 4$, $\mathbf{F}$ becomes

$$\mathbf{F} = \begin{bmatrix} .7183 & 3.6708 & 11.6965 & 33.5126 \\ -1.7183 & -4.6708 & -12.6965 & -34.5126 \end{bmatrix}$$

Thus

$$\mathbf{F}\mathbf{F}^T = \begin{bmatrix} 1273.8924 & -1323.4916 \\ -1323.4916 & 1377.0897 \end{bmatrix}$$

Therefore,

$$(\mathbf{F}\mathbf{F}^T)^{-1} = \begin{bmatrix} .5225 & .5022 \\ .5022 & .4833 \end{bmatrix}$$

and the minimal right inverse becomes

$$\mathbf{F}^{RM} = \begin{bmatrix} -.4876 & -.4698 \\ -.4275 & -.4143 \\ -.2643 & -.2632 \\ .1794 & .1473 \end{bmatrix}$$

The minimal energy solution for $N = 4$ becomes

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} = \begin{bmatrix} -.4876 & -.4698 \\ -.4275 & -.4143 \\ -.2643 & -.2632 \\ .1794 & .1473 \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} \tag{7.6-19}$$

Similarly, for $N = 3$

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} -.7910 & -.7191 \\ -.5000 & -.4738 \\ .2910 & .1929 \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} \tag{7.6-20}$$

and finally for $N = 2$

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} -1.5820 & -1.2433 \\ .5820 & .2433 \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} \tag{7.6-21}$$

A specific initial condition will now be considered.

$$\mathbf{x}(0) = \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} -40.9067 \\ 43.5067 \end{bmatrix}$$

Substituting this initial condition into the matrix equations (7.6-19), (7.6-20), and (7.6-21), we obtain

$N = 4$

$u(0) = -.4963, \quad u(1) = -.5352,$

$u(2) = -.6408, \quad u(3) = -.9277$

$N = 3$

$u(0) = 1.0732, \quad u(1) = -.1602, \quad u(2) = -3.5130$

$N = 2$

$u(0) = 10.6225, \quad u(1) = -13.2225$

Although each of the above control sequences will force the plant with the specific initial condition to the zero state vector, it will be noted that by making N progressively larger the energy requirements are drastically reduced; i.e.,

$N = 2$

$$\sum_{i=0}^{1} u_i^2 = 287.6720$$

$N = 3$

$$\sum_{i=0}^{2} u_i^2 = 13.5186$$

$N = 4$

$$\sum_{i=0}^{3} u_i^2 = 1.8040$$

7.6-5 Minimum Energy Control with Amplitude Constraint

There are many practical control systems in which there is an amplitude limitation on the control that may be applied. A problem is now proposed for a sampled-data control system with a control amplitude limitation that may be solved by utilizing the techniques developed in this section.

Problem. Force any controllable plant from some initial state to the zero state in the minimum number of sampling periods subject to the condition that for that number of sampling periods it is the minimum energy solution and in addition none of the components of u_i exceeds a certain positive number a in absolute value; e.g.,

$$|u_i| \leq a$$

Solution. Since we are to force the plant to the zero state, use of the minimal right inverse matrix will be made. Precalculate $\mathbf{F}_N^{RM}$ for $N = n, n + 1, \ldots$ and store these matrices in the optimal controller. First compute

$$\mathbf{u}_n^0 = \mathbf{F}_n^{RM}\mathbf{x}(0)$$

If none of the components of $\mathbf{u}$ exceeds a in absolute value, then the optimal control sequence is obtained. If some do, then compute

$$\mathbf{u}_{n+1}^0 = \mathbf{F}_{n+1}^{RM}\mathbf{x}(0)$$

and continue this process until a $\mathbf{F}_k^{RM}\mathbf{x}(0)$ is obtained which satisfies the absolute magnitude constraint. The $\mathbf{u}_k^0$ obtained is the solution to the problem.

As a practical example, consider the system studied in section 7.6-4 with constraints

$$|u_i| \leq 1, \qquad i = 1, 2, \ldots$$

and the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} -40.9067 \\ 43.5067 \end{bmatrix}$$

It was shown that the optimal solution for the problem proposed in this section is given by

$$u_1 = -.4693, \qquad u_2 = -.5352, \qquad u_3 = -.6408, \qquad u_4 = -.9277$$

7.7 Tracking Test Inputs

A frequent requirement of discrete systems is the ability to track certain deterministic test input signals. The standard test input for such purposes is the discrete step of amplitude Q; that is

$$\begin{aligned} r(k) &= Q \quad \text{for } k \geq 0 \\ r(k) &= 0 \quad \text{for } k < 0 \end{aligned} \tag{7.7-1}$$

By *tracking*, we mean the ability of the system to respond to test inputs, such as the discrete step, so that the system's output is equal to the system's input with possibly some delay involved. Expressing this mathematically, we have

$$c(k) = r(k - m) \tag{7.7-2}$$

where $c(k)$ denotes the system's output and the integer m is the delay interval given in discrete-time iterations. If $m = 0$, this implies that the system's output exactly equals its input when the test signal is applied.

As most practical discrete systems have nonzero time constants, equalities such as that given in (7.7-2) are possible only after the transient terms have decayed to zero. Therefore, we shall initially investigate the case when (7.7-2)

holds for very large values of k(i.e., as $k \to \infty$). It is assumed that the system to which the input is applied is linear, so the relationship between $C(z)$ and $R(z)$ is of the form

$$C(z) = H(z)R(z) \tag{7.7-3}$$

The error signal is defined by

$$e(k) = c(k) - r(k - m) \tag{7.7-4}$$

and measures the amount by which the desired relationship (7.7-2) is incorrect. Taking the z-transform of (7.7-4) gives

$$E(z) = C(z) - z^{-m}R(z)$$

and using (7.7-3) results in

$$E(z) = [H(z) - z^{-m}]R(z) \tag{7.7-5}$$

Since we wish $e(k)$ to be zero for large k, we apply the final value theorem to (7.7-5) and set the result to zero; that is,

$$e(\infty) = \lim_{z \to 1} (1 - z^{-1})[H(z) - z^{-m}]R(z) = 0 \tag{7.7-6}$$

Relationship (7.7-6) is the necessary condition that must be satisfied in order for the discrete system with transfer function $H(z)$ to track the input $r(k)$.

An investigation of the properties required by $H(z)$ in order to track specific test input will now be made.

7.7-1 Step Input

This test input is characterized by equation (7.7-1) so that

$$R(z) = \frac{R}{1 - z^{-1}}$$

which when inserted into (7.7-6) yields

$$\lim_{z \to 1} [H(z) - z^{-m}] = 0 \tag{7.7-7}$$

Therefore, $H(1) - 1 = 0$, which implies that a linear discrete system will properly track a step input only if its transfer function evaluated at $z = 1$ equals unity. Since $H(z)$ is related to its weighting sequence $h(n)$ by

$$H(z) = \sum_{k=0}^{\infty} h(k)z^{-k}$$

requirement (7.7-7) is seen to be equivalent to

$$\sum_{k=0}^{\infty} h(k) = 1 \tag{7.7-8}$$

that is, the sum of the terms in the weighting sequence equals unity. For example, the discrete system with weighting sequence

$$\begin{aligned} h(0) &= 1 \\ h(1) &= -2 \\ h(2) &= 2 \\ h(k) &= 0 \quad \text{for } k \neq 0, 1, 2 \end{aligned}$$

will properly track a step input.

To demonstrate the potential control applications available, consider the digitally controlled system shown in Figure 7.7-1. Suppose it is desired that this control system track a step input of amplitude R; that is,

$$c(kT) \longrightarrow R \quad \text{for large } k$$

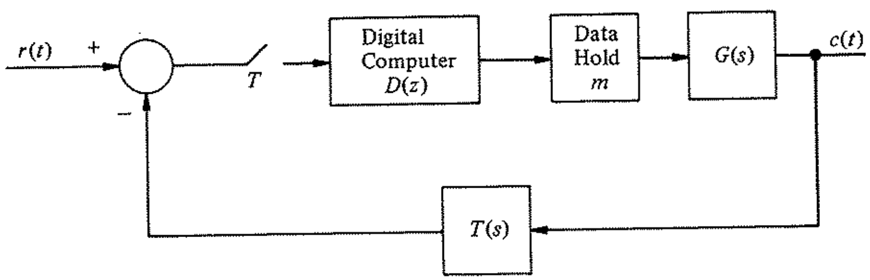

Figure 7.7-1

What constraint does this imply on the digital computer? From (6.5-13) it was previously shown that

$$H(z) = \frac{C(z)}{R(z)} = \frac{D(z)\mathscr{Z}[G_m(s)G(s)]}{1 + D(z)\mathscr{Z}[G_m(s)G(s)T(s)]} \tag{7.7-9}$$

For proper tracking of a step input, we must have $H(1) = 1$, which from (7.7-9) gives

$$D(z)\Big|_{z=1} = \sum_{k=0}^{\infty} d(k) = \frac{1}{\mathscr{Z}[G_m(s)G(s)] - \mathscr{Z}[G_m(s)G(s)T(s)]}\Bigg|_{z=1} \tag{7.7-10}$$

There is obviously an uncountable number of discrete systems that satisfy

(7.7-10), all of which will guarantee $c(nT) \longrightarrow R$ for sufficiently large n. It is up to the design engineer to select that $D(z)$ which satisfies (7.7-10) and meets other criteria. For example, since it is desired to force $c(t) = R$ identically for sufficiently large time t(and not just at the sampling times nT) we select a $D(z)$ that both satisfies (7.7-10) and meets this requirement. Techniques for making such a selection are treated in references 2, 3, and 4.

An interesting special case occurs when $T(s) = 1$ (i.e., unity feedback). In this case

$$D(z)\big|_{z=1} = \infty$$

which implies that the transfer function of the digital computer has a pole at $z = 1$.

We return to expression (7.7-7); a linear system with transfer function $H(z)$ will track a step input if $H(z) - z^{-m}$ has a zero of at least order one at $z = 1$. Putting this into an explicit form gives

$$H(z) = z^{-m} + (1 - z^{-1})S(z)$$

where $S(z)$ is a ratio of polynomials in z which has a finite value at $z = 1$ [i.e., $S(z)$ has no pole at $z = 1$].

7.7-2 Ramp Input

A test ramp input signal is characterized by

$$\begin{aligned} r(k) &= Vk \quad \text{for } k \geq 0 \\ r(k) &= 0 \quad \text{for } k < 0 \end{aligned} \tag{7.7-11}$$

which has the z-transform

$$R(z) = \frac{Vz^{-1}}{(1 - z^{-1})^2} \tag{7.7-12}$$

Inserting (7.7-12) into (7.7-6) gives

$$\lim_{z\to 1}\left[\frac{H(z) - z^{-m}}{1 - z^{-1}}\right] = 0 \tag{7.7-13}$$

Expression (7.7-13) is the necessary condition that the discrete system with transfer function $H(z)$ must satisfy in order to track a ramp input. An inspection of (7.7-13) reveals that $H(z) - z^{-m}$ must have a zero of at least order two at $z = 1$; that is,

$$H(z) - z^{-m} = (1 - z^{-1})^2 S(z) \tag{7.7-14}$$

where $S(z)$ is a ratio of polynomials which is analytic (no pole) at $z = 1$. Evaluating (7.7-14) at $z = 1$ reveals that $H(1) = 1$, which implies that a discrete system that tracks a ramp input will also track a step input.

If it is desired that the discrete system perfectly follow a ramp input, i.e.,

$$c(k) = r(k) \quad \text{as } k \longrightarrow \infty$$

we set $m = 0$ in (7.7-14), obtaining

$$H(z) = 1 - (1 - z^{-1})^2 S(z)$$

Examples of two systems that will track a ramp input are

(a) $H(z) = 1$; set $S(z) = 0$.
(b) $H(z) = 2z^{-1} - z^{-2}$; set $S(z) = 1$.

7.7-3 Acceleration Input

An acceleration input of amplitude A is characterized by the time sequence

$$r(k) = \frac{A}{2}k^2 \quad \text{for } k \geq 0$$

$$r(k) = 0 \qquad \text{for } k < 0$$

Therefore,

$$R(z) = \frac{A}{2}\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

The necessary condition for proper tracking becomes

$$\lim_{z \to 1}\left[\frac{H(z) - z^{-m}}{(1 - z^{-1})^2}\right] = 0$$

which indicates that $H(z) - z^{-m}$ must have a zero of order at least three at $z = 1$. Therefore, the form of $H(z)$ is given by

$$H(z) = z^{-m} + (1 - z^{-1})^3 S(z)$$

with $S(z)$ being a ratio of polynomials in z analytic at $z = 1$.

7.8 Controller with a Quadratic Performance Index

A useful criterion for measuring the performance of a control system is the quadratic performance index. This index, typically, will have the form

$$J = \sum_{k=0}^{N-1} [\tfrac{1}{2}\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \tfrac{1}{2}\mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k)] \tag{7.8-1}$$

where $\mathbf{x}(k)$ is the $n \times 1$ state vector and $\mathbf{u}(k)$ is the $p \times 1$ control vector at the kth iteration time. Q is an $n \times n$ positive semidefinite symmetric matrix, and $\mathbf{R}$ is a $p \times p$ positive definite symmetric matrix. These matrices may be selected to weight the magnitudes of the state vector and control vector.

The quadratic control problem is the following: Given a system characterized by the vector difference equation

$$\mathbf{x}(k + 1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{7.8-2}$$

which is at some arbitrary initial state $\mathbf{x}(0)$, determine the control sequence $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(N - 1)$ that minimizes the quadratic performance (7.8-1).

This problem may be treated as a minimization problem involving a function of several variables. The objective here is to minimize J of (7.8-1) subject to the constraint equations specified by (7.8-2). By use of a set of Lagrange multipliers $\boldsymbol{\lambda}(0), \boldsymbol{\lambda}(1), \ldots, \boldsymbol{\lambda}(N)$, we may recast this problem as one in which the augmented performance index

$$H = \sum_{k=0}^{N-1} \{\tfrac{1}{2}\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \tfrac{1}{2}\mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) + \boldsymbol{\lambda}^T(k + 1)[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{x}(k + 1)]\} \tag{7.8-3}$$

is to be minimized. It is known that the recast problem is equivalent to the original problem.

The minimization of (7.8-3) is carried out as an ordinary problem of finding the maximum or minimum of a function of several variables. We merely obtain the partial derivatives of J with respect to $\mathbf{x}(k)$, $\mathbf{u}(k)$, and $\boldsymbol{\lambda}(k)$, for all values of k, and equate these relations to zero.

$$\frac{\partial H}{\partial \mathbf{x}_k} = \mathbf{0} \quad k = 0, 1, \ldots, N - 1 \tag{7.8-4}$$

$$\frac{\partial H}{\partial \mathbf{u}_k} = \mathbf{0} \quad k = 0, 1, \ldots, N - 1 \tag{7.8-5}$$

$$\frac{\partial H}{\partial \boldsymbol{\lambda}_k} = \mathbf{0} \quad k = 0, 1, \ldots, N - 1 \tag{7.8-6}$$

Equations (7.8-4) through (7.8-6) constitute the necessary conditions for H to have a minimum.

The individual relations involve the differentiation of such quadratic expressions as $\mathbf{z}^T\mathbf{W}_1\mathbf{v}$, $\mathbf{v}^T\mathbf{W}_2$, and $\mathbf{z}^T\mathbf{W}_3\mathbf{z}$. Each one of these expressions is a scalar, but is differentiated with respect to the vector variables $\mathbf{z}$ and $\mathbf{v}$.

For the purpose of this problem, we are interested in the following results:

$$\frac{\partial \mathbf{z}^T\mathbf{W}_1\mathbf{v}}{\partial \mathbf{v}} = \mathbf{W}_1^T\mathbf{z} \tag{7.8-7}$$

$$\frac{\partial \mathbf{v}^T\mathbf{W}_2\mathbf{z}}{\partial \mathbf{v}} = \mathbf{W}_2\mathbf{z} \tag{7.8-8}$$

$$\frac{\partial \mathbf{z}^T\mathbf{W}_3\mathbf{z}}{\partial \mathbf{z}} = \mathbf{W}_3^T\mathbf{z} + \mathbf{W}_3\mathbf{z} \tag{7.8-9}$$

It is left as an exercise for the reader to verify these results.

The differentiation as indicated by the necessary conditions may now be carried out.

For $k = 0$ we have

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{x}(0)} &= \mathbf{Q}\mathbf{x}(0) + \mathbf{A}^T\boldsymbol{\lambda}(1) - \boldsymbol{\lambda}(0) = \mathbf{0} && \text{(a)} \\ \frac{\partial H}{\partial \mathbf{u}(0)} &= \mathbf{R}\mathbf{u}(0) + \mathbf{B}^T\boldsymbol{\lambda}(1) = \mathbf{0} && \text{(b)} \\ \frac{\partial H}{\partial \boldsymbol{\lambda}(0)} &= \mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{u}(0) - \mathbf{x}(1) = \mathbf{0} && \text{(c)} \end{aligned} \tag{7.8-10}$$

For $k = 1, 2, \ldots, N - 1$

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{x}(k)} &= \mathbf{Q}\mathbf{x}(k) + \mathbf{A}^T\boldsymbol{\lambda}(k+1) - \boldsymbol{\lambda}(k) = \mathbf{0} && \text{(a)} \\ \frac{\partial H}{\partial \mathbf{u}(k)} &= \mathbf{R}\mathbf{u}(k) + \mathbf{B}^T\boldsymbol{\lambda}(k+1) = \mathbf{0} && \text{(b)} \\ \frac{\partial H}{\partial \boldsymbol{\lambda}(k)} &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{x}(k+1) = \mathbf{0} && \text{(c)} \end{aligned} \tag{7.8-11}$$

Note that equations (c) in (7.8-10) and (7.8-11) are the system equations.

When $k = N - 1$, a term with the index N is present in H. We must also include this in our conditions.

$$\frac{\partial H}{\partial_{\mathbf{x}} N} = \boldsymbol{\lambda}(N) = \mathbf{0} \tag{7.8-12}$$

This condition specifies a fixed value for the last number of the set of Lagrange multipliers which will help us in the solution of equations (7.8-10) and (7.8-11) for the control vector $\mathbf{u}(k)$.

From (7.8-10a) and (7.8-11a), we obtain

$$\boldsymbol{\lambda}(k) = \mathbf{Q}\mathbf{x}(k) + \mathbf{A}^T\mathbf{x}(k+1) \tag{7.8-13}$$

Next we solve for $\mathbf{u}(k)$ in equations (7.8-10b) and (7.8-11b)

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(k+1) \tag{7.8-14}$$

and substitute this result into equations (7.8-2).

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(k+1) \tag{7.8-15}$$

We next stipulate an important relationship between the state vector and the Lagrange multiplier.

$$\boldsymbol{\lambda}(k) = \mathbf{P}(k)\mathbf{x}(k) \tag{7.8-16}$$

This linear transformation is called a Riccati transformation and is of fundamental importance in the solution of this problem. An investigation of the validity of this transformation is beyond the scope of this book. Related discussions may be found in any advanced text on optimal control theory. Utilizing the Riccati transformation in equations (7.8-13) and (7.8-15) enables us to eliminate $\boldsymbol{\lambda}(k)$ with the result

$$\mathbf{P}(k)\mathbf{x}(k) = \mathbf{Q}\mathbf{x}(k) + \mathbf{A}^T\mathbf{P}(k+1)\mathbf{x}(k+1) \tag{7.8-17}$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)\mathbf{x}(k+1) \tag{7.8-18}$$

Solving for $\mathbf{x}(k+1)$ in (7.8-18) yields

$$\mathbf{x}(k+1) = [\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)]^{-1}\mathbf{A}\mathbf{x}(k) \tag{7.8-19}$$

Substituting equation (7.8-19) into (7.8-17) yields

$$\mathbf{P}(k)\mathbf{x}(k) = \mathbf{Q}\mathbf{x}(k) + \mathbf{A}^T\mathbf{P}(k+1)[\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)]^{-1}\mathbf{A}\mathbf{x}(k) \tag{7.8-20}$$

Since equation (7.8-20) must hold for all $\mathbf{x}(k)$, it simplifies to

$$\mathbf{P}(k) = \mathbf{Q} + \mathbf{A}^T\mathbf{P}(k+1)[\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)]^{-1}\mathbf{A} \tag{7.8-21}$$

This is a recursive relationship for the matrix $\mathbf{P}(k)$ used in the Riccati transformation. It must be solved backwards, starting with $\mathbf{P}(N)$. Since $\boldsymbol{\lambda}(N) = \mathbf{P}(N)\mathbf{x}(N)$ and $\boldsymbol{\lambda}(N) = \mathbf{0}$, we have $\mathbf{P}(N) = \mathbf{0}$. With $\mathbf{P}(k)$ determined the problem is essentially solved.

To compute the control vector, we eliminate $\boldsymbol{\lambda}(k+1)$ from equations (7.8-14) and (7.8-13). This results in

$$\begin{aligned} \mathbf{u}(k) &= -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{A}^T)^{-1}[\boldsymbol{\lambda}(k) - \mathbf{Q}\mathbf{x}(k)] \\ &= -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{A}^T)^{-1}[\mathbf{P}(k) - \mathbf{Q}]\mathbf{x}(k) \end{aligned} \tag{7.8-22}$$

This is the desired expression for the optimal control law. We note that it is of the form

$$\mathbf{u}(k) = \mathbf{H}(k)\mathbf{x}(k) \tag{7.8-23}$$

which indicates the components of the control vector are proportional to the state vector. Indeed, this expression prescribes a feedback control law with time-varying feedback gains $\mathbf{H}(k)$. This is a very useful and convenient result.

EXAMPLE 7.8-1

Compute the feedback gain matrix $\mathbf{H}(k)$ for the following problem: The system to be controlled is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 1.0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} u(k)$$

The performance index to be minimized is

$$J = \sum_{k=0}^{4} [\tfrac{1}{2}(x_1^2(k) + x_2^2(k)) + \tfrac{1}{2}u^2(k)]$$

The initial conditions are given as

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 100.0 \\ 0.0 \end{bmatrix}$$

Thus it is seen that

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{R} = 1 \quad \text{and} \quad N = 5$$

The required calculations to determine the feedback gain matrix may best be carried out by a computer program such as that presented in Appendix 7B of this chapter. This program solves equations (7.8-2), (7.8-21), and (7.8-22). Let the solution for the control law be represented as

$$u(k) = h_1(k)x_1(k) + h_2(k)x_2(k)$$

Table 7.8-1 shows the values of the control variable, the feedback gains, and the state variables.

Table 7.8-1

k	$h_1(k)$	$h_2(k)$	$u(k)$	$x_1(k)$	$x_2(k)$
0	−.395	−.687	−39.5	100.0	.0
1	−.395	−.687	−2.4	40.46	19.7
2	−.395	−.677	3.48	10.19	−11.9
3	−.355	−.555	1.42	.54	−3.8
4	.0	.0	.0	−1.45	−9.42

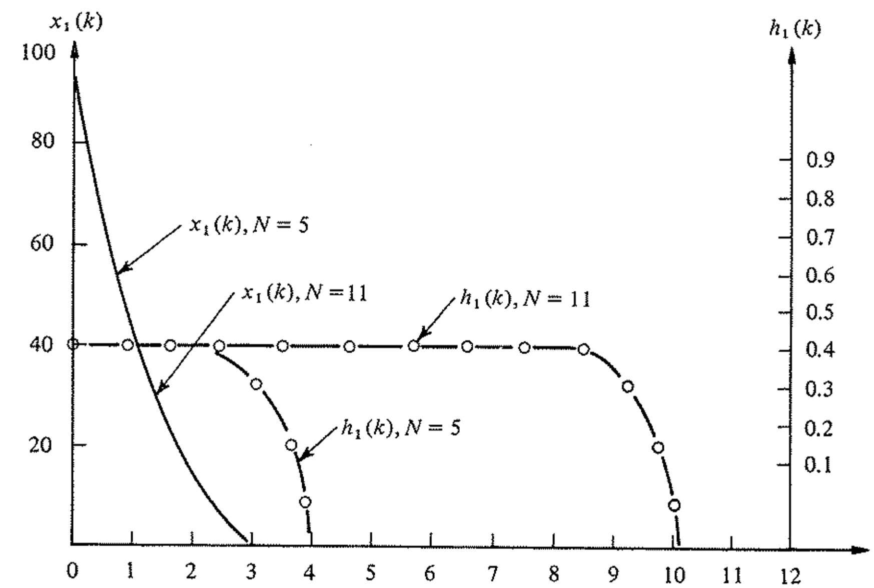

Figure 7.8-1. Response of linear regulator.

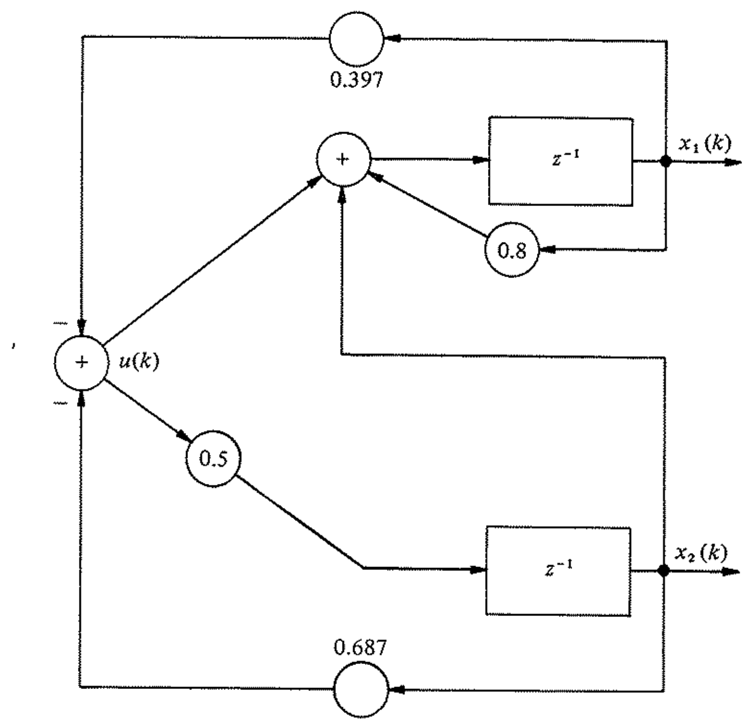

Figure 7.8-2. Block diagram for linear regulator, where $N \to$ large.

An interesting property of the linear regulator is revealed by plotting these variables as a function of k. This is shown in Figure 7.8-1. The feedback gains [only $h_1(k)$ is shown] are constant for low values of k and decrease rapidly to zero as k approaches $N-1$. Note that they are a function of $\mathbf{P}(k)$, which is solved backwards in time. The solutions for $N=5$ and $N=11$ are shown. In both cases identically shaped curves result for $h_1(k)$, except for a lateral displacement, while the responses of $x_1(k)$ are identical. This result carries the important implication that if N is made sufficiently large, the feedback gains in the linear regulator become constants, permitting a closed-loop control system design with constant feedback gains, as displayed by Figure 7.8-2.

7.9 DC Gain of a Discrete System

A meaningful characteristic in the design of linear continuous systems is the DC gain. It applies to systems that generate a steady-state constant output in response to a step input. The DC gain is defined as the ratio of the system's steady-state output to the amplitude of the input step.

The DC gain concept may be extended to discrete systems. For example, consider a system whose transfer is $H(z)$ and which has applied to it a discrete step input of amplitude R. What is the system's resultant response $c(n)$? Writing the familiar transfer function relationship gives

$$C(z) = H(z)R(z) = H(z)\left(\frac{R}{1-z^{-1}}\right) \tag{7.9-1}$$

Expanding (7.9-1) by partial-fraction expansion gives

$$C(z) = \frac{RH(1)}{1-z^{-1}} + \hat{C}(z) \tag{7.9-2}$$

where $H(1) = \lim_{z\to 1} H(z)$

$$\hat{C}(z) = \frac{R}{1-z^{-1}}[H(z) - H(1)]$$

The term $\hat{C}(z)$ contains the transient response terms that depend on the poles of $H(z)$. If the system is stable, these terms will decay to zero for sufficiently large values of discrete time n. It is, therefore, possible to express the system's steady-state response to a discrete step of amplitude R as

$$c_{ss} = \lim_{n\to\infty} c(n) = RH(1) \tag{7.9-3}$$

The DC gain of this system is defined as

$$K = \frac{\text{steady-state response}}{\text{amplitude of step input}} = \frac{RH(1)}{R} = H(1)$$

Some comments on the partial-fraction expansion (7.9-2) should now be made. In this expansion, it has been assumed that the transfer function $H(z)$ has no poles at $z=1$. If this is not the case, then the expansion as given is incorrect. To demonstrate this, assume that $H(z)$ has a simple pole at $z=1$. The proper partial-fraction expansion of $C(z)$ would be of the form

$$C(z) = \frac{RH(1)}{(1-z^{-1})^2} + \frac{A}{1-z^{-1}} + \hat{\hat{C}}(z) \tag{7.9-4}$$

Expression (7.9-4) indicates that the system's response, in part, will be a ramp of slope $RH(1)$. This ramp was generated because the $H(z)$ has a simple pole at $z=1$.

If, as is standard, a stable discrete system is defined as one whose transfer function has all its poles inside the unit circle, then the partial-fraction expansion as given by (7.9-2) will be proper for all stable systems. With this in mind, the DC gain of a stable system with transfer function $H(z)$ is given by

$$K = H(1) = \lim_{z\to 1} H(z) \tag{7.9-5}$$

Rewriting (7.9-5) in the standard expansion of $H(z)$ in terms of its weighting sequence $h(n)$, we have

$$K = \lim_{z\to 1} H(z) = \lim_{z\to 1}\left\{\sum_{n=0}^{\infty} h(n)z^{-n}\right\} = \sum_{n=0}^{\infty} h(n) \tag{7.9-6}$$

It has been shown in the section on the tracking of test signals that the value of the DC gain, as given by (7.9-6), plays an important role in the system's tracking ability.

EXAMPLE 7.9-1

Determine the DC gain for the system characterized by

$$c(n+2) + \tfrac{3}{4}c(n+1) + \tfrac{1}{8}c(n) = r(n+1) + 2r(n)$$

This system has the transfer function

$$H(z) = \frac{z+2}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

which has poles at $z = -\frac{1}{4}$, $z = -\frac{1}{2}$, so it is stable. Utilizing (7.9-5), we find

$$K = H(1) = \frac{8}{5}$$

Conclusions

Several selected optimization problems have been investigated in this chapter. Many of the optimal control laws that resulted involved the solution of a system of linear equations. Under very minor assumptions, it is guaranteed that this system of equations has a unique solution.

The reader is reminded that only a very limited number of design techniques has been treated here. For example, the concepts of dynamic programming and the discrete maximum principle have not been discussed. For a more extensive treatment of optimal design techniques for discrete systems numerous texts are available.

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PROBLEMS

7.1 For the plant with transfer function

$$G(s) = \frac{1}{s^2}$$

and step input make a time domain:
(a) Minimal prototype design
(b) Ripple-free design

Let the sampling period T be an arbitrary parameter in the design process. The initial state is zero.

7.2 For the plant with transfer function

$$G(s) = \frac{1}{(s + a)(s + 1)}$$

and step input make a time domain:
(a) Minimal prototype design
(b) Ripple-free design
with $T = 1$ second and initial state zero. Investigate the comparison of results with those of Section 6.2 as $a \longrightarrow 0$.

7.3 For the system in Problem 7.1 and ramp input make a time domain:
(a) Minimal prototype design
(b) Ripple-free design

Check to verify that it has desirable characteristics for a unit step input.

7.4 For the system in Problem 7.1, carry out a z-domain synthesis for a minimal prototype design. Check with the results of Problem 7.1.

7.5 For the system in Problem 7.3, carry out a z-domain synthesis for a minimal prototype design. Check with the results of Problem 7.3.

7.6 For the system of Problem 7.1 and a step input make a time domain:
(a) Minimal prototype design
(b) Ripple-free design
under the assumption that the initial state is not zero.

7.7 Determine the controllability characteristics of the discrete systems with transfer functions

(a) $$\frac{C(z)}{U(z)} = \frac{z + 2}{(z + 1)(z + 3)}$$

(b) $$\frac{C(z)}{U(z)} = \frac{z}{(z + 1)^2}$$

7.8 Design a digital regulator for the system characterized by

$$\frac{C(s)}{U(s)} = \frac{1}{s^3}$$

if the input $u(t)$ is constrained to be constant over one-second time intervals.

7.9 Repeat Problem 7.8 for the system with transfer function

$$\frac{C(s)}{U(s)} = \frac{1}{s(s + a)}$$

Check the results with (7.5-5) by letting $a = 1$.

7.10 Design a minimum energy regulator (Section 7.6) for the system with transfer function

$$\frac{C(s)}{U(s)} = \frac{1}{s^2}$$