8

Introduction to Optimization Theory

8.1 INTRODUCTION

The Second World War provided a great impetus to the development of the feedback control systems area. After a somewhat dormant period in the 1950's, the area again received a strong stimulus. This was caused by the interest in industrial automation and, even more, by the advent of the space age. Modern control system design has become exceedingly complex because of the desire to control large-scale, inherently nonlinear processes which sometimes operate in widely changing environments, and because of extremely stringent specifications on the performance of such systems. Optimization theory appears to offer the control system engineer a means of combating the complexities of modern control system design. It is an excellent example of the usage of linear vector space concepts. Although much research is presently being performed in the optimization area, this chapter is limited to attempting to provide the reader with the basics of optimization theory, and to indicating the nature of some of the difficulties involved with its application.

The philosophy of optimization theory is to design the "best" system. This, of course, implies some criterion or *performance index* for judging what is "best." The determination of a suitable performance index is often a problem in itself. Performance indices are discussed in later sections.

In comparison with more conventional methods for feedback control system design, the advantages of optimization theory include:

1. The design procedure is more direct, because of the inclusion of all the important aspects of performance in a single design index.
2. The best the designer can hope to achieve with respect to the performance index is apparent. Thus the ultimate performance limitations, and the extent to which these limitations affect a given design problem, are indicated.
3. Inconsistent sets of performance specifications are revealed.
4. Prediction is naturally included in the procedure, because the design index evaluates performance over the future interval of control.
5. The resulting control system is adaptive, if the design index is reformulated and the controller parameters recomputed on-line.
6. Time-varying processes do not cause any added difficulty, assuming that a computer is used to determine the optimum.
7. Nonlinear processes can be treated directly, however, at the expense of increased computational complexity.

The difficulties of optimization theory include:

1. The conversion of prescribed design specifications into a meaningful mathematical performance index is not a straightforward process, and it may involve trial and error.
2. Existing algorithms for the computation of the optimum control signals in nonlinear cases require complex computer programs and, in some cases, a large amount of computer time.
3. Proven techniques for the design of controllers for large regions of state space, rather than merely for small regions about nominal trajectories, are presently unavailable for nonlinear cases.
4. The resulting control system performance is highly sensitive to erroneous assumptions about and/or changes in the values of the parameters of the controlled elements.

Considerable research is presently being devoted to these limitations.

The subject of system optimization had its birth in the optimum linear filter theory of Wiener.[1] This theory was extended to the time-varying case by Booton.[2] Neither of these is directly applicable to control system optimization, however, since the limitations of physical components are not considered. On the basis of Wiener's optimum filter theory, Newton considered the limitations of physical components by introducing constraints on functions of signals in the system.[3] With reference to Fig. 8.1-1*a*, Newton's method can be viewed as determining the transfer function of the optimum compensation for the system. As such, it is necessarily restricted to linear systems. Furthermore, this method neglects the effect of the configuration of the system on its performance.

A departure from the preceding procedure is indicated with respect to Fig. 8.1-1*b*, by seeking the optimum control signals for the controlled elements. The optimum *control law*, i.e., the dependence of these optimum control signals on the state variables of the controlled elements and the

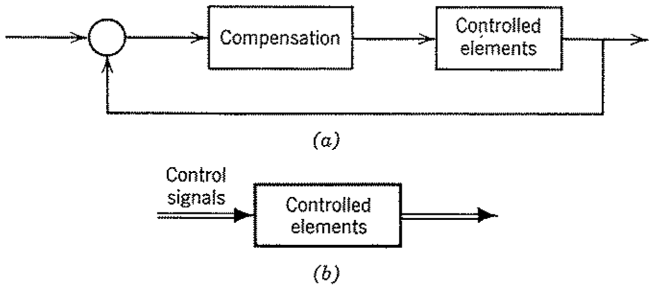

Fig. 8.1-1

desired system behavior, must also be determined in order to realize the system. The optimum control law indicates the optimum system configuration.

The latter approach to system optimization is utilized in the modern procedures developed around the dynamic programing concepts of Bellman, and the extended variational calculus methods of Pontryagin. Numerous others, some of whom are mentioned later in this chapter, have also made many important contributions to optimization theory. Notable among these is Merriam, who has been particularly concerned with making optimization theory of practical value to the control engineer. Much of the material of this chapter has been taken from his writings and those of Ellert, one of his associates.

8.2 DESIGN REQUIREMENTS AND PERFORMANCE INDICES

The primary task of the control engineer is to design practical control systems for physical processes. The application of optimization theory to this problem, as considered here, consists of three fundamental steps. They are:

1. Formulation of mathematical models for both the behavior of the physical process to be controlled and the performance requirements. The mathematical model of the performance requirements is the performance index.
2. Computation of the optimum control signals.
3. Synthesis of a controller to generate the optimum control signals.

This section considers various performance indices and their relationship to the performance requirements.

Control systems must satisfy numerous requirements relating to the

performance of the system and its implementation. For example, system performance requirements may include:

1. Desired system response.
2. Desired control effort.
3. Limits on the control effort.
4. Limits on the system response, dictated either by the nature of the system mission or by saturation limits.
5. Desired system response at some future terminal time.
6. Minimization or maximization of some function of a process variable or time.
7. Disturbances, initial conditions, parameter variations, etc., which must be tolerated.
8. Damping ratio.
9. Undamped natural frequency.

Requirements 1 through 7 are objective requirements, since they can be mathematically described for any system. The last two requirements are subjective, however, because they have a precise meaning for linear, second order, time-invariant systems only. Nevertheless, they are useful for approximately characterizing the relative stability and the speed of response of more general feedback systems.

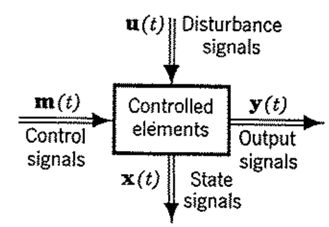

Fig. 8.2-1

The system implementation requirements may include the specification of

1. Available sensors.
2. Available controller components.
3. System size, weight, cost, and reliability.

Implementation requirements are exceedingly difficult to include directly in any design procedure.

The performance index is a mathematical model of the performance requirements. It is expressed in terms of the inputs, outputs, and state signals of the controlled elements. These are indicated in Fig. 8.2-1. Many performance indices have been proposed in the literature.[4–16] A substantial portion of these are special cases of the performance index

$$I = \int_0^\infty f(t)g[e(t)]\,dt$$

where $f(t)$ is a factor which weights $g[e(t)]$ as a function of time. $g[e(t)]$ is a function of the error $e(t)$. $f(t)$ is usually one of the functions $1, t, t^2, \ldots,$ or t^n, and $g[e(t)]$ is usually $e^2(t)$ or $|e(t)|$. In particular, the integral square error index

$$I = \int_0^\infty e^2(t)\, dt \qquad (8.2\text{-}1)$$

leads to responses which tend not to be sufficiently damped, because large errors are counted more heavily than small errors. Thus minimization of the index requires that large errors be removed rapidly. However, this performance index is often used because of its analytical tractability.

Performance indices of the form of Eq. 8.2-1 are not suitable when multiple design specifications are encountered, since the error may be only one of these specifications. For this reason, performance indices of more general forms have been proposed. For example, Ellert uses the form[16]

$$I = \int_{t_0}^{t_f} \left\{ \sum_{i=1}^{n} \left(\phi_{ii}(t) \left[\frac{y_i^d(t) - y_i(t)}{l_{y_i}} \right]^2 + \xi_{ii}(t) \left| \frac{y_i^d(t) - y_i(t)}{l_{y_i}} \right|^{\gamma_i} \right) + \sum_{i=1}^{M} \left(\psi_{ii}(t) \left[\frac{m_i^d(t) - m_i(t)}{l_{m_i}} \right]^2 + \beta_{ii}(t) \left| \frac{m_i^d(t) - m_i(t)}{l_{m_i}} \right|^{\mu_i} \right) \right\} dt \qquad (8.2\text{-}2)$$

where y_i^d and m_i^d are the desired output and control effort, respectively; l_{y_i} and l_{m_i} are related to limits on y_i and m_i, respectively; $\phi_{ii}(t)$, $\xi_{ii}(t)$, $\psi_{ii}(t)$ and $\beta_{ii}(t)$ are time-dependent weighting factors; and γ_i and μ_i are integers. The performance index considers the system behavior during the future time interval $t_0 \leqslant t \leqslant t_f$, where t_f may be a constant, a variable, or infinity.

The *weighting factors* permit the various terms of the performance index to be emphasized or weighted in time, depending upon the relative importance of these terms. The terms raised to the powers γ_i and μ_i are *penalty functions*, which tend to maintain output and control signals within prescribed limits. This is accomplished by heavily weighting these signals, if they exceed their limits.

A unique set of weighting factors and penalty functions to satisfy prescribed design specifications generally does not exist. Furthermore, the selection of these quantities is unfortunately not a straightforward matter. However, the lack of uniqueness of the weighting factors and penalty functions does introduce a flexibility which makes their selection simpler. From an engineering viewpoint, an efficient procedure for selecting weighting factors and penalty functions is needed. As discussed in later sections, Ellert has partially answered this need.

The performance indices above, and many of the specialized indices found in the literature, can be put in the form

$$I = \int_{t_0}^{t_f} q[\mathbf{y}(t), \mathbf{m}(t), t]\, dt \qquad (8.2\text{-}3)$$

For example, in the flight of a vehicle from one point to another with least fuel consumption, *minimization* of Eq. 8.2-3 is desirable, if $q(\mathbf{y}, \mathbf{m}, t)$ is chosen as the fuel consumption per unit time. In chemical process control, one might seek a *maximum* of Eq. 8.2-3. In the latter case, however, $q[\mathbf{y}, \mathbf{m}, t]$ typically would represent the instantaneous yield of the process. As a final illustration, minimization of the time required for a system to go from one state to another can be accomplished by minimizing Eq. 8.2-3, with $q[\mathbf{y}, \mathbf{m}, t]$ chosen to be a constant. In such a case, constraints would exist on the maximum velocities and accelerations which can be tolerated.

Many more examples of optimization problems could be listed. However, the important aspect of this discussion is that, even though these problems are different, they are all closely related mathematically by the objective of finding a maximum or a minimum of Eq. 8.2-3. Problems of this type can be solved by Pontryagin's method, or by the dynamic programming techniques of Bellman.

8.3 NECESSARY CONDITIONS FOR AN EXTREMUM—VARIATIONAL CALCULUS APPROACH

The problem to be considered is one of determining the control signals $\mathbf{m}(t)$ which minimize (or maximize) the performance index of Eq. 8.2-3. The controlled elements are described by the equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{m}, t)$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, t) \qquad (8.3\text{-}1)$$

The elements of $\mathbf{f}$ are assumed to be continuous with respect to the elements of $\mathbf{x}$ and $\mathbf{m}$, and continuously differentiable with respect to the elements of $\mathbf{x}$. The controlled elements are assumed to be observable and controllable, i.e., all state variables are measureable, and it is possible to excite every state of the controlled elements. The presentation here is further limited to the special case for which there are no restrictions on the amplitudes of the control signals or state variables. A more general presentation is given by Pontryagin et al.[17]

Before considering minimization (maximization) of a functional, as Eq. 8.2-3, it is worthwhile to consider the more familiar case of minimization (maximization) of a function. All engineers have encountered problems of trying to minimize (maximize) a function of a finite number of independent variables, say $\theta(\mathbf{x})$. Points at which all the first partial derivatives of the function are zero are known as *stationary points*. If the function is a minimum (maximum) at a stationary point, then that point is called an *extremum*.

If the variables of the function are not independent but are subject to equality constraints, e.g., $\mathbf{w}(\mathbf{x}) = \mathbf{0}$, necessary conditions for an extremum can be determined by Lagrange's method of multipliers. This method consists in introducing as many new parameters (*Lagrange multipliers*) $p_1, p_2, \ldots$ (which may be regarded as the components of a vector $\mathbf{p}$) as there are constraint equations, forming the function $\theta_c = \theta(\mathbf{x}) + \langle \mathbf{p}, \mathbf{w} \rangle$ and determining necessary conditions for an extremum from

$$\mathbf{grad}_x\, \theta_c = \mathbf{0} \quad \text{and} \quad \mathbf{grad}_p\, \theta_c = \mathbf{0}.$$

Thus these conditions are

$$\frac{\partial \theta_c}{\partial x_1} = \frac{\partial \theta_c}{\partial x_2} = \cdots = 0$$

$$\frac{\partial \theta_c}{\partial p_1} = w_1(\mathbf{x}) = \frac{\partial \theta_c}{\partial p_2} = w_2(\mathbf{x}) = \cdots = 0$$

Lagrange's method avoids having to solve the constraint equations for the x's and substituting the results into $\theta(\mathbf{x})$. This is accomplished by introducing the above additional restrictions.

The calculus of variations is also concerned with the determination of extrema.† Rather than extrema of functions, however, the object of the calculus of variations is to determine extrema of functionals. Section 1.4 indicates that, if x has a unique value corresponding to each value of t lying in some domain, then $x(t)$ is said to be a function of t for that domain; to each *value* of t, there corresponds a *value* of x. In essence, a *functional* is a function of a function, rather than of a variable. For example, $f[x(t)]$ is a functional if, to each *function* $x(t)$, there corresponds a *value* of f. The performance index I of Eq. 8.2-3 is also a functional.

If the second of Eqs. 8.3-1 is substituted into Eq. 8.2-3, the result can be written as‡

$$I = \int_{t_0}^{t_f} f_0(\mathbf{x}, \mathbf{m}, t)\, dt \tag{8.3-2}$$

Then the problem of determining an extremum of Eq. 8.3-2 for the controlled elements of Eqs. 8.3-1, is one of determining the function $\mathbf{m}(t)$ which makes I an extremum, subject to the n equality constraints $\mathbf{f}(\mathbf{x}, \mathbf{m}, t) - \dot{\mathbf{x}} = \mathbf{0}$. The method of Lagrange multipliers is also useful for

† Reference 18 is a particularly readable presentation of the calculus of variations. Reference 19 provides a higher degree of rigor.

‡ In order to exclude degenerate problems, it is assumed that all state variables contribute to the value of the performance index. This may be due to the state variables appearing explicitly in f_0, or through their effect on other state variables which appear in f_0.

minimizing (maximizing) functionals, subject to functional equality constraints, which is the problem of interest.

Thus the functional

$$I_c = \int_{t_0}^{t_f} (f_0 + \langle \mathbf{p}, \mathbf{f} - \dot{\mathbf{x}} \rangle)\, dt \tag{8.3-3}$$

is formed. The components of $\mathbf{p}$ are Lagrange multipliers. If the optimum values (i.e., those furnishing the extremum of I) of $\mathbf{x}$, $\mathbf{m}$, and $\mathbf{p}$ are denoted by $\mathbf{x}^0$, $\mathbf{m}^0$, and $\mathbf{p}^0$, respectively, then perturbations in these variables from their optimum values are indicated by

$$\begin{aligned} \mathbf{x} &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^a \\ \mathbf{m} &= \mathbf{m}^0 + \mathbf{B}\mathbf{m}^a \\ \mathbf{p} &= \mathbf{p}^0 + \mathbf{\Gamma}\mathbf{p}^a \end{aligned} \tag{8.3-4}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{\Gamma}$ are diagonal matrices with elements α_i, β_i and γ_i, respectively. α_i, β_i, and γ_i are parameters which adjust the amount of perturbation that the quantities x_i^a, m_i^a, and p_i^a introduce into x_i, m_i and p_i, respectively. It is assumed that these perturbations are unrestricted.

From the first of Eqs. 8.3-4, it is apparent that

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}^0 + \mathbf{A}\dot{\mathbf{x}}^a \tag{8.3-5}$$

If Eqs. 8.3-4 and 8.3-5 are substituted into Eq. 8.3-3, I_c has its optimum value I_c^0 for $\boldsymbol{\alpha} = \mathbf{A1} = \mathbf{0}$, $\boldsymbol{\beta} = \mathbf{B1} = \mathbf{0}$, $\boldsymbol{\gamma} = \mathbf{\Gamma 1} = \mathbf{0}$, since $\mathbf{x}$, $\mathbf{m}$, and $\mathbf{p}$ then have their optimum values $\mathbf{x}^0$, $\mathbf{m}^0$, and $\mathbf{p}^0$, respectively. Thus Eq. 8.3-3 has a stationary point at $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma} = \mathbf{0}$, and necessary conditions for the optimum are

$$\begin{aligned} \mathbf{grad}_\alpha\, I_c \big|_{\alpha=\beta=\gamma=0} &= \mathbf{0} \\ \mathbf{grad}_\beta\, I_c \big|_{\alpha=\beta=\gamma=0} &= \mathbf{0} \\ \mathbf{grad}_\gamma\, I_c \big|_{\alpha=\beta=\gamma=0} &= \mathbf{0} \end{aligned} \tag{8.3-6}$$

Application of Eq. 8.3-6 to Eq. 8.3-3, after substitution of Eqs. 8.3-4 and 8.3-5, yields

$$\begin{aligned} \int_{t_0}^{t_f} (\mathbf{X}^a\, \mathbf{grad}_{\mathbf{x}^0} H_c^0 + \dot{\mathbf{X}}^a\, \mathbf{grad}_{\dot{\mathbf{x}}^0} H_c^0)\, dt &= \mathbf{0} \\ \int_{t_0}^{t_f} (\mathbf{M}^a\, \mathbf{grad}_{\mathbf{m}^0} H_c^0)\, dt &= \mathbf{0} \\ \int_{t_0}^{t_f} (\mathbf{P}_a\, \mathbf{grad}_{\mathbf{p}^0} H_c^0)\, dt &= \mathbf{0} \end{aligned} \tag{8.3-7}$$

where $\mathbf{X}^a$, $\mathbf{M}^a$, and $\mathbf{P}^a$ are diagonal matrices whose elements are the elements of $\mathbf{x}^a$, $\mathbf{m}^a$, and $\mathbf{p}^a$, respectively, and H_c^0 is the optimum value of the integrand of Eq. 8.3-3, i.e., $H_c^0 = f_0^0 + \langle \mathbf{p}^0, \mathbf{f}^0 - \dot{\mathbf{x}}^0 \rangle$. Integration by parts of the second term in the first of Eqs. 8.3-7 allows that equation to be written as

$$\int_{t_0}^{t_f} \mathbf{X}^a \left[\mathbf{grad}_{\mathbf{x}^0} H_c^0 - \frac{d}{dt} (\mathbf{grad}_{\dot{\mathbf{x}}^0} H_c^0) \right] dt + \mathbf{X}^a \, \mathbf{grad}_{\dot{\mathbf{x}}^0} H_c^0 \Big|_{t=t_0}^{t=t_f} = \mathbf{0}$$

$\mathbf{x}^a$ is an arbitrary perturbation, except at $t = t_0$, and possibly at $t = t_f$. At $t = t_0$, $\mathbf{x}^a = \mathbf{0}$, so that $\mathbf{x}^0(t_0) = \mathbf{x}(t_0)$ in order for the optimum solution to apply to the problem of interest. For a problem with specified terminal conditions $\mathbf{x}(t_f)$, $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$ and $\mathbf{x}^a(t_f) = \mathbf{0}$. If the terminal conditions on $\mathbf{x}$ are not specified, $\mathbf{x}^a(t_f)$ is arbitrary. Thus the preceding equation requires that

$$\mathbf{grad}_{\mathbf{x}^0} H_c^0 - \frac{d}{dt} (\mathbf{grad}_{\dot{\mathbf{x}}^0} H_c^0) = \mathbf{0} \tag{8.3-8}$$

and either $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$ or

$$\mathbf{grad}_{\dot{\mathbf{x}}^0} H_c^0 \Big|_{t=t_f} = \mathbf{0} \tag{8.3-9}$$

The last two of Eqs. 8.3-7 are satisfied if

$$\begin{aligned} \mathbf{grad}_{\mathbf{m}^0} H_c^0 &= \mathbf{0} \\ \mathbf{grad}_{\mathbf{p}^0} H_c^0 &= \mathbf{0} \end{aligned} \tag{8.3-10}$$

Equations 8.3-8 and 8.3-10, together with the boundary conditions $\mathbf{x}^0(t_0) = \mathbf{x}(t_0)$, and either $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$ or Eq. 8.3-9, constitute the first necessary condition for an optimum.

Pontryagin's equations are usually written in a form analogous to Hamilton's equations of analytical mechanics. This can be accomplished by defining H, analogous to the Hamiltonian, as

$$H(\mathbf{x}, \mathbf{m}, \mathbf{p}, t) = \langle \mathbf{p}, \mathbf{f} \rangle \tag{8.3-11}$$

The vector $\mathbf{p}$ in Eq. 8.3-11 differs from the one of Eq. 8.3-3 in that it has a zeroth component equal to unity. Likewise, $\mathbf{f}$ in Eq. 8.3-11 differs from the one of Eq. 8.3-1 in that it has a zeroth component equal to $f_0(\mathbf{x}, \mathbf{m}, t)$ of Eq. 8.3-2. Thus $\mathbf{p}$ and $\mathbf{f}$ are now vectors with $n + 1$ components.

In terms of the optimum H, the first necessary condition for an optimum for the case of unspecified terminal conditions on the state variables can be written as

$$\begin{aligned} \mathbf{grad}_{\mathbf{x}^0} H^0 &= -\dot{\mathbf{p}}^0 \\ \mathbf{grad}_{\mathbf{m}^0} H^0 &= \mathbf{0} \\ \mathbf{grad}_{\mathbf{p}^0} H^0 &= \dot{\mathbf{x}}^0 \end{aligned} \tag{8.3-12}$$

subject to the boundary conditions $\mathbf{x}^0(t_0) = \mathbf{x}(t_0)$ and $\mathbf{p}^0(t_f) = \mathbf{0}$.† For specified terminal conditions on $\mathbf{x}^0$, the latter boundary condition is replaced by $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$.† The $\dot{\mathbf{x}}^0$ equation above is equivalent to Eq. 8.3-2 and the first of Eqs. 8.3-1, and hence is always part of the problem statement. Equations 8.3-12 are called the *Euler* or *Hamilton* equations in canonic form. Their simultaneous solution yields the control signal $\mathbf{m}^0(t)$ which makes I stationary.

† The latter boundary conditions are a special case of the so-called *transversality condition.*

In the calculus of variations, a distinction is made between weak and strong maxima and minima. In addition to the Euler equations, two other necessary conditions for a weak maximum or minimum must be satisfied. They are the Legendre condition and the Jacobi condition.[18] The Legendre condition for a weak minimum requires that the matrix (see Eq. 4.11-11) $\mathbf{grad}_{\mathbf{m}^0} > <\mathbf{grad}_{\mathbf{m}^0} H^0$ be positive definite. This is analogous to the requirement that the second derivative be positive in the minimization of a function by the usual techniques of calculus. As indicated in the next section, proper formulation of f_0 guarantees satisfaction of the Legendre condition, if the controlled elements are linear.

The Jacobi condition requires that no conjugate points exist for $t_0 < t \leqslant t_f$. A *conjugate point* is one at which $x_i^a(t_1)$ is restricted, where $t_0 < t_1 \leqslant t_f$.‡ Hence, at such a point, the perturbation in $x_i(t)$ is restricted. If a conjugate point existed, the controller parameters could become unbounded. Thus, to ensure bounded controller parameters, the Jacobi condition must be satisfied. If the controlled elements are linear, proper formulation of f_0 also guarantees satisfaction of the Jacobi condition. This is indicated in Section 8.4.

Coordinate Optimization Interpretation

Pontryagin's formulation of the optimization problem restates the problem as one of optimization of a coordinate. In essence, a zeroth coordinate of $\mathbf{x}$ is introduced as

$$x_0(t) = \int_{t_0}^{t} f_0(\mathbf{x}, \mathbf{m}, \tau) \, d\tau$$

so that $\dot{x}_0(t) = f_0(\mathbf{x}, \mathbf{m}, t)$. Optimization of $x_0(t)$ at $t = t_f$ is optimization of the performance index, since

$$I = x_0(t_f) = \int_{t_0}^{t_f} f_0(\mathbf{x}, \mathbf{m}, t) \, dt \tag{8.3-13}$$

which is Eq. 8.3-2.

For first order controlled elements described by $\dot{x}_1 = f_1(x_1, m_1, t)$ the optimization problem with a specified terminal condition $x_1^0(t_f) = x_1(t_f)$ can be interpreted in the x_0x_1-plane of Fig. 8.3-1. (The generalization to

† Problems in which some, but not all, of the state variables have specified terminal conditions are also considered in the literature.[17]

‡ Conjugate points have very interesting geometrical interpretations.[18,19] However, these are beyond the scope intended here.

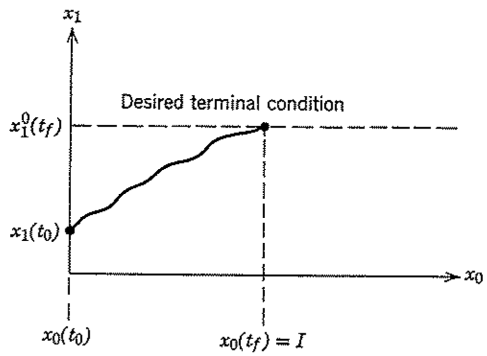

Fig. 8.3-1

higher order controlled elements is straightforward, but difficult to picture.) The desired terminal condition is the line $x_1 = x_1{}^0(t_f)$. For various $m_1(t)$, corresponding values of $I = x_0(t_f)$ can be determined from Eq. 8.3-13. Assuming that minimum I is desired, the optimum control signal $m_1{}^0(t)$ is the one for which $x_0(t_f) = I$ has the smallest coordinate $x_0{}^0(t_f) = I^0$. If I does not depend upon $m_1(t)$, a desirable $m_1{}^0(t)$ is an impulse. Then $x_1(t)$ would be transferred from $x_1(t_0)$ to $x_1{}^0(t_f)$ in zero time, and hence with zero I. However, impulses in $m_1(t)$ could not be realized physically. $m_1{}^0(t)$ must be chosen from a set of *admissible* control signals, which are defined to be bounded, and also continuous for all $t_0 \leqslant t \leqslant t_f$, except possibly at a finite number of t.

In the case of unspecified terminal conditions on $\mathbf{x}^0(t_f)$, all components of $\mathbf{p}^0$ are zero at $t = t_f$ except for the $p_0{}^0(t_f)$ component, which is unity. Thus the problem of minimizing (maximizing) $I = x^0(t_f)$ can be viewed as minimizing (maximizing) $\langle \mathbf{p}, \mathbf{x} \rangle$ at $t = t_f$. In other words, starting from the initial conditions $\mathbf{x}^0(t_0)$, $\mathbf{m}^0(t)$ is to be chosen to move the state of the system (including the x_0 component) as little (much) as possible in the direction of the vector $\mathbf{p}$. But the first and last of Eqs. 8.3-12 are the same as Hamilton's equations of analytical mechanics. H is analogous to the Hamiltonian, or total energy, and $\mathbf{p}$ and $\mathbf{x}$ are analogous to the momenta and generalized coordinates, respectively. Since H is the total energy for moving the state, $\mathbf{x}$, $\mathbf{m}(t)$ should be chosen at each instant of time to minimize (maximize) H. This is indicated by the second of Eqs. 8.3-12.

8.4 LINEAR OPTIMIZATION PROBLEMS

In this section, it is assumed that the controlled elements are described by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{m} \tag{8.4-1}$$

and the performance index is given by substituting

$$f_0(\mathbf{x}, \mathbf{m}, t) = \tfrac{1}{2}[\langle(\mathbf{x}^d - \mathbf{x}), \mathbf{\Omega}(\mathbf{x}^d - \mathbf{x})\rangle + \langle \mathbf{m}, \mathbf{Z}\mathbf{m}\rangle]$$

into Eq. 8.3-2. $\mathbf{x}^d$ is the desired state behavior, and $\mathbf{\Omega}$ and $\mathbf{Z}$ are symmetric matrices which are possibly time-varying.† The dimensions of $\mathbf{\Omega}$ are less than $(n \times n)$, unless all components of $(\mathbf{x}^d - \mathbf{x})$ are included in f_0.‡ The objective is to determine $\mathbf{x}^0$, $\mathbf{m}^0$ and the dependence of $\mathbf{m}^0$ on $\mathbf{x}^0$ and $\mathbf{x}^d$.

From Eq. 8.3-10,

$$H^0 = \tfrac{1}{2}[\langle \mathbf{x}^d - \mathbf{x}^0, \mathbf{\Omega}(\mathbf{x}^d - \mathbf{x}^0)\rangle + \langle \mathbf{m}^0, \mathbf{Z}\mathbf{m}^0\rangle] + \langle \mathbf{p}^0, \mathbf{A}\mathbf{x}^0 + \mathbf{B}\mathbf{m}^0\rangle$$

From the second of Eqs. 8.3-12, $\mathbf{Z}\mathbf{m}^0 + \mathbf{B}^T\mathbf{p}^0 = \mathbf{0}$ since $\mathbf{Z}$ is symmetric. Then

$$\mathbf{m}^0 = -\mathbf{Z}^{-1}\mathbf{B}^T\mathbf{p}^0 \tag{8.4-2}$$

This is an expression for the optimum control signal, but it is in terms of $\mathbf{p}^0$. The control law requires $\mathbf{m}^0(t)$ in terms of $\mathbf{x}^0(t)$.

From the first of Eqs. 8.3-12, $\dot{\mathbf{p}}^0 = -\mathbf{A}^T\mathbf{p}^0 + \mathbf{\Omega}(\mathbf{x}^d - \mathbf{x}^0)$. From Eq. 8.4-1, after substituting Eq. 8.4-2, $\dot{\mathbf{x}}^0 = \mathbf{A}\mathbf{x}^0 - \mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T\mathbf{p}^0$. The last two equations can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}^0 \\ \dot{\mathbf{p}}^0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T \\ -\mathbf{\Omega} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{p}^0 \end{bmatrix} + \mathbf{\Omega} \begin{bmatrix} \mathbf{0} \\ \mathbf{x}^d \end{bmatrix} \tag{8.4-3}$$

Equation 8.4-3 represents $2n$ linear, first order differential equations in the $2n$ unknowns $x_1{}^0, x_2{}^0, \ldots, x_n{}^0, p_1{}^0, p_2{}^0, \ldots, p_n{}^0$. They are subject to n boundary conditions at $t = t_0$, i.e., $\mathbf{x}^0(t_0) = \mathbf{x}(t_0)$, and n boundary conditions at $t = t_f$, i.e.. either $\mathbf{p}^0(t_f) = \mathbf{0}$ or $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$, depending upon the nature of the problem. Equation 8.4-3, subject to the preceding boundary conditions, is a *two-point boundary value problem.* Its solution yields the optimum control signal $\mathbf{m}^0(t)$ and the corresponding behavior of the controlled elements $\mathbf{x}^0(t)$ for $t_0 \leqslant t \leqslant t_f$.

Conversion of the Two-Point Boundary Value Problem

For the case under consideration, the two-point boundary value problem can be converted into two one-point boundary value problems. Equation 8.4-3 consists of a set of interrelated linear differential equations for $\mathbf{x}^0$ and

† The Euler equations together with a positive semidefinite $\mathbf{\Omega}$ and a positive definite $\mathbf{Z}$ constitute necessary and sufficient conditions for a minimum of the performance index, for the class of problems considered here. Furthermore, the corresponding linear optimum control system is stable (asymptotically stable if $\mathbf{\Omega}$ is positive definite).[25]

‡ A similar statement holds with respect to $\mathbf{Z}$ in terms of the dimensions of $\mathbf{m}$.

$\mathbf{p}^0$. Thus $\mathbf{x}^0$ and $\mathbf{p}^0$ must be related by a linear transformation. This transformation may be expressed by

$$\mathbf{p}^0 = \mathbf{K}\mathbf{x}^0 - \mathbf{v}^0 \tag{8.4-4}$$

where $\mathbf{K}$ is a square matrix of time-varying gains and $\mathbf{v}$ is a time-varying vector. Substitution of Eq. 8.4-4 into the second of Eqs. 8.4-3 yields

$$\mathbf{K}\dot{\mathbf{x}}^0 + \dot{\mathbf{K}}\mathbf{x}^0 - \dot{\mathbf{v}}^0 = -\boldsymbol{\Omega}\mathbf{x}^0 - \mathbf{A}^T\mathbf{K}\mathbf{x}^0 + \mathbf{A}^T\mathbf{v}^0 + \boldsymbol{\Omega}\mathbf{x}^d$$

Then substituting for $\dot{\mathbf{x}}^0$ from the first of Eqs. 8.4-3 and using Eq. 8.4-4 results in

$$(\dot{\mathbf{K}} + \mathbf{KA} + \mathbf{A}^T\mathbf{K} - \mathbf{KBZ}^{-1}\mathbf{B}^T\mathbf{K} + \boldsymbol{\Omega})\mathbf{x}^0 = \dot{\mathbf{v}}^0 + (\mathbf{A}^T - \mathbf{BZ}^{-1}\mathbf{B}^T)\mathbf{v}^0 + \boldsymbol{\Omega}\mathbf{x}^d$$

Since this expression must be valid for all possible $\mathbf{x}$, the conditions are

$$\begin{aligned} \dot{\mathbf{K}} + \mathbf{KA} + \mathbf{A}^T\mathbf{K} - \mathbf{KBZ}^{-1}\mathbf{B}^T\mathbf{K} + \boldsymbol{\Omega} &= [0] \\ \dot{\mathbf{v}}^0 + (\mathbf{A}^T - \mathbf{KBZ}^{-1}\mathbf{B}^T)\mathbf{v}^0 + \boldsymbol{\Omega}\mathbf{x}^d &= \mathbf{0} \end{aligned} \tag{8.4-5}$$

The first of Eqs. 8.4-5 is a set of first order nonlinear differential equations of the Riccati type.[20] The second of Eqs. 8.4-5 is a set of linear, time-varying, first order differential equations.† In the case of unspecified terminal conditions on $\mathbf{x}^0$, $\mathbf{p}^0(t_f) = \mathbf{0}$. Thus the boundary conditions on $\mathbf{K}$ and $\mathbf{v}^0$ for this case are that each of the elements of $\mathbf{K}$ and $\mathbf{v}^0$ is zero at $t = t_f$, as indicated by Eq. 8.4-4.

Once $\mathbf{K}$ and $\mathbf{v}^0$ are determined, the control law for the optimum system is given by substituting Eq. 8.4-4 into Eq. 8.4-2 to obtain

$$\mathbf{m}^0 = -\mathbf{Z}^{-1}\mathbf{B}^T(\mathbf{K}\mathbf{x}^0 - \mathbf{v}^0) \tag{8.4-6}$$

Thus, for this case, the control law is linear, and the controller feedback gains $\mathbf{K}$ are independent of the state of the controlled elements. Furthermore, since the control law is independent of the initial conditions of the state variables, the system configuration as defined by Eq. 8.4-6 is optimum for all initial conditions. Merriam, who first noted this property, refers to this as the *optimum configuration*.[14] Figure 8.4-1 illustrates this configuration for the general linear case.

Once $\mathbf{m}^0$ is determined, the response of the optimum system can be obtained from

$$\dot{\mathbf{x}}^0 = (\mathbf{A} - \mathbf{BZ}^{-1}\mathbf{B}^T\mathbf{K})\mathbf{x}^0 + \mathbf{BZ}^{-1}\mathbf{B}^T\mathbf{v}^0 \tag{8.4-7}$$

which results from substituting Eq. 8.4-6 into Eq. 8.4-1. Thus the two-point boundary value problem has been converted into two one-point

† These equations are adjoint to the equations of the closed-loop (controlled) system.

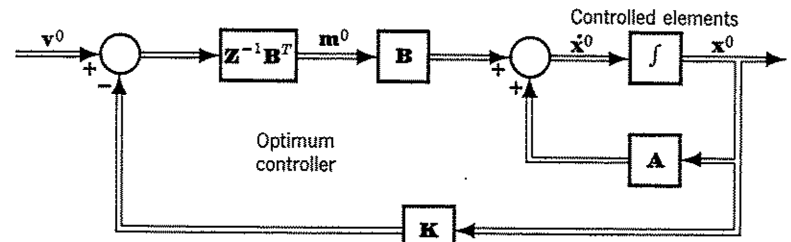

Fig. 8.4-1

boundary value problems. These are the solution of Eq. 8.4-5 backward in time from $t = t_f$ to $t = t_0$, and subsequently solving Eq. 8.4-7 forward in time from $t = t_0$ to $t = t_f$.

In the nonlinear case, where the controlled elements are nonlinear and/or the performance index is nonquadratic, it is not possible to convert the two point boundary value problem in the above manner. Also, the optimum control law is not linear. These aspects then generally demand computer solution of the equations defining the optimum system, as is considered in later sections.

Example 8.4-1. Determine the optimum controller according to the performance index

$$I = \tfrac{1}{2}\int_{t_0}^{t_f} [x_1\omega_{11}x_1 + m_1\zeta_{11}m_1]\, dt$$

for the first order controlled elements described by $\dot{x}_1(t) = a_{11}x_1(t) + b_{11}m_1(t)$. The system is assumed to be a regulator, so that $x_1^d = 0$. $x_1(t_f)$ is unspecified.

For $\mathbf{x}^d = \mathbf{0}$, Eq. 8.4-5 indicates $\mathbf{v}^0 = \mathbf{0}$. Therefore $m_1^0(t) = -\zeta_{11}^{-1}b_{11}k_{11}(t)x_1^0(t)$, where $k_{11}(t)$ is given by the solution to

$$\dot{k}_{11} + 2k_{11}a_{11} - \left(\frac{b_{11}^2}{\zeta_{11}}\right)k_{11}^2 + \omega_{11} = 0, \quad k_{11}(t_f) = 0$$

from Eqs. 8.4-6 and 8.4-5. In order to determine $k_{11}(t)$, let $\tau = t_f - t$ and $k_{11}(t_f - \tau) = k_{11}^0(\tau)$. Then

$$\dot{k}_{11}^0 = 2a_{11}k_{11}^0 - \left(\frac{b_{11}^2}{\zeta_{11}}\right)k_{11}^{02} + \omega_{11}, \quad k_{11}^0(0) = 0$$

Let $k_{11}^0(\tau) = \zeta_{11}\dot{z}/b_{11}^2 z$, which yields

$$\ddot{z} - 2a_{11}\dot{z} - \frac{\omega_{11}b_{11}^2}{\zeta_{11}}z = 0$$

This is a linear differential equation with constant coefficients; it can be solved by classical or transform methods. The solution is $z = c_1\epsilon^{\lambda_1\tau} + c_2\epsilon^{\lambda_2\tau}$, where c_1 and c_2 are constants, and $\lambda_1 = a_{11} + \beta$, $\lambda_2 = a_{11} - \beta$, and

$$\beta = \left(a_{11}^2 + \omega_{11}\frac{b_{11}^2}{\zeta_{11}}\right)^{1/2}$$

Thus

$$k_{11}^0(\tau) = \frac{\zeta_{11}(p_1c_1\epsilon^{\beta\tau} + p_2c_2\epsilon^{-\beta\tau})}{b_{11}^2(c_1\epsilon^{\beta\tau} + c_2\epsilon^{-\beta\tau})}$$

Since $k_{11}^0(0) = 0$, $c_2 = -c_1\lambda_1/\lambda_2$. Then

$$k_{11}^0(\tau) = \frac{\omega_{11}\sinh\beta\tau}{\beta\cosh\beta\tau - a_{11}\sinh\beta\tau}$$

Therefore

$$k_{11}(t) = \frac{\omega_{11}\sinh\beta(t_f - t)}{\beta\cosh\beta(t_f - t) - a_{11}\sinh\beta(t_f - t)}$$

and $\mathbf{m}^0(t)$ is given by

$$m_1^0(t) = -\frac{\omega_{11}b_{11}}{\zeta_{11}}\left[\frac{\sinh\beta(t_f - t)}{\beta\cosh\beta(t_f - t) - a_{11}\sinh\beta(t_f - t)}\right]x_1^0(t)$$

The resultant regulator is shown in Fig. 8.4-2.

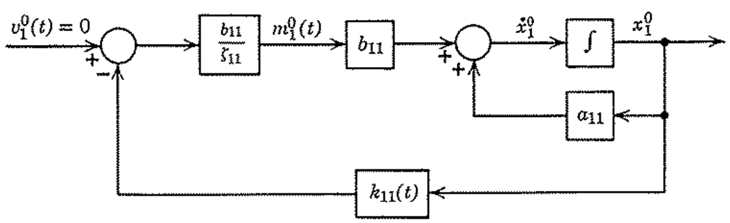

Fig. 8.4-2

If t_f is a constant, the terminal time of the performance index becomes nearer as real time advances, assuming $t < t_f$. In this so-called *shrinking interval* problem, the optimum system is time-varying. If t_f is a fixed time T in the future relative to real time, i.e., $t_f = t + T$, the terminal time of the performance index slides ahead in time as real time advances. This is called a *sliding interval* problem, and, if $\mathbf{x}^d$, $\mathbf{\Omega}$, $\mathbf{Z}$, and the linear controlled elements are time-invariant, the resultant system is stationary. A special case of these is given by infinite t_f. This is the *infinite interval* problem.† If $\mathbf{x}^d$, $\mathbf{\Omega}$, $\mathbf{Z}$, and the controlled elements are time-invariant, the resultant system designed according to an infinite interval performance criterion is stationary. In this example, $m_1^0(t)$ becomes

$$m_1^0(t) = -\frac{\omega_{11}b_{11}}{\zeta_{11}(\beta - a_{11})}x_1^0(t)$$

corresponding to a stationary system.

For the case in which the controlled elements consist of an integrator without feedback, $a_{11} = 0$. Then, for the infinite interval case, $m_1^0(t)$ is

$$m_1^0(t) = -\left(\frac{\omega_{11}}{\zeta_{11}}\right)^{1/2} x_1^0(t)$$

As ω_{11}/ζ_{11} is increased, so that the performance index emphasizes the system error relative to the cost of reducing it, the loop gain increases. Also, the speed of response as

† These names were coined by Merriam.

indicated by

$$x_1^0(t) = x_1^0(t_0)\exp\left[-b_{11}\left(\frac{\omega_{11}}{\zeta_{11}}\right)^{1/2}(t - t_0)\right]$$

increases. This agrees with one's intuition based on conventional feedback control theory.

If the optimum system is stationary, $\dot{\mathbf{K}} = [0]$ and the Riccati equation given in Eq. 8.4-5 reduces to a set of nonlinear algebraic equations defining the elements of $\mathbf{K}$. Even in this special case, however, it generally is not possible to determine $\mathbf{K}$ in closed form for controlled elements above second order. Thus the preceding discussion was presented to indicate that the control law of Eq. 8.4-6 exists for the linear case, rather than to provide a general method of determining it. Since the control law is of the form of Eq. 8.4-6, it is important to choose, as the state variables, variables which can be measured with available sensors.

In the general case, *analytical* determination of the optimum system makes use of direct solution of the two-point boundary value problem, rather than converting it to two one-point boundary value problems.

The Time-Invariant Case

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Omega}$, and $\mathbf{Z}$ are time-invariant, Eq. 8.4-3 can be solved by means of Laplace transforms. Assuming $t_0 = 0$, the transform of Eq. 8.4-3 is

$$\begin{bmatrix} \mathbf{I} & \mathbf{\Phi}(s)\mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T \\ -\mathbf{\Phi}^T(-s)\mathbf{\Omega} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{X}^0(s) \\ \mathbf{P}^0(s) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}(s)\mathbf{x}^0(0) \\ -\mathbf{\Phi}^T(-s)[\mathbf{p}^0(0) + \mathbf{\Omega}\mathbf{X}^d(s)] \end{bmatrix}$$

where $\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ and $-\mathbf{\Phi}^T(-s) = (s\mathbf{I} + \mathbf{A}^T)^{-1}$, and $\mathbf{X}^0(s)$, $\mathbf{P}^0(s)$ and $\mathbf{X}^d(s)$ are vectors. $\mathbf{\Phi}(s)$ and $-\mathbf{\Phi}(-s)$ are the Laplace transforms of the state transmission matrices for Eq. 8.4-1 and its adjoint, respectively. Since

$$\begin{bmatrix} \mathbf{I} & \boldsymbol{\alpha}_{12} \\ \boldsymbol{\alpha}_{21} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{I} - \boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_{21})^{-1} & -(\mathbf{I} - \boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_{21})^{-1}\boldsymbol{\alpha}_{12} \\ -(\mathbf{I} - \boldsymbol{\alpha}_{21}\boldsymbol{\alpha}_{12})^{-1}\boldsymbol{\alpha}_{21} & (\mathbf{I} - \boldsymbol{\alpha}_{21}\boldsymbol{\alpha}_{12})^{-1} \end{bmatrix}$$

$\mathbf{X}^0(s)$ and $\mathbf{P}^0(s)$ can be written as

$$\begin{aligned} \mathbf{X}^0(s) &= [\mathbf{I} + \mathbf{\Phi}(s)\mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T\mathbf{\Phi}^T(-s)\mathbf{\Omega}]^{-1} \\ &\qquad \times \mathbf{\Phi}(s)\{\mathbf{x}^0(0) + \mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T\mathbf{\Phi}^T(-s)[\mathbf{p}^0(0) + \mathbf{\Omega}\mathbf{X}^d(s)]\} \\ \mathbf{P}^0(s) &= [\mathbf{I} + \mathbf{\Phi}^T(-s)\mathbf{\Omega}\mathbf{\Phi}(s)\mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T]^{-1} \\ &\qquad \times \mathbf{\Phi}^T(-s)\{\mathbf{\Omega}\mathbf{\Phi}(s)\mathbf{x}^0(0) - [\mathbf{p}^0(0) + \mathbf{\Omega}\mathbf{X}^d(s)]\} \end{aligned} \tag{8.4-8}$$

Equations 8.4-8 and 8.4-2 can be utilized to determine the optimum control law and the response of the optimum system. However, the procedure is less direct than the previous method.

Example 8.4-2. Repeat Example 8.4-1, using the Laplace transform method.

For this case,

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{s - a_{11}}$$

and $\mathbf{X}^d(s) = \mathbf{0}$. Then

$$P_1{}^0(s) = -\left[1 - \frac{\omega_{11}b_{11}{}^2}{\zeta_{11}(s - a_{11})(s + a_{11})}\right]^{-1}\left[\frac{1}{s + a_{11}}\right]\left[\frac{\omega_{11}x_1{}^0(0)}{s - a_{11}} - p_1{}^0(0)\right]$$

$$= -\frac{\omega_{11}x_1{}^0(0)}{(s + \beta)(s - \beta)} + \frac{p_1{}^0(0)(s - a_{11})}{(s + \beta)(s - \beta)}$$

where β is defined in Example 8.4-1. Inverse transformation leads to

$$p_1{}^0(t) = -\frac{\omega_{11}x_1{}^0(0)}{\beta}\sinh\beta t + \frac{p_1{}^0(0)}{\beta}(\beta\cosh\beta t - a_{11}\sinh\beta t)$$

The boundary condition on $p_1{}^0(t)$ is $p_1{}^0(t_f) = 0$. Since the problem is linear, this may be accomplished by adjustment of $p_1{}^0(0)$. Thus

$$p_1{}^0(0) = \frac{\omega_{11}x_1{}^0(0)\sinh\beta t_f}{\beta\cosh\beta t_f - a_{11}\sinh\beta t_f}$$

Then $p_1{}^0(t)$ can be written as

$$p_1{}^0(t) = \frac{\omega_{11}x_1{}^0(0)}{\beta}\left[-\sinh\beta t + \frac{\sinh\beta t_f(\beta\cosh\beta t - a_{11}\sinh\beta t)}{\beta\cosh\beta t_f - a_{11}\sinh\beta t_f}\right]$$

In a similar fashion,

$$x_1{}^0(t) = x_1{}^0(0)\left[\frac{\beta\cosh\beta(t_f - t) - a_{11}\sinh\beta(t_f - t)}{\beta\cosh\beta t_f - a_{11}\sinh\beta t_f}\right]$$

This expression can be solved for $x_1{}^0(0)$, and the result substituted for $x_1{}^0(0)$ in the equation for $p_1{}^0(t)$. This yields

$$p_1{}^0(t) = \left[\frac{\omega_{11}\sinh\beta(t_f - t)}{\beta\cosh\beta(t_f - t) - a_{11}\sinh\beta(t_f - t)}\right]x_1{}^0(t)$$

Then, from Eq. 8.4-2,

$$m_1{}^0(t) = -\frac{\omega_{11}b_{11}}{\zeta_{11}}\left[\frac{\sinh\beta(t_f - t)}{\beta\cosh\beta(t_f - t) - a_{11}\sinh\beta(t_f - t)}\right]x_1{}^0(t)$$

which is the result previously obtained by the more direct method. This result is also illustrated by Fig. 8.4-2.

Example 8.4-3. Determine the optimum system for the controlled elements of Fig. 8.4-3. The performance index is described by

$$\mathbf{\Omega} = \begin{bmatrix}\omega_{11} & 0\\ 0 & 0\end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix}0 & 0\\ 0 & \zeta_{22}\end{bmatrix}$$

and t_f is infinite, corresponding to an infinite interval problem. Again a regulator problem is assumed, so that $\mathbf{x}^d(t) = \mathbf{0}$. Also, $\mathbf{x}(t_f)$ is unspecified.

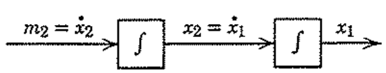

Fig. 8.4-3

Since $\mathbf{Z}$ is singular, the first of Eqs. 8.4-8 cannot be used directly. The factor $\mathbf{Z}^{-1}$ in Eqs. 8.4-8 is due to solving

$$\mathbf{Z}\mathbf{m}^0 + \mathbf{B}^T\mathbf{p}^0 = \mathbf{0} \tag{8.4-9}$$

for $\mathbf{m}^0$ and substituting the result into Eq. 8.4-1. In this case, $\mathbf{Z}$ is as given above anc

$$\mathbf{B} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Thus the only information contained in Eq. 8.4-9 is $m_2{}^0 = -(p_2{}^0/\zeta_{22})$. But, by the problem definition, $m_1{}^0 = 0$. Then Eq. 8.4-9 is unchanged if $\mathbf{Z}$ is replaced by

$$\mathbf{Z}_1 = \begin{bmatrix}1 & 0\\ 0 & \zeta_{22}\end{bmatrix}$$

Therefore Eqs. 8.4-8 can be used if $\mathbf{Z}^{-1}$ is replaced by $\mathbf{Z}_1{}^{-1}$.† Then since

$$\mathbf{\Phi}(s) = \begin{bmatrix}s^{-1} & s^{-2}\\ 0 & s^{-1}\end{bmatrix}$$

$\mathbf{P}^0(s)$ can be determined to be

$$\mathbf{P}^0(s) = \frac{\omega_{11}\begin{bmatrix}-s^2 & -s\\ s & 1\end{bmatrix}\mathbf{x}^0(0) - \begin{bmatrix}-s^3 & -\dfrac{\omega_{11}}{\zeta_{22}}\\ s^2 & -s^3\end{bmatrix}\mathbf{p}^0(0)}{s^4 + (\omega_{11}/\zeta_{22})}$$

where $\mathbf{p}^0(0)$ is to be adjusted so that $\mathbf{p}^0(\infty) = \mathbf{0}$. But $s^4 + (\omega_{11}/\zeta_{22}) = G(s)G(-s)$, where

$$G(s) = s^2 + (2)^{1/2}\left(\frac{\omega_{11}}{\zeta_{22}}\right)^{1/4}s + \left(\frac{\omega_{11}}{\zeta_{22}}\right)^{1/2}$$

This shows that $\mathbf{P}^0(s)$ has two right-half-plane poles and two left half-plane poles, symmetrically located with respect to the origin. In order to have $\mathbf{p}^0(\infty) = \mathbf{0}$, the residue in the right half-plane poles must be zero. A partial fraction expansion of either $P_1{}^0(s)$ or $P_2{}^0(s)$ reveals that the requirements on $\mathbf{p}^0(0)$ for zero residue in each of the right half-plane poles of $\mathbf{P}^0(s)$ are

$$\mathbf{p}^0(0) = \begin{bmatrix}(4\omega_{11}{}^3\zeta_{22})^{1/4} & (\omega_{11}\zeta_{22})^{1/2}\\ (\omega_{11}\zeta_{22})^{1/2} & (4\omega_{11}\zeta_{22}{}^3)^{1/4}\end{bmatrix}\mathbf{x}^0(0)$$

Since

$$\mathbf{X}^0(s) = \frac{\begin{bmatrix}s^3 & s^2\\ -\dfrac{\omega_{11}}{\zeta_{22}} & s_3\end{bmatrix}\mathbf{x}^0(0) + \zeta_{22}{}^{-1}\begin{bmatrix}1 & -s\\ s & -s^2\end{bmatrix}\mathbf{p}^0(0)}{s^4 + (\omega_{11}/\zeta_{22})}$$

† Note that $\mathbf{Z}_1$ is positive definite and hence the resulting linear optimum system is asymptotically stable, since the other requirements previously given for this are also satisfied.

An obvious alternative to this procedure is to rederive Eq. 8.4-8, but for the case in which $\mathbf{B}$ is a vector.

$\mathbf{x}^0(t)$, for the above values of $\mathbf{p}^0(0)$, is $\mathbf{x}^0(t) = \boldsymbol{\phi}(t)\mathbf{x}^0(0)$, where

$$\boldsymbol{\phi}(t) = \epsilon^{-\alpha t}\begin{bmatrix} (2)^{1/2}\sin\left(\alpha t + \dfrac{\pi}{4}\right) & \alpha^{-1}\sin\alpha t \\ -2\alpha\sin\alpha t & -(2)^{1/2}\sin\left(\alpha t - \dfrac{\pi}{4}\right)\end{bmatrix}$$

and $\alpha = (\omega_{11}/4\zeta_{22})^{1/4}$. Similarly,

$$m_2{}^0(t) = -\zeta_{22}{}^{-1}p_2{}^0(t) = -2\alpha^2\epsilon^{-\alpha t}\left[-x_1{}^0(0)(2)^{1/2}\sin\left(\alpha t - \frac{\pi}{4}\right) - \frac{x_2{}^0(0)}{\alpha}\sin\left(\alpha t + \frac{\pi}{2}\right)\right]$$

Substitution of $\mathbf{x}^0(0) = \boldsymbol{\phi}^{-1}(t)\mathbf{x}^0(t)$ yields

$$m_2{}^0(t) = -2\alpha[\alpha x_1{}^0(t) + x_2{}^0(t)]$$

This is the control law for the controlled elements of Fig. 8.4-3. As expected, it is a linear function of the state variables.

As α is increased, the performance index emphasizes the error relative to the cost of reducing it. From $\boldsymbol{\phi}(t)$ or $G(s)$, it can be seen that the effect is to increase the speed of response and the natural frequency of the system. The damping ratio, however, remains constant at 0.707. Increased damping would have been obtained if $\omega_{22} > 0$ had been chosen in the performance index.

The Time-Varying Case[21]

If any of the elements of **A**, **B**, **Ω**, or **Z** are time-varying, a time-domain solution of Eq. 8.4-3 is generally necessary. The solution of Eq. 8.4-3 is given by its state transition matrix, which is defined by

$$\dot{\boldsymbol{\phi}}(t, t_0) = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{Z}^{-1}\mathbf{B}^T \\ -\boldsymbol{\Omega} & -\mathbf{A}^T\end{bmatrix}\boldsymbol{\phi}(t, t_0)$$

The state transition matrix has $2n$ rows and $2n$ columns and can be partitioned into four $(n \times n)$ submatrices

$$\boldsymbol{\phi}(t, t_0) = \begin{bmatrix} \boldsymbol{\phi}_{11}(t, t_0) & \boldsymbol{\phi}_{12}(t, t_0) \\ \boldsymbol{\phi}_{21}(t, t_0) & \boldsymbol{\phi}_{22}(t, t_0)\end{bmatrix} \qquad (8.4\text{-}10)$$

Since $\boldsymbol{\phi}(t_0, t_0) = \mathbf{I}$,

$$\boldsymbol{\phi}_{11}(t_0, t_0) = \boldsymbol{\phi}_{22}(t_0, t_0) = \mathbf{I}$$
$$\boldsymbol{\phi}_{12}(t_0, t_0) = \boldsymbol{\phi}_{21}(t_0, t_0) = [0]$$

In terms of $\boldsymbol{\phi}(t, t_0)$, the solution of Eq. 8.4-3 is

$$\begin{bmatrix}\mathbf{x}^0(t) \\ \mathbf{p}^0(t)\end{bmatrix} = \boldsymbol{\phi}(t, t_0)\begin{bmatrix}\mathbf{x}^0(t_0) \\ \mathbf{p}^0(t_0)\end{bmatrix} + \begin{bmatrix}\mathbf{b}_1(t, t_0) \\ \mathbf{b}_2(t, t_0)\end{bmatrix} \qquad (8.4\text{-}11)$$

where

$$\begin{bmatrix}\mathbf{b}_1(t, t_0) \\ \mathbf{b}_2(t, t_0)\end{bmatrix} = \int_{t_0}^{t}\boldsymbol{\phi}(t, \tau)\begin{bmatrix}\mathbf{0} \\ \boldsymbol{\Omega}(\tau)\mathbf{x}^d(\tau)\end{bmatrix} d\tau \qquad (8.4\text{-}12)$$

Substitution of $\boldsymbol{\phi}(t, \tau)$ yields

$$\mathbf{b}_1(t, t_0) = \int_{t_0}^{t}\boldsymbol{\phi}_{12}(t, \tau)\boldsymbol{\Omega}(\tau)\mathbf{x}^d(\tau)\, d\tau$$
$$\mathbf{b}_2(t, t_0) = \int_{t_0}^{t}\boldsymbol{\phi}_{22}(t, \tau)\boldsymbol{\Omega}(\tau)\mathbf{x}^d(\tau)\, d\tau$$

For unspecified terminal conditions on $\mathbf{x}^0(t)$, the terminal boundary conditions are $\mathbf{p}^0(t_f) = \mathbf{0}$. Thus

$$\mathbf{p}^0(t_f) = \boldsymbol{\phi}_{21}(t_f, t_0)\mathbf{x}^0(t_0) + \boldsymbol{\phi}_{22}(t_f, t_0)\mathbf{p}^0(t_0) + \mathbf{b}_2(t_f, t_0) = \mathbf{0}$$

Solving for $\mathbf{p}^0(t_0)$,

$$\mathbf{p}^0(t_0) = -\boldsymbol{\phi}_{22}{}^{-1}(t_f, t_0)[\boldsymbol{\phi}_{21}(t_f, t_0)\mathbf{x}^0(t_0) + \mathbf{b}_2(t_f, t_0)] \qquad (8.4\text{-}13)$$

Substitution of t for t_0 yields

$$\mathbf{p}^0(t) = -\boldsymbol{\phi}_{22}{}^{-1}(t_f, t)[\boldsymbol{\phi}_{21}(t_f, t)\mathbf{x}^0(t) + \mathbf{b}_2(t_f, t)]$$

Then, from Eq. 8.4-2, the definition of $\mathbf{b}_2(t_f, t)$, and the fact that

$$\boldsymbol{\phi}_{22}{}^{-1}(t_f, t)\boldsymbol{\phi}_{22}(t_f, \tau) = \boldsymbol{\phi}_{22}(t, \tau)$$

the control law for unspecified terminal conditions on $\mathbf{x}^0(t)$ is

$$\mathbf{m}^0(t) = \mathbf{Z}^{-1}\mathbf{B}^T\left[\boldsymbol{\phi}_{22}{}^{-1}(t_f, t)\boldsymbol{\phi}_{21}(t_f, t)\mathbf{x}^0(t) + \int_{t}^{t_f}\boldsymbol{\phi}_{22}(t, \tau)\boldsymbol{\Omega}(\tau)\mathbf{x}^d(\tau)\, d\tau\right] \qquad (8.4\text{-}14)$$

The resulting response $\mathbf{x}^0(t)$ can be found from Eqs. 8.4-11 and 8.4-13.

For specified terminal conditions on $\mathbf{x}^0(t)$, i.e., $\mathbf{x}^0(t_f) = \mathbf{x}(t_f)$, Eq. 8.4-11 gives

$$\mathbf{x}^0(t_f) = \boldsymbol{\phi}_{11}(t_f, t_0)\mathbf{x}^0(t_0) + \boldsymbol{\phi}_{12}(t_f, t_0)\mathbf{p}^0(t_0) + \mathbf{b}_1(t_f, t_0)$$

Then

$$\mathbf{p}^0(t_0) = -\boldsymbol{\phi}_{12}{}^{-1}(t_f, t_0)[\boldsymbol{\phi}_{11}(t_f, t_0)\mathbf{x}^0(t_0) - \mathbf{x}^0(t_f) + \mathbf{b}_1(t_f, t_0)] \qquad (8.4\text{-}15)$$

Substituting t for t_0, using Eq. 8.4-2, the definition of $\mathbf{b}_1(t_f, t)$ and the fact that

$$\boldsymbol{\phi}_{12}{}^{-1}(t_f, t)\boldsymbol{\phi}_{12}(t_f, \tau) = \boldsymbol{\phi}_{12}(t, \tau)$$

yields for the control law, in the case of specified terminal conditions on $\mathbf{x}^0(t)$,

$$\mathbf{m}^0(t) = \mathbf{Z}^{-1}\mathbf{B}^T\left\{\boldsymbol{\phi}_{12}{}^{-1}(t_f, t)[\boldsymbol{\phi}_{11}(t_f, t)\mathbf{x}^0(t) - \mathbf{x}^0(t_f)] + \int_{t}^{t_f}\boldsymbol{\phi}_{12}(t, \tau)\boldsymbol{\Omega}(\tau)\mathbf{x}^d(\tau)\, d\tau\right\} \qquad (8.4\text{-}16)$$

The resulting response $\mathbf{x}^0(t)$ can be found from Eqs. 8.4-11 and 8.4-15.

The reader should note that these control laws are of the form of Eq. 8.4-6, where

$$\mathbf{K} = -\boldsymbol{\phi}_{22}^{-1}(t_f, t)\boldsymbol{\phi}_{21}(t_f, t)$$

for unspecified $\mathbf{x}^0(t_f)$, and

$$\mathbf{K} = -\boldsymbol{\phi}_{12}^{-1}(t_f, t)\boldsymbol{\phi}_{11}(t_f, t)$$

for specified $\mathbf{x}^0(t_f)$. Also, the control law requires knowledge of $\mathbf{x}^d$ in the future interval of control, i.e., $\mathbf{x}^d(\tau)$ for $t \leqslant \tau \leqslant t_f$. This is a general requirement for optimization according to this performance criterion.

Example 8.4-4. Repeat Example 8.4-3 in the time domain.

Let

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & -\mathbf{BZ}^{-1}\mathbf{B}^T \\ -\boldsymbol{\Omega} & -\mathbf{A}^T \end{bmatrix} = (-1)\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{22}^{-1} \\ \hline \omega_{11} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so that $\boldsymbol{\phi}(t, t_0) = \epsilon^{\mathbf{G}(t-t_0)}$. Use of the Cayley-Hamilton technique gives

$$\boldsymbol{\phi}(t, t_0) = \left[\begin{array}{cc|cc} \alpha_0(t-t_0) & \alpha_1(t-t_0) & \zeta_{22}^{-1}\alpha_3(t-t_0) & -\zeta_{22}^{-1}\alpha_2(t-t_0) \\ -\omega_{11}\zeta_{22}^{-1}\alpha_3(t-t_0) & \alpha_0(t-t_0) & \zeta_{22}^{-1}\alpha_2(t-t_0) & -\zeta_{22}^{-1}\alpha_1(t-t_0) \\ \hline -\omega_{11}\alpha_1(t-t_0) & -\omega_{11}\alpha_2(t-t_0) & \alpha_0(t-t_0) & \omega_{11}\zeta_{22}^{-1}\alpha_3(t-t_0) \\ \omega_{11}\alpha_2(t-t_0) & \omega_{11}\alpha_3(t-t_0) & -\alpha_1(t-t_0) & \alpha_0(t-t_0) \end{array}\right]$$

where

$$\alpha_0(t) = \cosh \alpha t \cos \alpha t$$

$$\alpha_1(t) = \frac{\sinh \alpha t \cos \alpha t + \cosh \alpha t \sin \alpha t}{2\alpha}$$

$$\alpha_2(t) = \frac{\sinh \alpha t \sin \alpha t}{2\alpha^2}$$

$$\alpha_3(t) = \frac{\cosh \alpha t \sin \alpha t - \sinh \alpha t \cos \alpha t}{4\alpha^3}$$

and α is as defined in Example 8.4-3. Substitution into Eq. 8.4-14 yields, for infinite t_f,

$$m_2{}^0(t) = -2\alpha[\alpha x_1{}^0(t) + x_2{}^0(t)]$$

the same result obtained in Example 8.4-3.

Example 8.4-5. Determine the optimum system for the controlled elements characterized by

$$\dot{x}_1 = -\frac{1}{t}x_1 + \frac{1}{t}m_1$$

The performance index is

$$I = \tfrac{1}{2}\int_{t_0}^{t_f}(x_1{}^2 + m_1{}^2)\,dt$$

and $x_1{}^d(t) = 0$. The terminal conditions are unspecified.

From the problem statement,

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & -\mathbf{BZB}^T \\ -\boldsymbol{\Omega} & -\mathbf{A}^T \end{bmatrix} = \begin{bmatrix} -t^{-1} & -t^{-2} \\ -1 & t^{-1} \end{bmatrix}$$

Then the components $\phi_{11}(t, t_0)$ and $\phi_{21}(t, t_0)$ of $\boldsymbol{\phi}(t, t_0)$ must satisfy

$$\dot{\phi}_{11}(t, t_0) = -\frac{1}{t}\phi_{11}(t, t_0) - \frac{1}{t^2}\phi_{21}(t, t_0)$$

$$\dot{\phi}_{21}(t, t_0) = -\phi_{11}(t, t_2) + \frac{1}{t}\phi_{21}(t, t_0)$$

Solving for $\phi_{11}(t, t_0)$ in the second equation and substituting the result into the first yields

$$\ddot{\phi}_{21}(t, t_0) - \frac{1}{t^2}\phi_{21}(t, t_0) = 0$$

This is a form of Euler's equation, considered in Example 2.8-2. The change of variable $t = \epsilon^z$ gives

$$\phi_{21}''(\epsilon^z, \epsilon^{z_0}) - \phi_{21}'(\epsilon^z, \epsilon^{z_0}) - \phi_{21}(\epsilon^z, \epsilon^{z_0}) = 0$$

where the primes denotes differentiation with respect to z. The differential equation for $\phi_{21}(\epsilon^z, \epsilon^{z_0})$ has the solution

$$\phi_{21}(\epsilon^z, \epsilon^{z_0}) = k_1(z_0)\exp\left(\frac{\sqrt{5}+1}{2}z\right) + k_2(z_0)\exp\left(\frac{1-\sqrt{5}}{2}z\right)$$

so that

$$\phi_{21}(t, t_0) = c_1(t_0)t^{(\sqrt{5}+1)/2} + c_2(t_0)t^{(1-\sqrt{5})/2}$$

Similarly,

$$\phi_{11}(t, t_0) = c_1(t_0)\left(\frac{1-\sqrt{5}}{2}\right)t^{(\sqrt{5}-1)/2} + c_2(t_0)\left(\frac{1+\sqrt{5}}{2}\right)t^{-(\sqrt{5}+1)/2}$$

Since $\phi_{11}(t_0, t_0) = 1$ and $\phi_{21}(t_0, t_0) = 0$, $c_1(t_0)$ and $c_2(t_0)$ can be determined to be

$$c_1(t_0) = -\frac{1}{\sqrt{5}}t_0^{(1-\sqrt{5})/2}$$

$$c_2(t_0) = \frac{1}{\sqrt{5}}t_0^{(1+\sqrt{5})/2}$$

Then

$$\phi_{11}(t, t_0) = \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{t_0}{t}\right)^{-(\sqrt{5}-1)/2} + \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{t_0}{t}\right)^{(\sqrt{5}+1)/2}$$

and

$$\phi_{21}(t, t_0) = \frac{t_0}{\sqrt{5}}\left(\frac{t_0}{t}\right)^{(\sqrt{5}-1)/2} - \frac{t_0}{\sqrt{5}}\left(\frac{t_0}{t}\right)^{-(\sqrt{5}+1)/2}$$

Similarly,

$$\phi_{12}(t, t_0) = -\frac{t^{-1}}{\sqrt{5}}\left(\frac{t_0}{t}\right)^{-(\sqrt{5}-1)/2} + \frac{t^{-1}}{\sqrt{5}}\left(\frac{t_0}{t}\right)^{(\sqrt{5}+1)/2}$$

$$\phi_{22}(t, t_0) = \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{t_0}{t}\right)^{-(\sqrt{5}-1)/2} + \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{t_0}{t}\right)^{(\sqrt{5}+1)/2}$$

From Eq. 8.4-14,

$$m_1^0(t) = \frac{1}{t}\frac{\phi_{21}(t_f, t)}{\phi_{22}(t, t_f)}x_1^0(t)$$

Thus

$$m_1^0(t) = -2\left[\frac{1-(t/t_f)^{\sqrt{5}}}{(\sqrt{5}+1)+(\sqrt{5}-1)(t/t_f)^{\sqrt{5}}}\right]x_1^0(t)$$

In the infinite interval problem, t_f is infinite. Then

$$m_1^0(t) = -\frac{2}{\sqrt{5}+1}x_1^0(t)$$

In this case, a time-invariant control law is obtained, even though the controlled are time-varying.

Example 8.4-6. Determine the optimum system for the controlled elements of Example 8.4-5, if the performance index is

$$I = \tfrac{1}{2}\int_{t_0}^{t_f} m_1^2\,dt$$

and the terminal condition $x_1^0(t_f) = 0$ is specified.

For this case,

$$\mathbf{G} = \begin{bmatrix} -t^{-1} & -t^{-2} \\ 0 & t^{-1} \end{bmatrix}$$

$\boldsymbol{\phi}(t, t_0)$ is the solution to $\dot{\boldsymbol{\phi}}(t, t_0) = \mathbf{G}\boldsymbol{\phi}(t, t_0)$. The equations can be integrated by separation of variables to yield

$$\boldsymbol{\phi}(t, t_0) = \begin{bmatrix} \dfrac{t_0}{t} & \dfrac{1}{t} - \dfrac{1}{t_0} \\ \dfrac{t}{t_0} & \dfrac{t}{t_0} \end{bmatrix}$$

From Eq. 8.4-16,

$$m_1^0(t) = -\left[\frac{t}{t_f - t}\right]x_1^0(t)$$

The resulting response, as found from Eqs. 8.4-11 and 8.4-15, is

$$x_1^0(t) = \frac{t_0(t_f - t)}{t(t_f - t_0)}x_1^0(t_0)$$

The response does satisfy the terminal condition $x_1^0(t_f) = 0$. For $t > t_f$, the response is not zero, however. This is to be expected, since the performance index does not consider this part of the response.

Although the time-varying feedback gain becomes infinite at $t = t_f$, $m_1^0(t)$ is always finite. In fact, from the expressions for $m_1^0(t)$ and $x_1^0(t)$,

$$m_1^0(t) = -\frac{t_0}{t_f - t_0}x_1^0(t_0)$$

a constant. For infinite t_f, corresponding to an infinite interval problem, $m_1^0(t) = 0$. Thus the system is open-loop, and the response is the open-loop initial condition response $x_1^0(t) = (t_0/t)x_1^0(t_0)$. This response satisfies all the requirements in the infinite interval case and obviously has the smallest possible value for the performance index. The system would be undesirable, however, owing to its poor performance with respect to unwanted disturbances.

The controlled elements in the examples of this section are rather simple, and yet considerable effort is required in some cases to determine the optimum system. This is true in spite of the fact that the systems are linear. In practical situations systems are nonlinear, and analytical determination of optimum systems is virtually impossible. For the most part, optimization theory is practical only when used in conjunction with computers, as considered later in this chapter.

8.5 SELECTION OF CONSTANT WEIGHTING FACTORS

The examples of the previous section indicate that the control law and system response are greatly influenced by the weighting factors $\boldsymbol{\Omega}$ and $\mathbf{Z}$ chosen in the performance index. Selection of these weighting factors is a difficult task, since the relationships between the weighting factors and the optimum system parameters or the system response are generally very complex. However, Ellert has developed a technique for the selection of weighting factors in the time-invariant case.[16]

Consider, as an example, the second order linear controlled elements described by

$$\dot{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & b_{22} \end{bmatrix}\mathbf{m}$$

The performance index is the one of Section 8.4, with infinite t_f and

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Using the method of Eqs. 8.4-5, the optimum control law is found to be

$$m_2^0(t) = -b_{22}[k_{21}x_1^0(t) + k_{22}x_2^0(t)] + b_{22}v_1^0(t) \qquad (8.5\text{-}1)$$

where the k's are defined by

$$\begin{aligned} \omega_{22} + 2a_{22}k_{22} + 2a_{12}k_{21} - b_{22}^2k_{22}^2 &= 0 \\ \omega_{11} + 2a_{21}k_{21} + 2a_{11}k_{11} - b_{22}^2k_{21}^2 &= 0 \qquad (8.5\text{-}2) \\ a_{21}k_{22} + a_{22}k_{21} + a_{11}k_{21} + a_{12}k_{11} - b_{22}^2k_{22}k_{21} &= 0 \end{aligned}$$

and v^0 is defined by

$$\begin{aligned} -\dot{v}_1^0 &= \omega_{22}x_2^d + a_{22}v_2^0 + a_{12}v_1^0 - b_{22}^2k_{22}v_2^0 \\ -\dot{v}_2^0 &= \omega_{11}x_1^d + a_{21}v_1^0 + a_{11}v_1^0 - b_{11}^2k_{21}v_2^0 \end{aligned} \tag{8.5-3}$$

Since this is a linear, time-invariant system, the closed-loop transfer function can be determined to be

$$\frac{X_1^0(s)}{V(s)} = \frac{\omega_0^2}{s^2 + z_1\omega_0 s + \omega_0^2} \tag{8.5-4}$$

where

$$z_1\omega_0 = b_{22}^2k_{22} - a_{11} - a_{22}$$

$$\omega_0^2 = a_{11}(a_{22} - b_{22}^2k_{22}) + a_{12}(b_{22}^2k_{21} - a_{21})$$

$$V(s) = \frac{a_{12}b_{22}^2}{\omega_0^2} V_1(s)$$

and $V_1(s)$ is the Laplace transform of the system input.

With these definitions, k_{22} and k_{21} can be written as

$$\begin{aligned} k_{22} &= \frac{1}{b_{22}^2}(z_1\omega_0 + a_{11} + a_{22}) \\ k_{21} &= \frac{1}{a_{21}b_{22}^2}(\omega_0^2 + a_{11}z_1\omega_0 + a_{12}a_{21} + a_{11}^2) \end{aligned} \tag{8.5-5}$$

From Eqs. 8.5-2 and 8.5-5,

$$\begin{aligned} \omega_{11} &= \frac{1}{a_{12}^2b_{22}^2}[\omega_0^4 + 3a_{11}z_1\omega_0^3 + a_{11}^2(3z_1^2 + 2)\omega_0^2 + 4a_{11}^3z_1\omega_0 \\ &\quad + (2a_{11}a_{12}a_{21}a_{22} + 2a_{11}^2a_{12}a_{21} + a_{12}^2a_{21}^2 + a_{11}^4)] \\ \omega_{22} &= \frac{1}{b_{22}^2}[(z_1^2 - 2)\omega_0^2 - a_{11}^2 - a_{22}^2 - 2a_{12}a_{21}] \end{aligned} \tag{8.5-6}$$

These expressions determine ω_{11} and ω_{22}, once values of z_1 and ω_0 have been selected.

Ellert's procedure is to choose z_1 to provide the desired relative stability of the system, assuming that none of the system variables exceed their prescribed limits. ω_0 is then chosen in accordance with the system bandwidth requirements or any limits on $m_2(t)$. The relationship between $m_2(t)$ and ω_0 is given by substituting Eq. 8.5-5 into Eq. 8.5-1. It is

$$\begin{aligned} m_2(t) = -\frac{1}{b_{22}}\Big\{&\frac{\omega_0^2}{a_{12}}x_1(t) + \left[\frac{a_{11}}{a_{12}}x_1(t) + x_2(t)\right]z_1\omega_0 \\ &+ \left(\frac{a_{11}^2}{a_{12}} + a_{21}\right)x_1(t) + (a_{11} + a_{22})x_2(t)\Big\} + b_{22}v_1(t) \end{aligned} \tag{8.5-7}$$

Specification of the maximum available value of $m_2(t)$, worst case values of $x_1(t)$ and $x_2(t)$, and solution of Eq. 8.5-3 permits Eq. 8.5-7 to be solved for ω_0.

When z_1 and ω_0 have been determined, the weighting factors ω_{11} and ω_{22} are given by Eqs. 8.5-6. Since the performance index should be convex, ω_{11} and ω_{22} must be non-negative. This requirement, in a sense, tests the compatibility of the design requirements, assuming that a quadratic performance index with constant weighting factors is a reasonable choice.[16]

For controlled elements of higher order, Eq. 8.5-4 becomes

$$\frac{X_1(s)}{V(s)} = \frac{N(s)}{s^n + z_{n-1}\omega_0 s^{n-1} + \cdots + z_1\omega_0^{n-1}s + \omega_0^n} \tag{8.5-8}$$

where

$$N(s) = \omega_0^n$$

$$N(s) = z_1\omega_0^{n-1}s + \omega_0^n$$

or

$$N(s) = z_2\omega_0^{n-2}s^2 + z_1\omega_0^{n-1}s + \omega_0^n$$

for types one, two, or three systems, respectively, i.e., systems with zero steady-state error to a unit step input, zero steady-state error to a unit ramp input, etc., respectively. Ellert's procedure for selection of the performance index weighting factors can be applied to these higher order cases, if the z's can be determined without undue trial and error. Criteria for selecting the z's to obtain acceptable responses have been presented in the literature. In fact, tabulations of numerical values of the z's, called standard forms, can be found.[6,10,22] Whiteley's standard forms for the characteristic equations are given in Table 8.5-1.[22] The corresponding step responses are shown in Fig. 8.5-1. Since many practical control systems

Table 8.5-1

System Type		Standard Forms	Maximum Percent Overshoot
Zero position error	(*a*)	$s^2 + 1.4\omega_0 s + \omega_0^2$	5
	(*b*)	$s^3 + 2\omega_0 s^2 + 2\omega_0^2 s + \omega_0^3$	8
	(*c*)	$s^4 + 2.6\omega_0 s^3 + 3.4\omega_0^2 s^2 + 2.6\omega_0^3 s + \omega_0^4$	10
Zero velocity error	(*d*)	$s^2 + 2.5\omega_0 s + \omega_0^2$	10
	(*e*)	$s^3 + 5.1\omega_0 s^2 + 6.3\omega_0^2 s + \omega_0^3$	10
	(*f*)	$s^4 + 7.2\omega_0 s^3 + 16\omega_0^2 s^2 + 12\omega_0^3 s + \omega_0^4$	10
	(*g*)	$s^5 + 9\omega_0 s^4 + 29\omega_0^2 s^3 + 38\omega_0^3 s^2 + 18\omega_0^4 s + \omega_0^5$	10
	(*h*)	$s^6 + 11\omega_0 s^5 + 43\omega_0^2 s^4 + 83\omega_0^3 s^3 + 73\omega_0^4 s^2 + 25\omega_0^5 s + \omega_0^6$	10
Zero acceleration error	(*i*)	$s^3 + 6.7\omega_0 s^2 + 6.7\omega_0^2 s + \omega_0^3$	10
	(*j*)	$s^4 + 7.9\omega_0 s^3 + 15\omega_0^2 s^2 + 7.9\omega_0^3 s + \omega_0^4$	20
	(*k*)	$s^5 + 18\omega_0 s^4 + 69\omega_0^2 s^3 + 69\omega_0^3 s^2 + 18\omega_0^4 s + \omega_0^5$	20
	(*l*)	$s^6 + 36\omega_0 s^5 + 251\omega_0^2 s^4 + 485\omega_0^3 s^3 + 251\omega_0^4 s^2 + 36\omega_0^5 s + \omega_0^6$	20

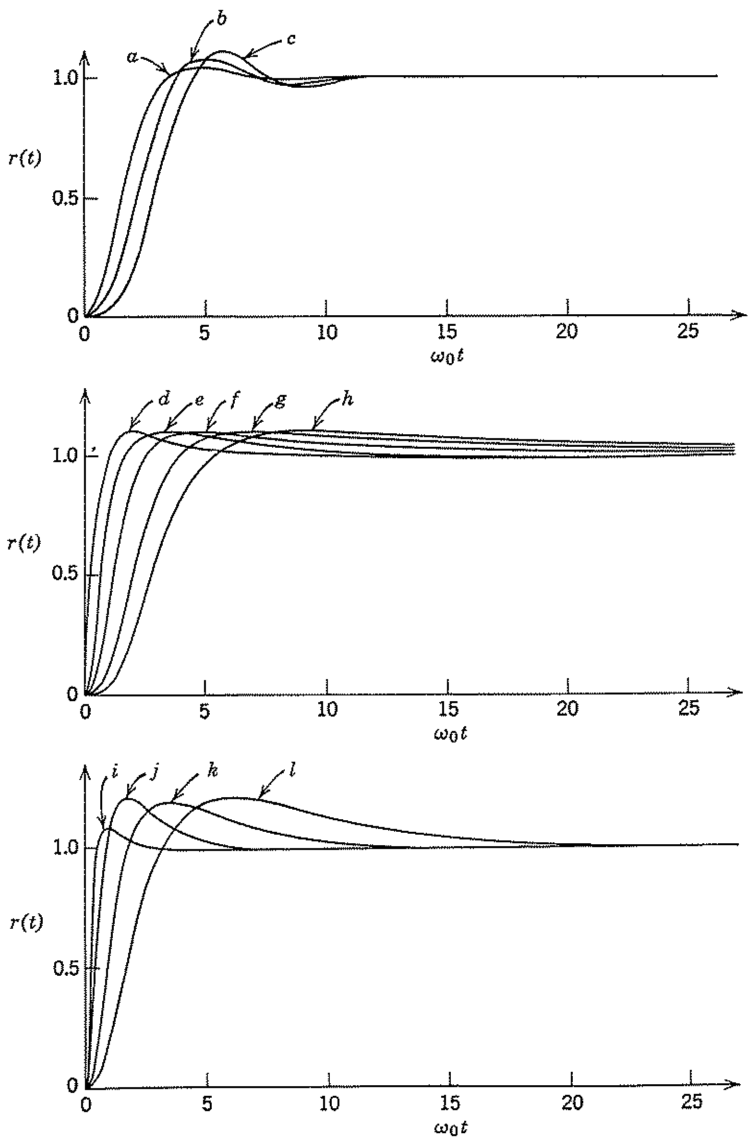

Fig. 8.5-1

have transfer functions of the form of Eq. 8.5-8, Whiteley's standard forms can often be used, in conjunction with Ellert's procedure, to determine the performance index weighting factors which satisfy the subjective design requirements. Objective design requirements, such as limits on the control signals or state variables, must be approached in a different fashion. This is considered in the next section.

8.6 PENALTY FUNCTIONS

The design specifications on most control systems require that some of the variables be constrained between prescribed limits. Such constraints may be imposed by saturation-type limits in the controlled elements, or they may be due to the mission requirements associated with the application of the system. For example, the maximum allowable stagnation temperature on the nose of a re-entry vehicle is often given as 3500° Fahrenheit. The temperature constraint, as stated, is a "hard" constraint, in the sense that it is a value not to be exceeded. Both "hard" and "soft" constraints are often stated, but in practice most constraints really are soft. For example, a temperature of 3600° would probably also be acceptable, since safety factors are usually included in such figures. No design procedure or subsequent implementation is precise, nor can any design procedure consider the uncertainties associated with the ultimate operation of the system. Although hard constraints are conceptually useful from a mathematical viewpoint, they normally do not physically exist. Furthermore, hard constraints cause considerable difficulty in obtaining computer solutions to optimization problems, and in controller realization. For these reasons, constraints are treated here by means of penalty functions. One can approach a hard constraint by making the penalty more severe.

A *penalty function* is a performance index term which increases the value of the index when the constrained variable approaches its limit. For example, the second and fourth terms of Eq. 8.2-2 are penalty function terms. Many other penalty functions have been proposed in the literature.[23–25]

Weighting factors, as contained in the first and third terms of Eq. 8.2-2, are selected to satisfy the subjective design requirements. This was discussed in the preceding section. Once the weighting factors are selected, the penalty functions may be selected to satisfy the constraints. If this is done, the system response satisfies the relative stability and speed of response specifications when none of the variables are at their limits, and, furthermore, the variables are properly constrained when they attempt to exceed these limits. This is the basis for Ellert's design philosophy.