10 / The Design and Compensation of Feedback Control Systems

10.1 INTRODUCTION

The performance of a feedback control system is of primary importance. This subject was discussed at length in Chapter 4 and quantitative measures of performance were developed. We have found that a suitable control system is stable and that it results in an acceptable response to input commands, is less sensitive to system parameter changes, results in a minimum steady-state error for input commands, and, finally, is able to eliminate the effect of undesirable disturbances. A feedback control system that provides an optimum performance without any necessary adjustments is rare indeed. Usually one finds it necessary to compromise among the many conflicting and demanding specifications and to adjust the system parameters to provide a suitable and acceptable performance when it is not possible to obtain all the desired optimum specifications.

We have considered at several points in the preceding chapters the question of design and adjustment of the system parameters in order to provide a desirable response and performance. In Chapter 4, we defined and established several suitable measures of performance. Then, in Chapter 5, we determined a method of investigating the stability of a control system, since we recognized that a system is unacceptable unless it is stable. In Chapter 6, we utilized the root locus method to effect a design of a self-balancing scale (Section 6.4) and then illustrated a method of parameter design by using the root locus method (Section 6.5). Furthermore, in Chapters 7 and 8, we developed suitable measures of performance in terms of the frequency variable ω and utilized them to design several suitable control systems. Finally, using time-domain methods in Chapter 9, we investigated the selection of feedback parameters in order to stabilize a system. Thus, we have been considering the problems of the design of feedback control systems as an integral part of the subjects of the preceding chapters. It is now our purpose to study the question somewhat further and to point out several significant design and compensation methods.

We have found in the preceding chapters that it is often possible to adjust the system parameters in order to provide the desired system response. However, we often find that we are not able to simply adjust a system parameter and thus obtain the desired performance. Rather we are forced to reconsider the structure of the system and redesign the system in order to obtain a suitable one. That is, we must examine the scheme or plan of the system and obtain a new design or plan which results in a suitable system. Thus, *the design of a control system is concerned with the arrangement, or the plan, of the system structure and the selection of suitable components and parameters*. For example, if one desires a set of performance measures to be less than some specified values, often one encounters a conflicting set of requirements. Thus, if we wish a system to have a percent overshoot less than 20% and $\omega_n T_p = 3.3$, we obtain a conflicting requirement on the system damping ratio, ζ, as can be seen by examining Fig. 4.8. Now, if we are unable to relax these two performance requirements, we must alter the system in some way. Often the alteration or adjustment of a control system, in order to provide a suitable performance, is called *compensation;* that is, compensation is the adjustment of a system in order to make up for deficiencies or inadequacies. It is the purpose of this chapter to consider briefly the issue of the design and compensation of control systems.

In redesigning a control system in order to alter the system response, an additional component is inserted within the structure of the feedback system. It is this additional component or device that equalizes or compensates for the performance deficiency. The compensating device may be an electric, mechanical, hydraulic, pneumatic, or other type of device or network, and is often called a *compensator.* Commonly, an electric circuit serves as a compensator in many control systems. The transfer function of the compensator is designated as $G_c(s) = E_{\text{out}}(s)/E_{\text{in}}(s)$ and the compensator may be placed in a suitable location within the structure of the system. Several types of compensation are shown in Fig. 10.1 for a simple single-loop feedback control system. The compensator placed in the feedforward path is

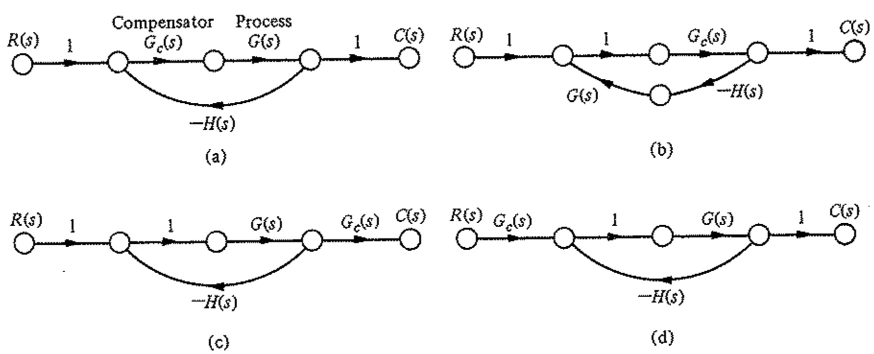


Fig. 10.1. Types of compensation. (a) Cascade compensation. (b) Feedback compensation. (c) Output or load compensation. (d) Input compensation.

called a cascade or series compensator. Similarly, the other compensation schemes are called feedback, output or load, and input compensation, as shown in Fig. 10.1(b), (c), and (d), respectively. The selection of the compensation scheme depends upon a consideration of the specifications, the power levels at various signal nodes in the system, and the networks available for use. It will not be possible for us to consider all the possibilities in this chapter, and the reader is referred to further work following the introductory material of this chapter [1, 2].

10.2 APPROACHES TO COMPENSATION

The performance of a control system may be described in terms of the time-domain performance measures or the frequency-domain performance measures. The performance of a system may be specified by requiring a certain peak time, T_p, maximum overshoot, and settling-time for a step input. Furthermore, it is usually necessary to specify the maximum allowable steady-state error for several test signal inputs and disturbance inputs. These performance specifications may be defined in terms of the desirable location of the poles and zeros of the closed-loop system transfer function, $T(s)$. Thus the location of the s-plane poles and zeros of $T(s)$ may be specified. As we found in Chapter 6, the locus of the roots of the closed-loop system may be readily obtained for the variation of one system parameter. However, when the locus of roots does not result in a suitable root configuration, one must add a compensating network (Fig. 10.1) in order to alter the locus of the roots as the parameter is varied. Therefore, one may utilize the root locus method and determine a suitable compensator network transfer function so that the resultant root locus results in the desired closed-loop root configuration.

Alternatively, one may describe the performance of a feedback control system in terms of frequency performance measures. Then a system may be described in terms of the peak of the closed-loop frequency response, M_{p_ω}, the resonant frequency, ω_r, the bandwidth, and the phase margin of the system. One may add a suitable compensation network, if necessary, in order to satisfy the system specifications. The design of the network $G_c(s)$, is developed in terms of the frequency response as portrayed on the polar plane, the Bode diagram, or the Nichols chart. Since a cascade transfer function is readily accounted for on a Bode plot by adding the frequency response of the network, we usually prefer to approach the frequency response methods by utilizing the Bode diagram.

Thus, the compensation of a system is concerned with the alteration of the frequency response or the root locus of the system in order to obtain a suitable system performance. For frequency response methods, we are concerned with altering the system so that the frequency response of the compensated system will satisfy the system specifications. Thus, in the case of the frequency response approach, one utilizes compensation networks to alter and reshape the frequency characteristics represented on the Bode diagram and Nichols chart.

Alternatively, the compensation of a control system may be accomplished in the

s-plane by root locus methods. For the case of the s-plane, the designer wishes to alter and reshape the root locus so that the roots of the system will lie in the desired position in the s-plane.

The time-domain method, expressed in terms of state variables, may also be utilized to design a suitable compensation scheme for a control system. Typically, one is interested in controlling the system with a control signal, $\mathbf{u}(t)$, which is a function of several measurable state variables. Then one develops a state-variable controller which operates on the information available in measured form. This type of system compensation is quite useful for system optimization and will be considered briefly in this chapter.

We have illustrated several of the aforementioned approaches in the preceding chapters. In Example 6.5, we utilized the root locus method in considering the design of a feedback network in order to obtain a satisfactory performance. In Chapter 8, we considered the selection of the gain in order to obtain a suitable phase margin and, therefore, a satisfactory relative stability. Also, in Example 9.6, we compensated for the unstable response of the pendulum by controllng the pendulum with a function of several of the state variables of the system.

Quite often, in practice, the best and simplest way to improve the performance of a control system is to alter, if possible, the process itself. That is, if the system designer is able to specify and alter the design of the process which is represented by the transfer function $G(s)$, then the performance of the system may be readily improved. For example, in order to improve the transient behavior of a servomechanism position controller, one can often choose a better motor for the system. In the case of an airplane control system, one might be able to alter the aerodynamic design of the airplane and thus improve the flight transient characteristics. Thus, a control system designer should recognize that an alteration of the process may result in an improved system. However, often the process is fixed and unalterable or has been altered as much as is possible and is still found to result in an unsatisfactory performance. Then the addition of compensation networks becomes useful for improving the performance of the system. In the following sections we will assume that the process has been improved as much as possible and the $G(s)$ representing the process is unalterable.

It is the purpose of this chapter to further describe the addition of several compensation networks to a feedback control system. First we shall consider the addition of a so-called phase-lead compensation network and describe the design of the network by root locus and frequency response techniques. Then, using both the root locus and frequency response techniques, we shall describe the design of the integration compensation networks in order to obtain a suitable system performance. Finally, we shall determine an optimum controller for a system described in terms of state variables. While these three approaches to compensation are not intended to be discussed in a complete manner, the discussion that follows should serve as a worthwhile introduction to the design and compensation of feedback control systems.

10.3 CASCADE COMPENSATION NETWORKS

In this section, we shall consider the design of a cascade or feedback network as shown in Fig. 10.1(a) and Fig. 10.1(b), respectively. The compensation network, $G_c(s)$, is cascaded with the unalterable process $G(s)$ in order to provide a suitable loop transfer function $G_c(s)G(s)H(s)$. Clearly, the compensator $G_c(s)$ may be chosen to alter the shape of the root locus or the frequency response. In either case, the network may be chosen to have a transfer function

$$G_c(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{N}(s+p_j)}. \tag{10.1}$$

Then the problem reduces to the judicious selection of the poles and zeros of the compensator. In order to illustrate the properties of the compensation network, we shall consider a first-order compensator. The compensation approach developed on the basis of a first-order compensator may then be extended to higher-order compensators.

Consider the first-order compensator with the transfer function

$$G_c(s) = \frac{K(s+z)}{(s+p)}. \tag{10.2}$$

The design problem becomes, then, the selection of z, p, and K in order to provide a suitable performance. When $|z| < |p|$, the network is called a *phase-lead network* and has a pole-zero s-plane configuration as shown in Fig. 10.2. If the pole was negligible, that is, $|p| \gg |z|$, and the zero occurred at the origin of the s-plane, we would have a differentiator so that

$$G_c(s) \simeq \left(\frac{K}{p}\right) s. \tag{10.3}$$

Thus a compensation network of the form of Eq. (10.2) is a differentiator type network. The differentiator network of Eq. (10.3) has a frequency characteristic as

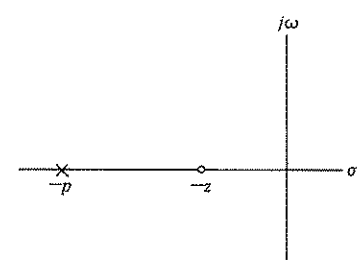


Fig. 10.2. The pole-zero diagram of the phase-lead network.

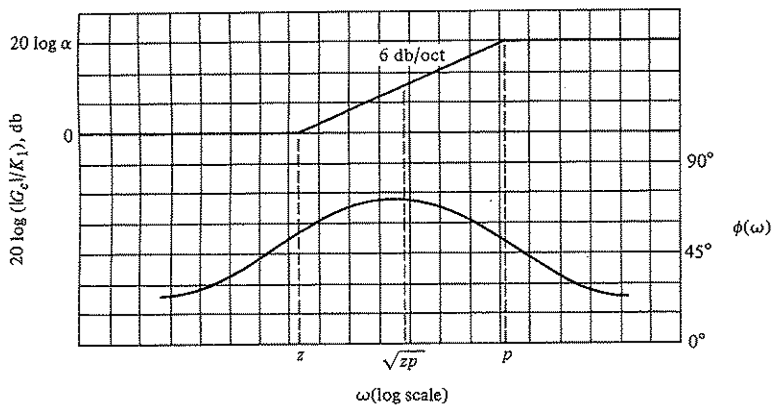


Fig. 10.3. The Bode diagram of the phase-lead network.

$$G_c(j\omega) = j\left(\frac{K}{p}\right)\omega = \left(\frac{K}{p}\omega\right)e^{+j90°} \tag{10.4}$$

and a phase angle of +90°, often called a phase-lead angle. Similarly, the frequency response of the differentiating network of Eq. (10.2) is

$$\begin{aligned} G_c(j\omega) &= \frac{K(j\omega + z)}{(j\omega + p)} = \frac{(Kz/p)(j(\omega/z) + 1)}{(j(\omega/p) + 1)} \\ &= \frac{K_1(1 + j\omega\alpha\tau)}{(1 + j\omega\tau)}, \end{aligned} \tag{10.5}$$

where $\tau = 1/p$, $p = \alpha z$, and $K_1 = K/\alpha$. The frequency response of this phase-lead network is shown in Fig. 10.3. The angle of the frequency characteristic is

$$\phi(\omega) = \tan^{-1}\alpha\omega\tau - \tan^{-1}\omega\tau. \tag{10.6}$$

Since the zero occurs first on the frequency axis, we obtain a phase-lead characteristic as shown in Fig. 10.3. The slope of the asymptotic magnitude curve is +6 db/octave.

The phase-lead compensation transfer function can be obtained with the network shown in Fig. 10.4. The transfer function of this network is

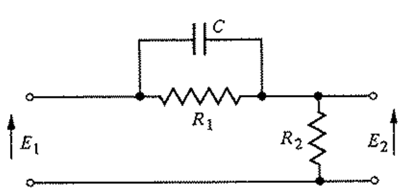


Fig. 10.4. A phase-lead network.

$$\begin{aligned} G_c(s) = \frac{E_2(s)}{E_1(s)} &= \frac{R_2}{R_2 + \{R_1(1/Cs)/[R_1 + (1/Cs)]\}} \\ &= \left(\frac{R_2}{R_1 + R_2}\right)\frac{(R_1Cs + 1)}{\{[R_1R_2/(R_1 + R_2)]Cs + 1\}}. \end{aligned} \tag{10.7}$$

Therefore, we let

$$\tau = \frac{R_1R_2}{R_1 + R_2}C \quad \text{and} \quad \alpha = \frac{R_1 + R_2}{R_2}$$

and obtain the transfer function

$$G_c(s) = \frac{(1 + \alpha\tau s)}{\alpha(1 + \tau s)}, \tag{10.8}$$

which is equal to Eq. (10.5) when an additional cascade gain K is inserted.

The maximum value of the phase lead occurs at a frequency ω_m, where ω_m is the geometric mean of $p = 1/\tau$ and $z = 1/\alpha\tau$; that is, the maximum phase lead occurs halfway between the pole and zero frequencies on the logarithmic frequency scale. Therefore,

$$\omega_m = \sqrt{zp} = \frac{1}{\tau\sqrt{\alpha}}.$$

In order to obtain an equation for the maximum phase-lead angle, we rewrite the phase angle of Eq. (10.5) as

$$\phi = \tan^{-1}\frac{\alpha\omega\tau - \omega\tau}{1 + (\omega\tau)^2\alpha}. \tag{10.9}$$

Then, substituting the frequency for the maximum phase angle, $\omega_m = 1/\tau\sqrt{\alpha}$ we have

$$\begin{aligned} \tan\phi_m &= \frac{(\alpha/\sqrt{\alpha}) - (1/\sqrt{\alpha})}{1 + 1} \\ &= \frac{\alpha - 1}{2\sqrt{\alpha}}. \end{aligned} \tag{10.10}$$

Since the $\tan\phi_m$ equals $(\alpha - 1)/2\sqrt{\alpha}$, we utilize the triangular relationship and note that

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1}. \tag{10.11}$$

Equation (10.11) is very useful for calculating a necessary α ratio between the pole and zero of a compensator in order to provide a required maximum phase lead. A plot of ϕ_m versus α is shown in Fig. 10.5. Clearly, the phase angle readily obtainable from this network is not much greater than 70°. Also, since $\alpha = (R_1 + R_2)/R_2$, there are practical limitations on the maximum value of α that one should attempt to

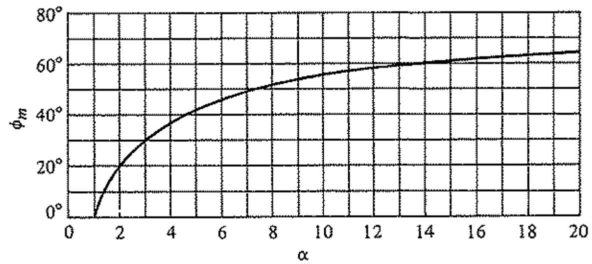


Fig. 10.5. The maximum phase angle ϕ_m versus α for a lead network.

obtain. Therefore, if one required a maximum angle of greater than 70°, two cascade compensation networks would be utilized. Then the equivalent compensation transfer function is $G_{c_1}(s)G_{c_2}(s)$ when the loading effect of $G_{c_2}(s)$ on $G_{c_1}(s)$ is negligible.

It is often useful to add a cascade compensation network which provides a phase-lag characteristic. The *phase-lag network* is shown in Fig. 10.6. The transfer function of the phase-lag network is

$$G_c(s) = \frac{E_2(s)}{E_1(s)} = \frac{R_2 + (1/Cs)}{R_1 + R_2 + (1/Cs)} = \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}. \tag{10.12}$$

When $\tau = R_2C$ and $\alpha = (R_1 + R_2)/R_2$, we have

$$G_c(s) = \frac{1 + \tau s}{1 + \alpha\tau s} = \frac{1}{\alpha}\frac{(s + z)}{(s + p)}, \tag{10.13}$$

where $z = 1/\tau$ and $p = 1/\alpha\tau$. In this case, since $\alpha > 1$, the pole lies closest to the origin of the s-plane as shown in Fig. 10.7. This type of compensation network is often called an integrating network. The Bode diagram of the phase-lag network is obtained from the transfer function

$$G_c(j\omega) = \frac{1 + j\omega\tau}{1 + j\omega\alpha\tau} \tag{10.14}$$

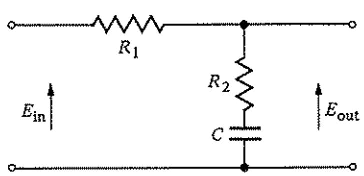


Fig. 10.6. A phase-lag network.

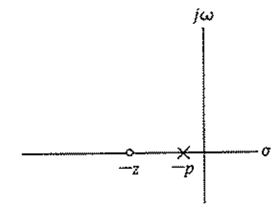


Fig. 10.7. The pole-zero diagram of the phase-lag network.

and is shown in Fig. 10.8. The form of the Bode diagram of the lag network is similar to that of the phase-lead network; the difference is the resulting attenuation and phase-lag angle instead of amplification and phase-lead angle. However, one notes that the shape of the diagrams of Figs. 10.3 and 10.8 are similar. Therefore, it can be shown that the maximum phase lag occurs at $\omega_m = \sqrt{zp}$.

In the succeeding sections, we wish to utilize these compensation networks in order to obtain a desired system frequency locus or s-plane root location. The lead network is utilized to provide a phase-lead angle and thus a satisfactory phase margin for a system. Alternatively, the use of the phase-lead network may be visualized on the s-plane as enabling one to reshape the root locus and thus provide the desired

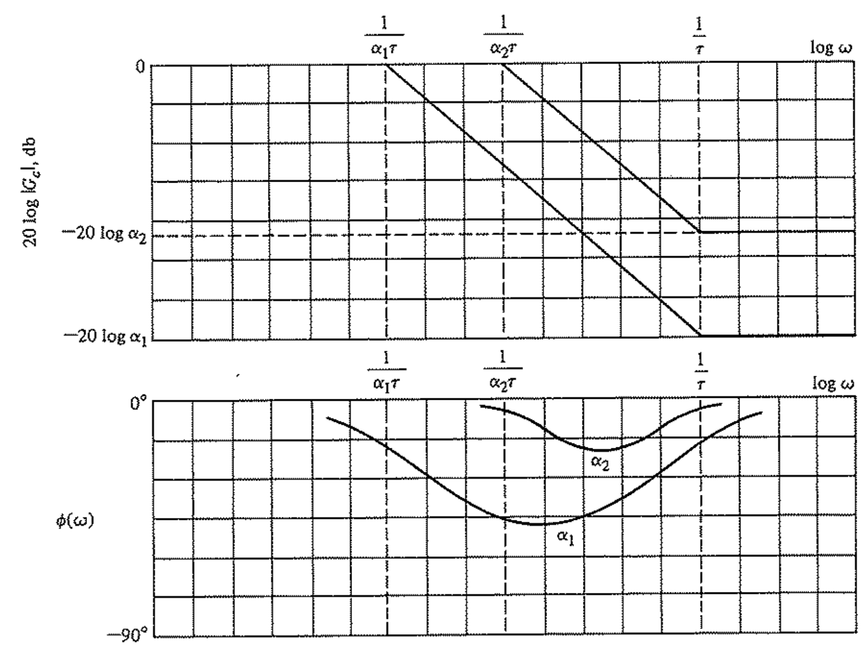


Fig. 10.8. The Bode diagram of the phase-lag network.

root locations. The phase-lag network is utilized not to provide a phase-lag angle, which is normally a destabilizing influence, but rather to provide an attenuation and increase the steady-state error constant [3]. These approaches to compensation utilizing the phase-lead and phase-lag networks will be the subject of the following four sections.

10.4 SYSTEM COMPENSATION ON THE BODE DIAGRAM USING THE PHASE-LEAD NETWORK

The Bode diagram is used in order to design a suitable phase-lead network in preference to other frequency response plots. The frequency response of the cascade compensation network is added to the frequency response of the uncompensated system. That is, since the total loop transfer function of Fig. 10.1(a) is $G_c(j\omega)G(j\omega)H(j\omega)$, we will first plot the Bode diagram for $G(j\omega)H(j\omega)$. Then one may examine the plot for $G(j\omega)H(j\omega)$ and determine a suitable location for p and z of $G_c(j\omega)$ in order to satisfactorily reshape the frequency response. The uncompensated $G(j\omega)$ is plotted with the desired gain to allow an acceptable steady-state error. Then the phase margin and the expected M_p are examined to find whether they satisfy the specifications. If the phase margin is not sufficient, phase lead may be added to the phase angle curve of the system by placing the $G_c(j\omega)$ in a suitable location. Clearly, in order to obtain maximum additional phase lead, we desire to place the network such that the frequency ω_m is located at the frequency where the magnitude of the compensated magnitude curves crosses the 0-db axis. (Recall the definition of phase margin.) The value of the added phase lead required allows us to determine the necessary value for α from Eq. (10.11) or Fig. 10.5. The zero $\omega = 1/\alpha\tau$ is located by noting that the maximum phase lead should occur at $\omega_m = \sqrt{zp}$, halfway between the pole and zero. Since the total magnitude gain for the network is $20 \log \alpha$, we expect a gain of $10 \log \alpha$ at ω_m. Thus we determine the compensation network by completing the following steps:

1. Evaluate the uncompensated system phase margin when the error constants are satisfied.
2. Allowing for a small amount of safety, determine the necessary additional phase lead, ϕ_m.
3. Evaluate α from Eq. (10.11).
4. Evaluate $10 \log \alpha$ and determine the frequency where the uncompensated magnitude curve is equal to $-10 \log \alpha$ db. This frequency is the new 0-db crossover frequency and ω_m simultaneously, since the compensation network provides a gain of $10 \log \alpha$ at ω_m.
5. Draw the compensated frequency response, check the resulting phase margin, and repeat the steps, if necessary. Finally, for an acceptable design, raise the gain of the amplifier in order to account for the attenuation $(1/\alpha)$.

Example 10.1. Let us consider a single-loop feedback control system as shown in Fig. 10.1(a), where

$$G(s) = \frac{K_1}{s^2} \tag{10.15}$$

and $H(s) = 1$. The uncompensated system is a type 2 system and at first appears to possess a satisfactory steady-state error for both step and ramp input signals. However, uncompensated, the response of the system is an undamped oscillation, since

$$T(s) = \frac{C(s)}{R(s)} = \frac{K_1}{s^2 + K_1}. \tag{10.16}$$

Therefore, the compensation network is added so that the loop transfer function is $G_c(s)G(s)H(s)$. The specifications for the system are

Settling time, $T_s \leq 4$ sec,
Percent overshoot for a step input $\leq 20\%$.

Using Fig. 4.8, we estimate that the damping ratio should be $\zeta \geq 0.45$. The settling time requirement is

$$T_s = \frac{4}{\zeta\omega_n} = 4, \tag{10.17}$$

and therefore

$$\omega_n = \frac{1}{\zeta} = \frac{1}{0.45} = 2.22.$$

Perhaps the simplest way to check the value of ω_n for the frequency response is to relate ω_n to the bandwidth and evaluate the bandwidth of the closed-loop system. For a closed-loop system with $\zeta = 0.45$, we estimate from Fig. 7.9 that $\omega_B = 1.36\omega_n$. Therefore, we require a closed-loop bandwidth $\omega_B = 1.36(2.22) = 3.02$. The bandwidth may be checked following compensation by utilizing the Nichols chart. For the uncompensated system, the bandwidth of the system is $\omega_B = 1.36\omega_n$ and $\omega_n = \sqrt{K}$. Therefore, a loop gain equal to $K = \omega_n^2 \approx 5$ would be sufficient. In order to provide a suitable margin for the settling time, we will select $K = 10$ in order to draw the Bode diagram of

$$GH(j\omega) = \frac{K}{(j\omega)^2}.$$

The Bode diagram of the uncompensated system is shown as solid lines in Fig. 10.9.

By using Eq. (8.58), the phase margin of the system is required to be approximately

$$\phi_{pm} = \frac{\zeta}{0.01} = \frac{0.45}{0.01} = 45°. \tag{10.18}$$

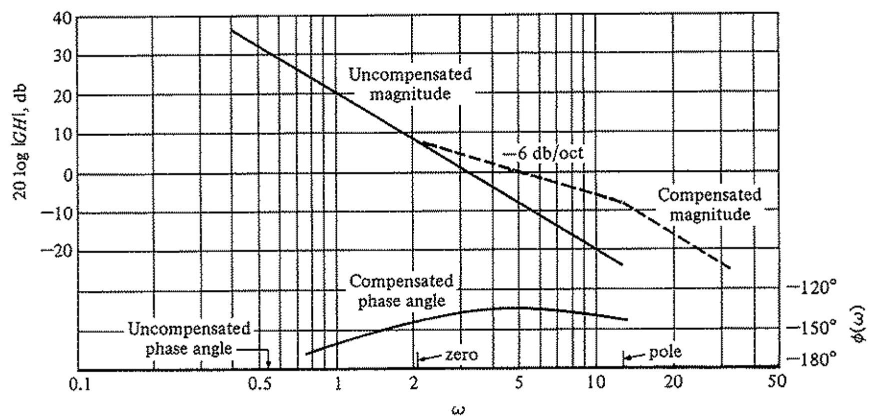


Fig. 10.9. Bode diagram for Example 10.1.

The phase margin of the uncompensated system is zero degrees since the double integration results in a constant 180° phase lag. Therefore we must add a 45° phase-lead angle at the crossover (0-db) frequency of the compensated magnitude curve. Evaluating the value of α, we have

$$\begin{aligned}\frac{\alpha - 1}{\alpha + 1} &= \sin \phi_m \\ &= \sin 45°,\end{aligned} \tag{10.19}$$

and therefore $\alpha = 5.8$. In order to provide a margin of safety, we will use $\alpha = 6$. The value of $10 \log \alpha$ is then equal to 7.78 db. Then the lead network will add an additional gain of 7.78 db at the frequency ω_m, and it is desired to have ω_m equal to the compensated system crossover frequency. This is accomplished by drawing the compensated slope near the 0-db axis (the dotted line) so that the new crossover is ω_m and the dotted magnitude curve is 7.78 db above the uncompensated curve at the crossover frequency. Thus the compensated crossover frequency is located by evaluating the frequency where the uncompensated magnitude curve is equal to −7.78 db, which, in this case, is $\omega = 4.9$. Then the maximum phase-lead angle is added at $\omega = \omega_m = 4.9$ as shown in Fig. 10.9. The bandwidth of the compensated system may be obtained from the Nichols chart. For estimating the bandwidth, one may simply examine Fig. 8.22 and note that the −3-db line for the closed-loop system occurs when the magnitude of $GH(j\omega)$ is −6-db and the phase shift of $GH(j\omega)$ is approximately −140°. Therefore, in order to estimate the bandwidth from the open-loop diagram we will approximate the bandwidth as the frequency for which $20 \log |GH|$ is equal to −6 db. Therefore the bandwidth of the uncompensated system is approximately equal to $\omega_B = 4.4$, while the bandwidth of the compensated system is equal to $\omega_B = 8.4$. The lead compensation doubles the bandwidth in this case and the specification that $\omega_B > 3.02$ is satisfied. Therefore the

compensation of the system is completed and the system specifications are satisfied. The total compensated loop transfer function is

$$G_c(j\omega)G(j\omega)H(j\omega) = \frac{10[(j\omega/2.1) + 1]}{(j\omega)^2[(j\omega/12.6) + 1]}. \tag{10.20}$$

The transfer function of the compensator is

$$\begin{aligned}G_c(s) &= \frac{(1 + \alpha\tau s)}{\alpha(1 + \tau s)} \\ &= \frac{1}{6}\frac{[1 + (s/2.1)]}{[1 + (s/12.6)]}\end{aligned} \tag{10.21}$$

in the form of Eq. (10.8). Since an attenuation of 1/6 results from the passive *RC* network, the gain of the amplifier in the loop must be raised by a factor of six so that the total dc loop gain is still equal to 10 as required in Eq. (10.20). When we add the compensation network Bode diagram to the uncompensated Bode diagram as in Fig. 10.9, we are assuming that we can raise the amplifier gain in order to account for this $1/\alpha$ attenuation. The pole and zero values are simply read from Fig. 10.9, noting that $p = \alpha z$.

Example 10.2. A feedback control system has a loop transfer function

$$GH(s) = \frac{K}{s(s + 2)}. \tag{10.22}$$

It is desired to have a steady-state error for a ramp input less than 5% of the magnitude of the ramp. Therefore, we require that

$$K_v = \frac{A}{e_{ss}} = \frac{A}{0.05A} = 20. \tag{10.23}$$

Furthermore, we desire that the phase margin of the system be at least 45°. The first step is to plot the Bode diagram of the uncompensated transfer function

$$\begin{aligned}GH(j\omega) &= \frac{K_v}{j\omega(0.5j\omega + 1)} \\ &= \frac{20}{j\omega(0.5j\omega + 1)}\end{aligned} \tag{10.24}$$

as shown in Fig. 10.10(a). The frequency at which the magnitude curve crosses the 0-db line is 6.2 rad/sec and the phase margin at this frequency is determined readily from the equation of the phase angle of $GH(j\omega)$ which is

$$\underline{/GH(j\omega)} = \phi(\omega) = -90° - \tan^{-1}(0.5\omega). \tag{10.25}$$

At the crossover frequency, $\omega = \omega_c = 6.2$ rad/sec, we have

$$\phi(\omega) = -162°, \tag{10.26}$$

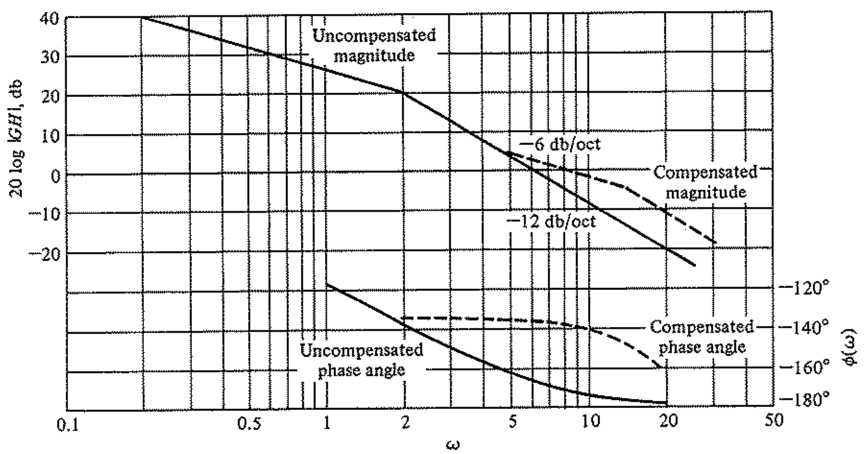


Fig. 10.10(a). Bode diagram for Example 10.2.

and therefore the phase margin is 18°. Using Eq. (10.25) to evaluate the phase margin is often easier than drawing the complete phase angle curve which is shown in Fig. 10.10(a). Thus we need to add a phase-lead network so that the phase margin is raised to 45° at the new crossover (0-db) frequency. Since the compensation crossover frequency is greater than the uncompensated crossover frequency, the phase lag of the uncompensated system is greater also. We shall account for this additional phase lag by attempting to obtain a maximum phase lead of 45° − 18° = 27° plus a small increment (10%) of phase lead to account for the added lag. Thus we will design a compensation network with a maximum phase lead equal to 27° + 3° = 30°. Then, calculating α, we obtain

$$\frac{\alpha - 1}{\alpha + 1} = \sin 30° = 0.5, \tag{10.27}$$

and therefore $\alpha = 3$.

The maximum phase lead occurs at ω_m, and this frequency will be selected so that the new crossover frequency and ω_m coincide. The magnitude of the lead network at ω_m is $10 \log \alpha = 10 \log 3 = 4.8$ db. The compensated crossover frequency is then evaluated where the magnitude of $GH(j\omega)$ is −4.8 db and thus $\omega_m = \omega_c = 8.4$. Drawing the compensated magnitude line so that it intersects the 0-db axis at $\omega = \omega_c = 8.4$, we find that $z = 4.8$ and $p = \alpha z = 14.4$. Therefore the compensation network is

$$G_c(s) = \frac{1}{3}\frac{(1 + s/4.8)}{(1 + s/14.4)}. \tag{10.28}$$

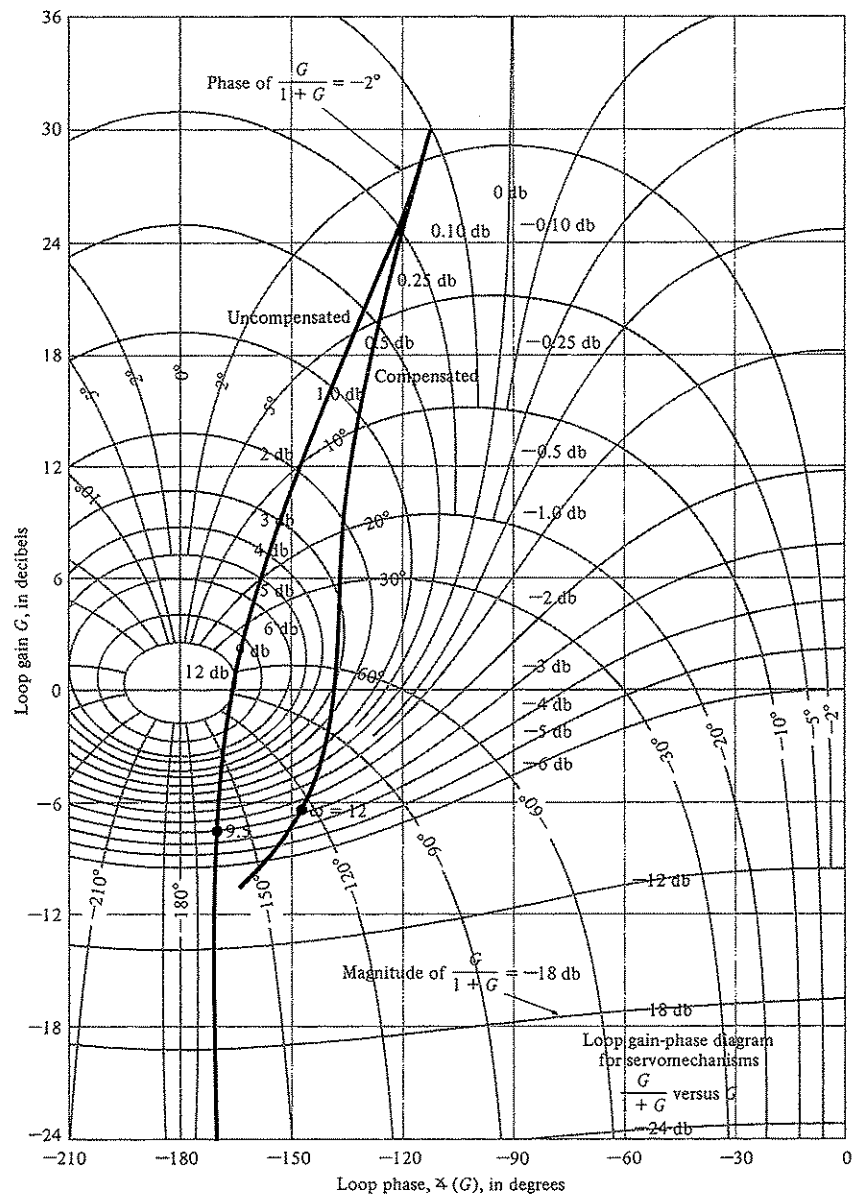


Fig. 10.10(b). Nichols diagram for Example 10.2.

The total dc loop gain must be raised by a factor of 3 in order to account for the factor $1/\alpha = 1/3$. Then the compensated loop transfer function is

$$G_c(s)GH(s) = \frac{20[(s/4.8) + 1]}{s(0.5s + 1)[(s/14.4) + 1]}. \tag{10.29}$$

In order to verify the final phase margin, we may evaluate the phase of $G_c(j\omega)GH(j\omega)$ at $\omega = \omega_c = 8.4$ and therefore obtain the phase margin. The phase angle is then

$$\begin{aligned} \phi(\omega_c) &= -90^\circ - \tan^{-1} 0.5\omega_c - \tan^{-1}\frac{\omega_c}{14.4} + \tan^{-1}\frac{\omega_c}{4.8} \\ &= -90^\circ - 76.5^\circ - 30.0^\circ + 60.2^\circ \\ &= -136.3^\circ. \end{aligned} \tag{10.30}$$

Therefore the phase margin for the compensated system is 43.7°. If we desire to have exactly 45° phase margin, we would repeat the steps with an increased value of α; for example, with $\alpha = 3.5$. In this case, the phase lag increased by 7° between $\omega = 6.2$ and $\omega = 8.4$, and therefore the allowance of 3° in the calculation of α was not sufficient.

The Nichols diagram for the compensated and uncompensated system is shown on Fig. 10.10(b). The reshaping of the frequency response locus is clear on this diagram. One notes the increased phase margin for the compensated system as well as the reduced magnitude of M_{p_ω}, the maximum magnitude of the closed-loop frequency response. In this case, M_{p_ω} has been reduced from an uncompensated value of +12 db to a compensated value of approximately +3.2 db. Also, we note that the closed-loop 3-db bandwidth of the compensated system is equal to 12 rad/sec compared with 9.5 rad/sec for the uncompensated system.

Examining both Examples 10.1 and 10.2 we note that the system design is satisfactory when the asymptotic curve for the magnitude 20 log $|GG_c|$ crosses the 0 db line with a slope of −6 db/octave.

10.5 COMPENSATION ON THE *s*-PLANE USING THE PHASE-LEAD NETWORK

The design of a phase-lead compensation network may also be readily accomplished on the s-plane. The phase-lead network has a transfer function

$$G_c(s) = \frac{[s + (1/\alpha\tau)]}{[s + (1/\tau)]} = \frac{(s + z)}{(s + p)}, \tag{10.31}$$

where α and τ are defined for the *RC* network in Eq. (10.7). The locations of the zero and pole are selected in order to result in a satisfactory root locus for the compensated system. The specifications of the system are used to specify the desired location of the dominant roots of the system. The s-plane root locus method is as follows:

1. List the system specifications and translate these specifications into a desired root location for the dominant roots.
2. Sketch the uncompensated root locus and determine whether the desired root locations can be realized with an uncompensated system.
3. If the compensator is necessary, place the zero of the phase-lead network directly below the desired root location.
4. Determine the pole location so that the total angle at the desired root location is 180° and therefore is on the compensated root locus.
5. Evaluate the total system gain at the desired root location and then calculate the error constant.
6. Repeat the steps if the error constant is not satisfactory.

Therefore, we first locate our desired dominant root locations so that the dominant roots satisfy the specifications in terms of ζ and ω_n as shown in Fig. 10.11(a). The root locus of the uncompensated system is sketched as illustrated in Fig. 10.11(b). Then the zero is added to provide a phase lead of +90° by placing it directly below the desired root location. Actually, some caution must be maintained

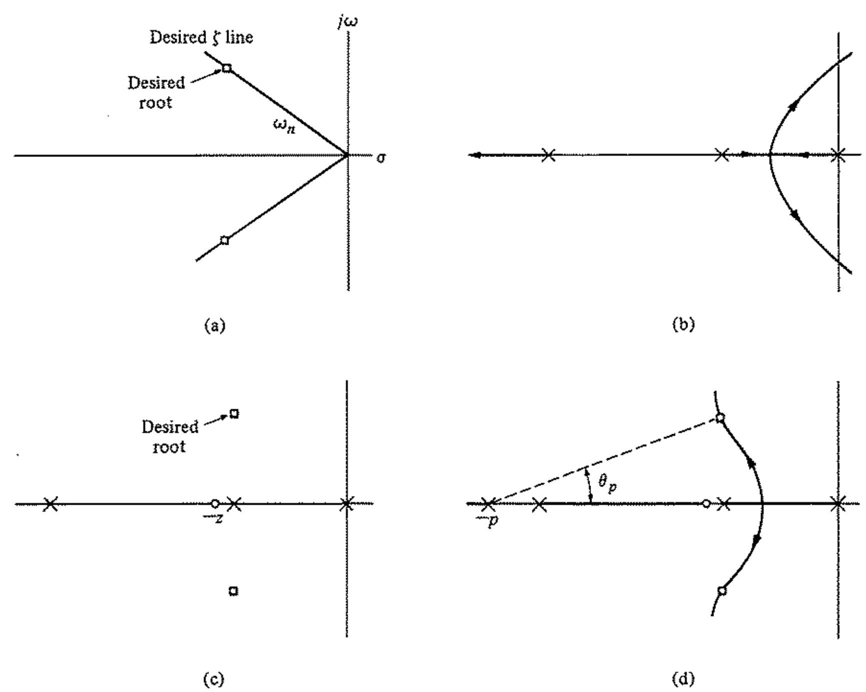


Fig. 10.11. Compensation on the s-plane using a phase-lead network.

since the zero must not alter the dominance of the desired roots; that is, the zero should not be placed nearer the origin than the second pole on the real axis or a real root near the origin will result and will dominate the system response. Thus, in Fig. 10.11(c), we note that the desired root is directly above the second pole, and we place the zero z somewhat to the left of the pole.

Then the real root will be near the real zero, the coefficient of this term of the partial fraction expansion will be relatively small, and thus the response due to this real root will have very little effect on the overall system response. Nevertheless, the designer must be continually aware that the compensated system response will be influenced by the roots and zeros of the system and the dominant roots will not by themselves dictate the response. It is usually wise to allow for some margin of error in the design and to test the compensated system using a digital simulation (e.g., CSMP simulation).

Since the desired root is a point on the root locus when the final compensation is accomplished, we expect the algebraic sum of the vector angles to be 180° at that point. Thus we calculate the angle from the pole of compensator, θ_p, in order to result in a total angle of 180°. Then, locating a line at an angle θ_p intersecting the desired root, we are able to evaluate the compensator pole, p, as shown in Fig. 10.11(d).

The advantage of the s-plane method is the ability of the designer to specify the location of the dominant roots and, therefore, the dominant transient response. The disadvantage of the method is that one cannot directly specify an error constant (for example, K_v) as in the Bode diagram approach. After the design is completed, one evaluates the gain of the system at the root location, which depends upon p and z, and then calculates the error constant for the compensated system. If the error constant is not satisfactory, one must repeat the design steps and alter the location of the desired root as well as the location of the compensator pole and zero. We shall reconsider the two examples we completed in the preceding section and design a compensation network using the root locus (s-plane) approach.

Example 10.3. Let us reconsider the system of Example 10.1 where the open-loop uncompensated transfer function is

$$GH(s) = \frac{K_1}{s^2}. \tag{10.32}$$

The characteristic equation of the uncompensated system is

$$1 + GH(s) = 1 + \frac{K_1}{s^2} = 0, \tag{10.33}$$

and the root locus is the $j\omega$-axis. Therefore we desire to compensate this system with a network, $G_c(s)$, where

$$G_c(s) = \frac{s + z}{s + p} \tag{10.34}$$

and $|z| < |p|$. The specifications for the system are

Settling time, $T_s \leq 4$ sec,
Percent overshoot for a step input $\leq 30\%$.

Therefore the damping ratio should be $\zeta \geq 0.35$. The settling time requirement is

$$T_s = \frac{4}{\zeta\omega_n} = 4,$$

and therefore $\zeta\omega_n = 1$. Thus we will choose a desired dominant root location as

$$r_1, \hat{r}_1 = -1 \pm j2 \tag{10.35}$$

as shown in Fig. 10.12 (thus $\zeta = 0.45$).

Now, we place the zero of the compensator directly below the desired location at $s = -z = -1$ as shown in Fig. 10.12. Then, measuring the angle at the desired root, we have

$$\phi = -2(116°) + 90° = -142°.$$

Therefore, in order to have a total of 180° at the desired root, we evaluate the angle from the undetermined pole, θ_p, as

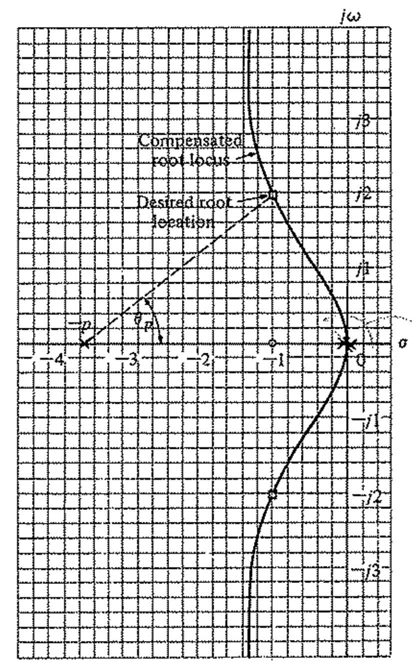


Fig. 10.12. Phase-lead compensation for Example 10.3.

$$-180° = -142° - \theta_p \tag{10.36}$$

or $\theta_p = 38°$. Then a line is drawn at an angle $\theta_p = 38°$ intersecting the desired root location and the real axis as shown in Fig. 10.12. The point of intersection with the real axis is then $s = -p = -3.6$. Therefore, the compensator is

$$G_c(s) = \frac{s+1}{s+3.6}, \tag{10.37}$$

and the compensated transfer function for the system is

$$GH(s)G_c(s) = \frac{K_1(s+1)}{s^2(s+3.6)}. \tag{10.38}$$

The gain K_1 is evaluated by measuring the vector lengths from the poles and zeros to the root location. Hence

$$K_1 = \frac{(2.23)^2(3.25)}{2} = 8.1. \tag{10.39}$$

Finally, the error constants of this system are evaluated. We find that this system with two open-loop integrations will result in a zero steady-state error for a step and ramp input signal. The acceleration constant is

$$K_a = \frac{8.1}{3.6} = 2.25. \tag{10.40}$$

The steady-state performance of this system is quite satisfactory and, therefore, the compensation is complete. When we compare the compensation network evaluated by the s-plane method with the network obtained by using the Bode diagram approach, we find that the magnitudes of the poles and zeros are different. However, the resulting system will have the same performance and we need not be concerned with the difference. In fact, the difference arises from the arbitrary design step (Number 3), which places the zero directly below the desired root location. If we placed the zero at $s = -2.1$, we would find that the pole evaluated by the s-plane method is approximately equal to the pole evaluated by the Bode diagram approach.

The specifications for the transient response of this system were originally expressed in terms of the overshoot and the settling time of the system. These specifications were translated, on the basis of an approximation of the system by a second-order system, to an equivalent ζ and ω_n and therefore a desired root location. However, the original specifications will be satisfied only if the roots selected are dominant. The zero of the compensator and the root resulting from the addition of the compensator pole result in a third-order system with a zero. The validity of approximating this system with a second-order system without a zero is dependent upon the validity of the dominance assumption. Often, the designer will simulate the final design by using an analog computer or a digital computer and obtain the

actual transient response of the system. In this case, an analog computer simulation of the system resulted in an overshoot of 40% and a settling time of 3.8 sec for a step input. These values compare moderately well with the specified values of 30% and 4 sec and justify the utilization of the dominant root specifications. The difference in the overshoot from the specified value is due to the third root which is not negligible. Thus, again we find that the specification of dominant roots is a useful approach, but must be utilized with caution and understanding. A second attempt to obtain a compensated system with an overshoot of 30% would utilize a compensator with a zero at −2 and then calculate the necessary pole location to yield the desired root locations for the dominant roots. This approach would move the third root farther to the left in the s-plane, reduce the effect of the third root on the transient response, and reduce the overshoot.

Example 10.4. Now let us reconsider the system of Example 10.2 and design a compensator based on the s-plane approach. The open-loop system transfer function is

$$GH(s) = \frac{K}{s(s+2)}. \tag{10.41}$$

It is desired that the damping ratio of the dominant roots of the system be $\zeta = 0.45$ and that the velocity error constant be equal to 20. In order to satisfy the error constant requirement, the gain of the uncompensated system must be $K = 40$. When $K = 40$, the roots of the uncompensated system are

$$s^2 + 2s + 40 = (s + 1 + j6.25)(s + 1 - j6.25). \tag{10.42}$$

The damping ratio of the uncompensated roots is approximately 0.16, and therefore a compensation network must be added. In order to achieve a rapid settling time, we will select the real part of the desired roots as $\zeta\omega_n = 4$ and therefore $T_s = 1$ sec. Also, the natural frequency of these roots is fairly large, $\omega_n = 9$; hence the velocity constant should be reasonably large. The location of the desired roots is shown on Fig. 10.13 for $\zeta\omega_n = 4$, $\zeta = 0.45$, and $\omega_n = 9$.

The zero of the compensator is placed at $s = -z = -4$, directly below the desired root location. Then the angle at the desired root location is

$$\phi = -116° - 104° + 90° = -130°. \tag{10.43}$$

Therefore the angle from the undetermined pole is determined from

$$-180° = -130° - \theta_p,$$

and thus $\theta_p = 50°$. This angle is drawn to intersect the desired root location, and p is evaluated as $s = -p = -10.6$ as shown in Fig. 10.13. The gain of the compensated system is then

$$K = \frac{9(8.25)(10.4)}{8} = 96.5 \tag{10.44}$$

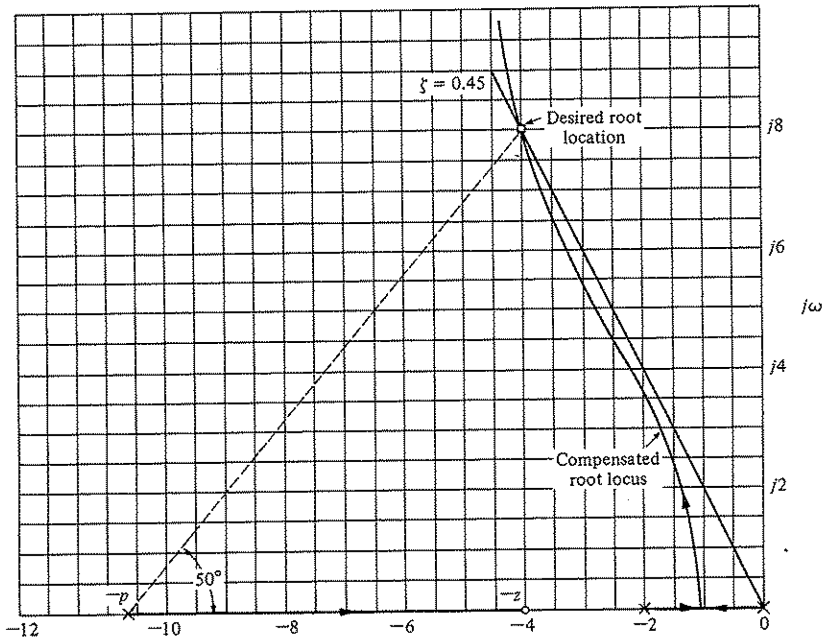


Fig. 10.13. The design of a phase-lead network on the s-plane for Example 10.4.

The compensated system is then

$$G_c(s)GH(s) = \frac{96.5(s+4)}{s(s+2)(s+10.6)}. \tag{10.45}$$

Therefore the velocity constant of the compensated system is

$$K_v = \lim_{s\to 0} s\{G(s)H(s)G_c(s)\} = \frac{96.5(4)}{2(10.6)} = 18.2. \tag{10.46}$$

The velocity constant of the compensated system is less than the desired value of 20. Therefore, one must repeat the design procedure for a second choice of a desired root. If we choose $\omega_n = 10$, the process may be repeated and the resulting gain K will be increased. The compensator pole and zero location will also be altered. Then the velocity constant may be again evaluated. We will leave it as an exercise for the reader to show that for $\omega_n = 10$, the velocity constant is $K_v = 22.7$ when $z = 4.5$ and $p = 11.6$.

Finally, for the compensation network of Eq. (10.45), we have

$$G_c(s) = \frac{s+4}{s+10.6} = \frac{(s+1/\alpha\tau)}{(s+1/\tau)}. \tag{10.47}$$

The design of an RC-lead network as shown in Fig. 10.4 follows directly from Eqs. (10.47) and (10.7), and is

$$G_c(s) = \left(\frac{R_2}{R_1+R_2}\right)\frac{(R_1Cs+1)}{(R_1R_2/(R_1+R_2))Cs+1}. \tag{10.48}$$

Thus, in this case, we have

$$\frac{1}{R_1C} = 4$$

and

$$\alpha = \frac{R_1+R_2}{R_2} = \frac{10.6}{4}.$$

Then, choosing $C = 1\ \mu$f, we obtain $R_1 = 250{,}000$ ohms and $R_2 = 152{,}000$ ohms.

The phase-lead compensation network is a useful compensator for altering the performance of a control system. The phase-lead network adds a phase-lead angle in order to provide adequate phase margin for feedback systems. Using an s-plane design approach, the phase-lead network may be chosen in order to alter the system root locus and place the roots of the system in a desired position in the s-plane. When the design specifications include an error constant requirement, the Bode diagram method is more suitable, since the error constant of a system designed on the s-plane must be ascertained following the choice of a compensator pole and zero. Therefore, the root locus method often results in an iterative design procedure when the error constant is specified. On the other hand, the root locus is a very satisfactory approach when the specifications are given in terms of overshoot and settling time, thus specifying the ζ and ω_n of the desired dominant roots in the s-plane. The use of a lead network compensator always extends the bandwidth of a feedback system which may be objectionable for systems subjected to large amounts of noise. Also, lead networks are not suitable for providing high steady-state accuracy systems requiring very high error constants. In order to provide large error constants, typically K_p and K_v, one must consider the use of integration-type compensation networks, and, therefore, this will be the subject of concern in the following section.

10.6 SYSTEM COMPENSATION USING INTEGRATION NETWORKS

For a large percentage of control systems, the primary objective is to obtain a high steady-state accuracy. Furthermore, it is desired to maintain the transient performance of these systems within reasonable limits. As we found in Chapters 3 and 4, the steady-state accuracy of many feedback systems may be increased by increasing the amplifier gain in the forward channel. However, the resulting transient response may be totally unacceptable, if not even unstable. Therefore it is often necessary to introduce a compensation network in the forward path of a feedback control system in order to provide a sufficient steady-state accuracy.

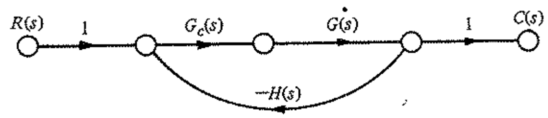


Fig. 10.14. A single-loop feedback control system.

Consider the single-loop control system shown in Fig. 10.14. The compensation network is to be chosen in order to provide a large error constant. The steady-state error of this system is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\left[\frac{R(s)}{1 + G_c(s)G(s)H(s)}\right]. \tag{10.49}$$

We found in Section 3.5 that the steady-state error of a system depends upon the number of poles at the origin for $G_c(s)G(s)H(s)$. A pole at the origin may be considered an integration and therefore the steady-state accuracy of a system ultimately depends upon the number of integrations in the transfer function $G_c(s)G(s)H(s)$. If the steady-state accuracy is not sufficient, we will introduce an integration-type network $G_c(s)$ in order to compensate for the lack of integration in the original transfer function $G(s)H(s)$.

One form of controller available and widely used in industrial process control is called a *three-mode controller* or *process controller*. This controller has a transfer function

$$\frac{U(s)}{E(s)} = G_c(s) = K_p + \frac{K_1}{s} + K_D s. \tag{10.50}$$

The controller provides a proportional term, an integration term, and a derivative term. The equation for the output in the time domain is

$$u(t) = K_p e(t) + K_1 \int e(t)\,dt + K_D \frac{de(t)}{dt}. \tag{10.51}$$

The three mode controller is also called a PID controller since it contains a proportional, an integration, and a derivative term. The transfer function of the derivative term is actually

$$G_d(s) = \frac{K_D s}{\tau_d s + 1}, \tag{10.52}$$

but usually τ_d is much smaller than the time constants of the process itself and may be neglected.

For an example, let us consider a temperature control system where the transfer function of the heat process is $G(s) = K_1/(\tau_1 s + 1)$ and the measurement transfer function is

$$H(s) = \frac{1}{\tau_2 s + 1}.$$

The steady-state error of the uncompensated system is then

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\left\{\frac{A/s}{1 + G(s)H(s)}\right\} = \frac{A}{1 + K_1}, \tag{10.53}$$

where $R(s) = A/s$, a step input signal. Clearly, in order to obtain a small steady-state error (less than 0.05 A, for example), the magnitude of the gain K_1 must be quite large. However, when K_1 is quite large, the transient performance of the system will very likely be unacceptable. Therefore, we must consider the addition of a compensation transfer function $G_c(s)$ as shown in Fig. 10.14. In order to eliminate the steady-state error of this system, we might choose the compensation as

$$G_c(s) = K_2 + \frac{K_3}{s} = \frac{K_2 s + K_3}{s}. \tag{10.54}$$

This compensation may be readily constructed by using an integrator and an amplifier and adding their output signals. Now, the steady-state error for a step input of the system is always zero, since

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} s\frac{A/s}{1 + G_c(s)GH(s)}\\ &= \lim_{s\to 0}\frac{A}{1 + [(K_2 s + K_3)/s]\{K_1/[(\tau_1 s + 1)(\tau_2 s + 1)]\}}\\ &= 0.\end{aligned} \tag{10.55}$$

The transient performance can be adjusted to satisfy the system specifications by adjusting the constants K_1, K_2, and K_3. The adjustment of the transient response is perhaps best accomplished by using the root locus methods of Chapter 6 and drawing a root locus for the gain K_2K_1 after locating the zero $s = -K_3/K_2$ on the s-plane by the method outlined for the s-plane in the preceding section.

The addition of an integration as $G_c(s) = K_2 + (K_3/s)$ may also be used to reduce the steady-state error for a ramp input, $r(t) = t$, $t \geq 0$. For example, if the uncompensated system $GH(s)$ possessed one integration, the additional integration due to $G_c(s)$ would result in a zero steady-state error for a ramp input. In order to illustrate the design of this type of integration compensation, we will consider a temperature control system in some detail.

Example 10.5. The uncompensated loop transfer function of a temperature control system is

$$GH(s) = \frac{K_1}{(2s + 1)(0.5s + 1)}, \tag{10.56}$$

where K_1 may be adjusted. In order to maintain zero steady-state error for a step input, we will add the compensation network

$$\begin{aligned} G_c(s) &= K_2 + \frac{K_3}{s} \\ &= K_2\left(\frac{s + K_3/K_2}{s}\right). \end{aligned} \tag{10.57}$$

Furthermore, the transient response of the system is required to have an overshoot less than or equal to 10%. Therefore, the dominant complex roots must be on (or below) the $\zeta = 0.6$ line as shown in Fig. 10.15. We will adjust the compensator zero so that the real part of the complex roots is $\zeta\omega_n = 0.75$ and thus the settling time is $T_s = 4/\zeta\omega_n = 16/3$ sec. Now, as in the preceding section, we will determine the location of the zero, $z = -K_3/K_2$, by assuring that the angle at the desired root is $-180°$. Therefore, the sum of the angles at the desired root is

$$-180° = -127° - 104° - 38° + \theta_z,$$

where θ_z is the angle from the undetermined zero. Therefore, we find that $\theta_z = +89°$ and the location of the zero is $z = -0.75$. Finally, in order to determine the gain at the desired root, we evaluate the vector lengths from the poles and zeros and obtain

$$K = K_1K_2 = \frac{1.25(1.06)1.6}{.95} = 2.23.$$

The compensated root locus and the location of the zero are shown in Fig. 10.15. It should be noted that the zero, $z = -K_3/K_2$, should be placed to the left of the pole at $s = -0.5$ in order to ensure that the complex roots dominate the transient response. In fact, the third root of the compensated system of Fig. 10.15 may be determined as $s = -1.0$, and therefore this real root is only $4/3$ times the real part of the complex roots. Thus, while complex roots dominate the response of the system,

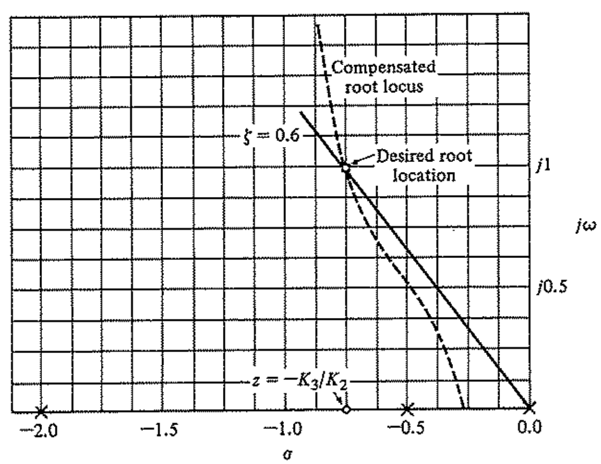


Fig. 10.15. The s-plane design of an integration compensator.

the equivalent damping of the system is somewhat less than $\zeta = 0.60$ due to the real root and zero.

10.7 COMPENSATION ON THE s-PLANE USING A PHASE-LAG NETWORK

The phase-lag RC network of Fig. 10.6 is an integration-type network and may be used to increase the error constant of a feedback control system. We found in Section 10.3 that the transfer function of the RC phase-lag network is of the form

$$G_c(s) = \frac{1}{\alpha}\frac{(s+z)}{(s+p)}, \tag{10.58}$$

as given in Eq. (10.13), where

$$z = \frac{1}{\tau} = \frac{1}{R_2C}, \qquad \alpha = \frac{R_1 + R_2}{R_2}, \qquad p = \frac{1}{\alpha\tau}.$$

The steady-state error of an uncompensated system is

$$\lim_{t\to 0} e(t) = \lim_{s\to 0} s\left\{\frac{R(s)}{1 + GH(s)}\right\}. \tag{10.59}$$

Then, for example, the velocity constant of a type-one system is

$$K_v = \lim_{s\to 0} s\{GH(s)\} \tag{10.60}$$

as shown in Section 4.4. Therefore, if $GH(s)$ is written as

$$GH(s) = \frac{K\prod_{i=1}^{M}(s + z_i)}{s\prod_{j=1}^{Q}(s + p_j)}, \tag{10.61}$$

we obtain the velocity constant

$$K_v = \frac{K\prod_{i=1}^{M} z_i}{\prod_{j=1}^{Q} p_j}. \tag{10.62}$$

We will now add the integration type phase-lag network as a compensator and determine the compensated velocity constant. If the velocity constant of the uncompensated system (Eq. 10.62) is designated as $K_{v_{\text{uncomp}}}$, we have

$$\begin{aligned} K_{v_{\text{comp}}} &= \lim_{s\to 0} s\{G_c(s)GH(s)\} = \lim_{s\to 0}\,(G_c(s))\,K_{v_{\text{uncomp}}} \\ &= \left(\frac{z}{p}\right)\left(\frac{1}{\alpha}\right)K_{v_{\text{uncomp}}} = \left(\frac{z}{p}\right)\left(\frac{K}{\alpha}\right)\left(\frac{\prod z_i}{\prod p_j}\right). \end{aligned} \tag{10.63}$$

The gain on the compensated root locus at the desired root location will be (K/α). Now, if the pole and zero of the compensator are chosen so that $|z| = \alpha|p| < 1$, the resultant K_v will be increased at the desired root location by the ratio $z/p = \alpha$. Then, for example, if $z = 0.1$ and $p = 0.01$, the velocity constant of the desired root location will be increased by a factor of 10. However, if the compensator pole and zero appear relatively close together on the s-plane, their effect on the location of the desired root will be negligible. Therefore the compensator pole-zero combination near the origin may be used to increase the error constant of a feedback system by the factor α while altering the root location very slightly. The factor α does have an upper limit, typically about 100, since the required resistors and capacitors of the network become excessively large for a higher α. For example, when $z = 0.1$ and $\alpha = 100$, we find from Eq. (10.58) that

$$z = 0.1 = \frac{1}{R_2 C}$$

and

$$\alpha = 100 = \frac{R_1 + R_2}{R_2}.$$

If we let $C = 10\ \mu f$, then $R_2 = 1$ megohm and $R_1 = 99$ megohms. As we increase α, we increase the magnitude of R_1 required. However, we should note that an attenuation, α, of 1000 or more may be obtained by utilizing pneumatic process controllers which approximate a phase-lag characteristic (Fig. 10.8).

The steps necessary for the design of a phase-lag network on the s-plane are as follows:

1. Obtain the root locus of the uncompensated system.
2. Determine the transient performance specifications for the system and locate suitable dominant root locations on the uncompensated root locus that will satisfy the specifications.
3. Calculate the loop gain at the desired root location and, thus, the system error constant.
4. Compare the uncompensated error constant with the desired error constant and calculate the necessary increase that must result from the pole-zero ratio of the compensator, α.
5. With the known ratio of the pole-zero combination of the compensator, determine a suitable location of the pole and zero of the compensator so that the compensated root locus will still pass through the desired root location.

The fifth requirement can be satisfied if the magnitude of the pole and zero is less than one and they appear to merge as measured from the desired root location. The pole and zero will appear to merge at the root location if the angles from the compensator pole and zero are essentially equal as measured to the root location. One method of locating the zero and pole of the compensator is based on the

requirement that the difference between the angle of the pole and the angle of the zero as measured at the desired root is less than 2°. An example will illustrate this approach to the design of a phase-lag compensator.

Example 10.6. Consider the uncompensated system of Example 10.2, where the uncompensated open-loop transfer function is

$$GH(s) = \frac{K}{s(s+2)}. \tag{10.64}$$

It is required that the damping ratio of the dominant complex roots is 0.45, while a system velocity constant equal to 20 is attained. The uncompensated root locus is a vertical line at $s = -1$ and results in a root on the $\zeta = 0.45$ line at $s = -1 \pm j2$ as shown in Fig. 10.16. Measuring the gain at this root, we have $K = (2.24)^2 = 5$. Therefore the velocity constant of the uncompensated system is

$$K_v = \frac{K}{2} = \frac{5}{2} = 2.5.$$

Thus the ratio of the zero to the pole of the compensator is

$$\left|\frac{z}{p}\right| = \alpha = \frac{K_{v_{\text{comp}}}}{K_{v_{\text{uncomp}}}} = \frac{20}{2.5} = 8. \tag{10.65}$$

Examining Fig. 10.17, we find that we might set $z = -0.1$ and then $p = -0.1/8$. The difference of the angles from p and z at the desired root is approximately one degree, and therefore, $s = -1 \pm j2$ is still the location of the dominant roots. A

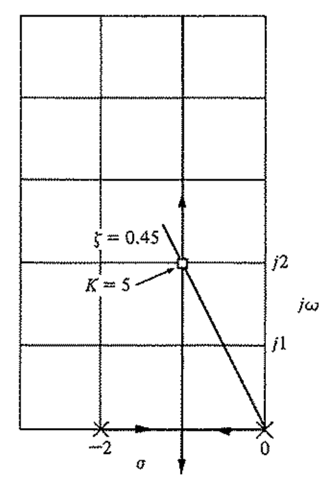


Fig. 10.16. Root locus of the uncompensated system of Example 10.6.

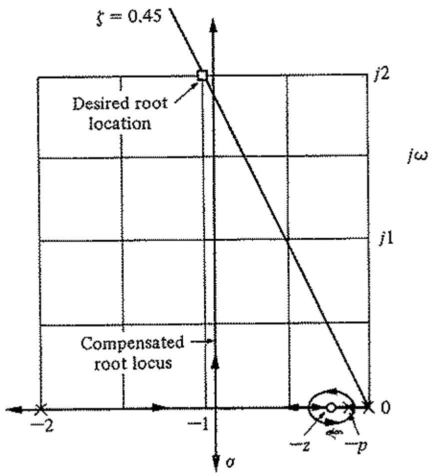


Fig. 10.17. Root locus of the compensated system of Example 10.6. Note that the actual root will differ from the desired root by a slight amount. The vertical portion of the locus leaves the σ axis at $\sigma = -0.95$.

sketch of the compensated root locus is shown as a heavy line in Fig. 10.17. Therefore the compensated system transfer function is

$$G_c(s)GH(s) = \frac{5(s + 0.1)}{s(s + 2)(s + 0.0125)}, \tag{10.66}$$

where $(K/\alpha) = 5$ or $K = 40$ in order to account for the attenuation of the lag network.

Example 10.7. Let us now consider a system which is difficult to compensate by a phase-lead network. The open-loop transfer function of the uncompensated system is

$$GH(s) = \frac{K}{s(s + 10)^2}. \tag{10.67}$$

It is specified that the velocity constant of this system be equal to 20, while the damping ratio of the dominant roots be equal to 0.707. The gain necessary for a K_v of 20 is

$$K_v = 20 = \frac{K}{(10)^2}$$

or $K = 2000$. However, using Routh's criterion, we find that the roots of the characteristic equation lie on the $j\omega$-axis at $\pm j10$ when $K = 2000$. Clearly, the roots of

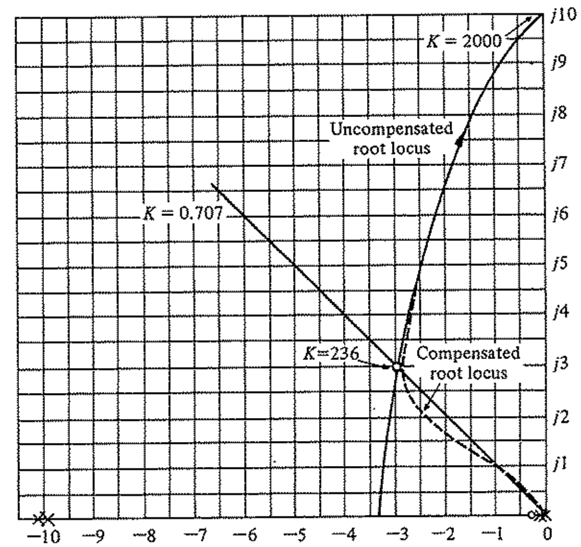


Fig. 10.18. Design of a phase-lag compensator on the s-plane.

the system when the K_v-requirement is satisfied are a long way from satisfying the damping ratio specification, and it would be dificult to bring the dominant roots from the $j\omega$-axis to the $\zeta = 0.707$ line by using a phase-lead compensator. Therefore, we will attempt to satisfy the K_v- and ζ-requirements by using a phase-lag network. The uncompensated root locus of this system is shown in Fig. 10.18 and the roots are shown when $\zeta = 0.707$ and $s = -2.9 \pm j2.9$. Measuring the gain at these roots, we find that $K = 236$. Therefore, the necessary ratio of zero to pole of the compensator is

$$\alpha = \left|\frac{z}{p}\right| = \frac{2000}{236} = 8.5.$$

Therefore we will choose $z = 0.1$ and $p = 0.1/9$ in order to allow a small margin of safety. Examining Fig. 10.18, we find that the difference between the angle from the pole and zero of $G_c(s)$ is negligible. Therefore the compensated system is

$$G_c(s)GH(s) = \frac{236(s + 0.1)}{s(s + 10)^2(s + 0.0111)}, \tag{10.68}$$

where $(K/\alpha) = 236$ and $\alpha = 9$.

The design of an integration compensator in order to increase the error constant of an uncompensated control system is particularly illustrative using s-plane and root locus methods. We shall now turn to similarly useful methods of designing integration compensation using Bode diagrams.

10.8 COMPENSATION ON THE BODE DIAGRAM USING A PHASE-LAG NETWORK

The design of a phase-lag *RC* network suitable for compensating a feedback control system may be readily accomplished on the Bode diagram. The advantage of the Bode diagram is again apparent for we will simply add the frequency response of the compensator to the Bode diagram of the uncompensated system in order to obtain a satisfactory system frequency response. The transfer function of the phase-lag network written in Bode diagram form is

$$G_c(j\omega) = \frac{1 + j\omega\tau}{1 + j\omega\alpha\tau} \tag{10.69}$$

as we found in Eq. (10.14). The Bode diagram of the phase-lag network is shown in Fig. 10.8 for two values of α. On the Bode diagram, the pole and zero of the compensator have a magnitude much smaller than the smallest pole of the uncompensated system. Thus the phase lag is not the useful effect of the compensator, but rather it is the attenuation $-20 \log \alpha$ which is the useful effect for compensation. The phase-lag network is used to provide an attenuation and, therefore, to lower the 0-db (crossover) frequency of the system. However, at lower crossover frequencies, we usually find that the phase margin of the system is increased and our specifications may be satisfied. The design procedure for a phase-lag network on the Bode diagram is as follows:

1. Draw the Bode diagram of the uncompensated system with the gain adjusted for the desired error constant.
2. Determine the phase margin of the uncompensated system and, if it is insufficient, proceed with the following steps.
3. Determine the frequency where the phase margin requirement would be satisfied if the magnitude curve crossed the 0-db line at this frequency, ω_c'. (Allow for 5° phase lag from the phase-lag network when determining the new crossover frequency.)
4. Place the zero of the compensator one decade below the new crossover frequency ω_c' and thus ensure only 5° of lag at ω_c' (see Fig. 10.8).
5. Measure the necessary attenuation at ω_c' in order to ensure that the magnitude curve crosses at this frequency.
6. Calculate α by noting that the attenuation is $-20 \log \alpha$.
7. Calculate the pole as $\omega_p = 1/\alpha\tau = \omega_z/\alpha$ and the design is completed.

An example of this design procedure will illustrate that the method is simple to carry out in practice.

Example 10.8. Let us reconsider the system of Example 10.6 and design a phase-lag network so that the desired phase margin is obtained. The uncompensated transfer function is

$$GH(j\omega) = \frac{K}{j\omega(j\omega + 2)} = \frac{K_v}{j\omega(0.5j\omega + 1)}, \tag{10.70}$$

where $K_v = K/2$. It is desired that $K_v = 20$ while a phase margin of 45° is attained. The uncompensated Bode diagram is shown as a solid line in Fig. 10.19. The uncompensated system has a phase margin of 20°, and the phase margin must be increased. Allowing 5° for the phase-lag compensator, we locate the frequency ω where $\phi(\omega) = -130°$, which is to be our new crossover frequency ω_c'. In this case, we find that $\omega_c' = 1.5$, which allows for a small margin of safety. The attenuation necessary to cause ω_c' to be the new crossover frequency is equal to 20 db, accounting for a 2-db difference between the actual and asymptotic curves. Then we find that 20 db $= 20 \log \alpha$, or $\alpha = 10$. Therefore the zero is one decade below the crossover, or $\omega_z = \omega_c'/10 = 0.15$, and the pole is at $\omega_p = \omega_z/10 = 0.015$. The compensated system is then

$$G_c(j\omega)GH(j\omega) = \frac{20(6.66j\omega + 1)}{j\omega(0.5j\omega + 1)(66.6j\omega + 1)}. \tag{10.71}$$

The frequency response of the compensated system is shown in Fig. 10.19 with dotted lines. It is evident that the phase lag introduces an attenuation which lowers the crossover frequency and, therefore, increases the phase margin. Note that the phase angle of the lag network has almost totally disappeared at the crossover frequency ω_c'. As a final check, we numerically evaluate the phase margin at $\omega_c' = 1.5$ and find that $\phi_{pm} = 45°$, which is the desired result. Using the Nichols chart, we

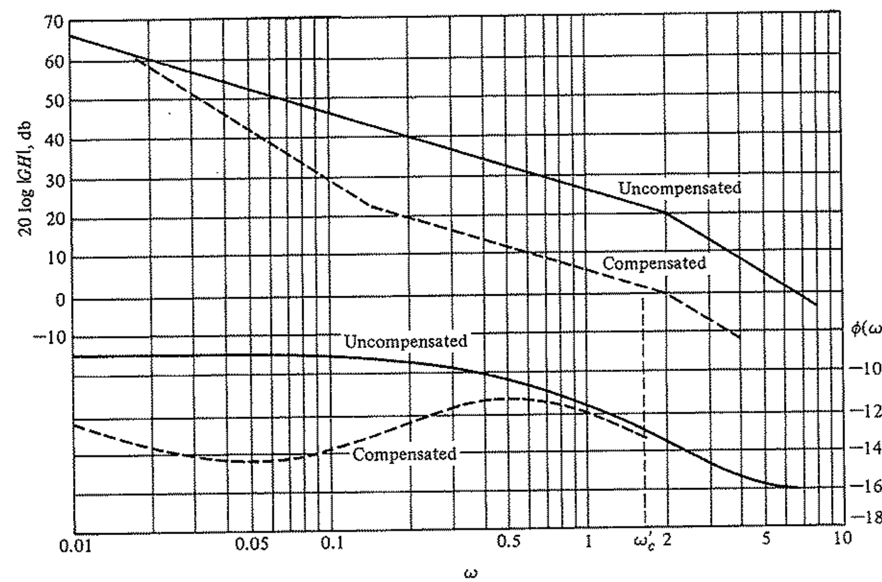


Fig. 10.19. Design of a phase-lag network on the Bode diagram for Example 10.8.

find that the closed-loop bandwidth of the system has been reduced from $\omega = 10$ rad/sec for the uncompensated system to $\omega = 2.5$ rad/sec for the compensated system.

Example 10.9. Let us reconsider the system of Example 10.7 which is

$$GH(j\omega) = \frac{K}{j\omega(j\omega + 10)^2} = \frac{K_v}{j\omega(0.1j\omega + 1)^2} \tag{10.72}$$

where $K_v = K/100$. A velocity constant of K_v equal to 20 is specified. Furthermore, a damping ratio of 0.707 for the dominant roots is required. From Fig. 8.18, we estimate that a phase margin of 65° is required. The frequency response of the uncompensated system is shown in Fig. 10.20. The phase margin of the uncompensated system is zero degrees. Allowing 5° for the lag network, we locate the frequency where the phase is −110°. This frequency is equal to 1.74, and therefore we shall attempt to locate the new crossover frequency at $\omega_c' = 1.5$. Measuring the necessary attenuation at $\omega = \omega_c'$, we find that 23 db is required; $23 = 20 \log \alpha$, or $\alpha = 14.2$. The zero of the compensator is located one decade below the crossover frequency and thus

$$\omega_z = \frac{\omega_c'}{10} = 0.15.$$

The pole is then

$$\omega_p = \frac{\omega_z}{\alpha} = \frac{0.15}{14.2}.$$

Therefore the compensated system is

$$G_c(j\omega)GH(j\omega) = \frac{20(6.66j\omega + 1)}{j\omega(0.1j\omega + 1)^2(94.6j\omega + 1)}. \tag{10.73}$$

The compensated frequency response is shown in Fig. 10.20. As a final check, we numerically evaluate the phase margin at $\omega_c' = 1.5$ and find that $\phi_{pm} = 67°$, which is within the specifications.

Therefore, a phase-lag compensation network may be used to alter the frequency response of a feedback control system in order to attain satisfactory system performance. Examining both Examples 10.8 and 10.9, we note again that the system design is satisfactory when the asymptotic curve for the magnitude of the compensated system crosses the 0 db line with a slope of −6 db/octave. The attenuation of the phase-lag network reduces the magnitude of the crossover (0-db) frequency to a point where the phase margin of the system is satisfactory. Thus, in contrast to the phase-lead network, the phase-lag network reduces the closed-loop bandwidth of the system as it maintains a suitable error constant.

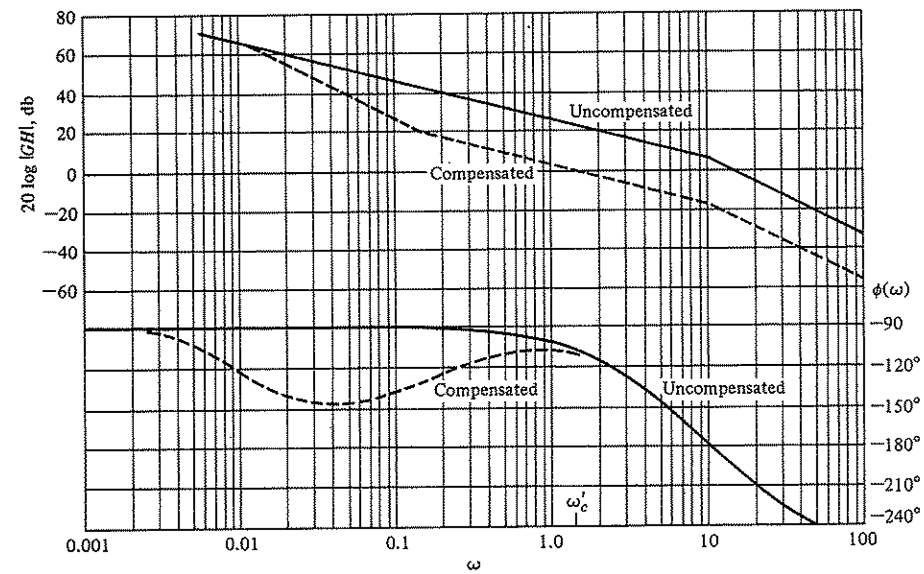


Fig. 10.20. Design of a phase-lag network on the Bode diagram for Example 10.9.

One might ask, why do we not place the compensator zero more than one decade below the new crossover ω_c' (see item 4 of the design procedure) and thus ensure less than 5° of lag at ω_c' due to the compensator? This question may be answered by considering the requirements placed on the resistors and capacitors of the lag network by the values of the poles and zeros (see Eq. 10.12). As the magnitudes of the pole and zero of the lag network are decreased, the magnitudes of the resistors and the capacitor required increase proportionately. The zero of the lag compensator in terms of the circuit components is $z = 1/R_2C$, and the α of the network is $\alpha = (R_1 + R_2)/R_2$. Thus, considering the preceding example, 10.9, we require a zero at $z = 0.15$ which can be obtained with $C = 1\ \mu$f and $R_2 = 6.66$ megohms. However, for $\alpha = 14$, we require a resistance R_1 of $R_1 = R_2(\alpha - 1) = 88$ megohms. Clearly, a designer does not wish to place the zero z further than one decade below ω_c' and thus require larger values of R_1, R_2, and C.

The phase-lead compensation network alters the frequency response of a network by adding a positive (leading) phase angle and, therefore, increases the phase margin at the crossover (0-db) frequency. It becomes evident that a designer might wish to consider using a compensation network which provided the attenuation of a phase-lag network and the lead-phase angle of a phase-lead network. Such a network exists and is called a lead-lag network and is shown in Fig. 10.21. The transfer function of this network is

$$\frac{E_2(s)}{E_1(s)} = \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2 + R_2C_2)s + 1}. \tag{10.74}$$

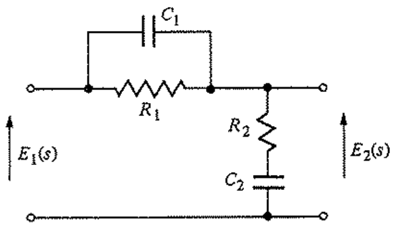


Fig. 10.21. An *RC* lead-lag network.

When $\alpha\tau_1 = R_1C_1$, $\beta\tau_2 = R_2C_2$, $\tau_1\tau_2 = R_1R_2C_1C_2$, we note that $\alpha\beta = 1$ and then Eq. (10.74) is

$$\frac{E_2(s)}{E_1(s)} = \frac{(1 + \alpha\tau_1 s)(1 + \beta\tau_2 s)}{(1 + \tau_1 s)(1 + \tau_2 s)}, \tag{10.75}$$

where $\alpha > 1$, $\beta < 1$. The first terms in the numerator and denominator, which are a function of τ_1, provide the phase-lead portion of the network. The second terms which are a function of τ_2, provide the phase-lag portion of the compensation network. The parameter β is adjusted to provide suitable attenuation of the low frequency portion of the frequency response, and the parameter α is adjusted to provide an additional phase lead at the new crossover (0-db) frequency. Alternatively, the compensation may be designed on the s-plane by placing the lead pole and zero compensation in order to locate the dominant roots in a desired location. Then the phase-lag compensation is used to raise the error constant at the dominant root location by a suitable ratio, $1/\beta$. The design of a phase lead-lag compensator follows the procedures already discussed, and the reader is referred to further literature illustrating the utility of lead-lag compensation [2, 3].

10.9. COMPENSATION ON THE BODE DIAGRAM USING ANALYTICAL AND COMPUTER METHODS

It is desirable to use computers, when appropriate, to assist the designer in the selection of the parameters of a compensator. The development of algorithms for computer-aided design is an important alternative approach to the trial-and-error methods considered in earlier sections. By the use of compensators, computer programs have been developed for the selection of suitable parameter values based on satisfaction of frequency response criteria such as phase margin [16, 17].

An analytical technique of selecting the parameters of a lead or lag network has been developed for Bode diagrams [18, 19]. For a single-stage compensator

$$G_c(s) = \frac{1 + \alpha\tau s}{1 + \tau s}, \tag{10.76}$$

where $\alpha < 1$ yields a lag compensator and $\alpha > 1$ yields a lead compensator. The phase contribution of the compensator at the desired crossover frequency ω_c (see Eq. 10.9) is

$$p = \tan\phi = \frac{\alpha\omega_c\tau - \omega_c\tau}{1 + (\omega_c\tau)^2\alpha}. \tag{10.77}$$

The magnitude M (in db) of the compensator at ω_c is

$$c = 10^{M/10} = \frac{1 + (\omega_c\alpha\tau)^2}{1 + (\omega_c\tau)^2}. \tag{10.78}$$

Eliminating $\omega_c\tau$ from Eqs. 10.77 and 10.78, we obtain the nontrivial solution equation for α as

$$(p^2 - c + 1)\alpha^2 + 2p^2c\alpha + p^2c^2 + c^2 - c = 0. \tag{10.79}$$

For a single-stage compensator, it is necessary that $c > p^2 + 1$. If we solve for α from Eq. 10.79, we can obtain τ from

$$\tau = \frac{\alpha}{\omega_c}\sqrt{\frac{1 - c}{c - \alpha^2}}. \tag{10.80}$$

The design steps for a lead compensator are:

1. Select the desired ω_c.
2. Determine the phase margin desired and therefore the required phase ϕ for Eq. 10.77.
3. Verify that the phase lead is applicable, $\phi > 0$ and $M > 0$.
4. Determine whether a single stage will be sufficient when $c > p^2 + 1$.
5. Determine α from Eq. 10.79.
6. Determine τ from Eq. 10.80.

If one needs to design a single lag compensator, then $\phi < 0$ and $M < 0$ (step 3). Also, step 4 will require $c < [1/(1 + p^2)]$. Otherwise the method is the same.

Example 10.10. Let us reconsider the system of Example 10.1 and design a lead network by the analytical technique. Examine the uncompensated curves in Fig. 10.9. We select $\omega_c = 5$. Then, as before, we desire a phase margin of 45°. The compensator must yield this phase, so

$$p = \tan 45° = 1. \tag{10.81}$$

The required magnitude contribution is 8 db or $M = 8$, so that

$$c = 10^{8/10} = 6.31. \tag{10.82}$$

Using c and p we obtain

$$-4.31\alpha^2 + 12.62\alpha + 73.32 = 0. \tag{10.83}$$

Solving for α we obtain $\alpha = 5.84$. Solving Eq. (10.80), we obtain $\tau = 0.510$. Therefore the compensator is

$$G_c(s) = \frac{1 + 2.98s}{1 + 0.51s}. \tag{10.84}$$

The pole is equal to 1.96 and the zero is 0.336. This design is similar to that obtained by the iteration technique of Section 10.4.

10.10 THE DESIGN OF CONTROL SYSTEMS IN THE TIME-DOMAIN

The design of automatic control systems is an important function of control engineering. The purpose of design is to realize a system with practical components which will provide the desired operating performance. The desired performance can be readily stated in terms of time-domain performance indices. For example, the maximum overshoot and rise time for a step input are valuable time-domain indices. In the case of steady-state and transient performance, the performance indices are normally specified in the time domain and, therefore, it is natural that we wish to develop design procedures in the time domain.

The performance of a control system may be represented by integral performance measures as we found in Section 4.5. Therefore, the design of a system must be based on minimizing a performance index such as the integral of the squared error (ISE) as in Section 4.5. Systems which are adjusted to provide a minimum performance index are often called *optimum control systems*. We shall consider, in this section, the design of an optimum control system where the system is described by a state variable formulation.

However, before proceeding to the specifics, we should note that we did design a system in the time domain in Example 9.6. In this example, we considered the unstable portion of an inverted pendulum system and developed a suitable feedback control so that the system was stable. This design was based on measuring the state variables of the system and using them to form a suitable control signal $u(t)$ so that the system was stable. In this section, we shall again consider the measurement of the state variables and their use in developing a control signal $u(t)$ so that the performance of the system is optimized.

The performance of a control system, written in terms of the state variables of a system, may be expressed in general as

$$J = \int_0^{t_f} g(\mathbf{x}, \mathbf{u}, t)\, dt, \tag{10.85}$$

where $\mathbf{x}$ equals the state vector and $\mathbf{u}$ equals the control vector.*

We are interested in minimizing the error of the system and, therefore, when the desired state vector is represented as $\mathbf{x}_d = \mathbf{0}$, we are able to consider the error as identically equal to the value of the state vector. That is, we desire the system to be

* Note that J is used to denote the performance index, instead of I, which was used in Chapter 4. This will enable the reader to readily distinguish the performance index from the identity matrix which is represented by the bold-faced capital $\mathbf{I}$.

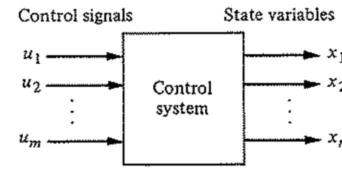


Fig. 10.22. A control system in terms of x and u.

at equilibrium, $\mathbf{x} = \mathbf{x}_d = \mathbf{0}$, and any deviation from equilibrium is considered an error. Therefore we will consider, in this section, the design of optimum control systems using *state-variable feedback* and error-squared performance indices [1, 2, 4, 5].

The control system which we will consider is shown in Fig. 10.22 and may be represented by the vector differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \tag{10.86}$$

We will select a feedback controller so that $\mathbf{u}$ is some function of the measured state variables $\mathbf{x}$ and therefore

$$\mathbf{u} = \mathbf{h}(\mathbf{x}). \tag{10.87}$$

For example, one might use

$$\begin{aligned} u_1 &= k_1 x_1, \\ u_2 &= k_2 x_2, \\ &\vdots \\ u_m &= k_m x_m. \end{aligned} \tag{10.88}$$

Alternatively, one might choose the control vector as

$$\begin{aligned} u_1 &= k_1(x_1 + x_2), \\ u_2 &= k_2(x_2 + x_3), \\ &\vdots \end{aligned} \tag{10.89}$$

The choice of the control signals is somewhat arbitrary and depends partially upon the actual desired performance and the complexity of the feedback structure allowable. Often we are limited in the number of state variables available for feedback, since we are only able to utilize measurable state variables.

Now, in our case, we limit the feedback function to a linear function so that $\mathbf{u} = \mathbf{H}\mathbf{x}$ where $\mathbf{H}$ is an $m \times n$ matrix. Therefore, in expanded form, we have

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & & \vdots \\ h_{m1} & \cdots & h_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{10.90}$$

Then, substituting Eq. (10.90) into Eq. (10.86), we obtain

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{BHx} = \mathbf{Dx}, \tag{10.91}$$

where **D** is the $n \times n$ matrix resulting from the addition of the elements of **A** and **BH**.

Now, returning to the error-squared performance index, we recall from Section 4.5 that the index for a single state variable, x_1, is written as

$$J = \int_0^{t_f} (x_1(t))^2\, dt. \tag{10.92}$$

A performance index written in terms of two state variables would then be

$$J = \int_0^{t_f} (x_1^2 + x_2^2)\, dt. \tag{10.93}$$

Therefore, since we wish to define the performance index in terms of an integral of the sum of the state variables squared, we will utilize the matrix operation

$$\mathbf{x}^T\mathbf{x} = [x_1, x_2, x_3, \ldots, x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2), \tag{10.94}$$

where $\mathbf{x}^T$ indicates the transpose of the **x** matrix.* Then the general form of the performance index, in terms of the state vector, is

$$J = \int_0^{t_f} (\mathbf{x}^T\mathbf{x})\, dt. \tag{10.95}$$

Again considering Eq. (10.95), we will let the final time of interest be $t_f = \infty$. In order to obtain the minimum value of J, we postulate the existence of an exact differential so that

$$\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) = -\mathbf{x}^T\mathbf{x}, \tag{10.96}$$

where **P** is to be determined. A symmetric **P** matrix will be used in order to simplify the algebra without any loss of generality. Then, for a symmetric **P** matrix, $p_{ij} = p_{ji}$. Completing the differentiation indicated on the left-hand side of Eq. (10.96), we have

$$\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}}.$$

*The matrix operation $\mathbf{x}^T\mathbf{x}$ is discussed in Appendix C, Section C.4.

Then, substituting Eq. (10.91), we obtain

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) &= (\mathbf{Dx})^T\mathbf{Px} + \mathbf{x}^T\mathbf{P}(\mathbf{Dx}) \\ &= \mathbf{x}^T\mathbf{D}^T\mathbf{Px} + \mathbf{x}^T\mathbf{PDx} \\ &= \mathbf{x}^T(\mathbf{D}^T\mathbf{P} + \mathbf{PD})\mathbf{x}, \end{aligned} \tag{10.97}$$

where $(\mathbf{Dx})^T = \mathbf{x}^T\mathbf{D}^T$ by the definition of the transpose of a product. If we let $(\mathbf{D}^T\mathbf{P} + \mathbf{PD}) = -\mathbf{I}$, then Eq. (10.97) becomes

$$\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) = -\mathbf{x}^T\mathbf{x}, \tag{10.98}$$

which is the exact differential we are seeking. Substituting Eq. (10.98) into Eq. (10.95), we obtain

$$\begin{aligned} J &= \int_0^{\infty} -\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x})\, dt \\ &= -\mathbf{x}^T\mathbf{P}\mathbf{x}\big|_0^{\infty} \\ &= \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0). \end{aligned} \tag{10.99}$$

In the evaluation of the limit at $t = \infty$, we have assumed that the system is stable and hence $\mathbf{x}(\infty) = 0$ as desired. Therefore, in order to minimize the performance index J, we consider the two equations

$$J = \int_0^{\infty} \mathbf{x}^T\mathbf{x}\, dt = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) \tag{10.100}$$

and

$$\mathbf{D}^T\mathbf{P} + \mathbf{PD} = -\mathbf{I}. \tag{10.101}$$

The design steps are then as follows:

1. Determine the matrix **P** which satisfies Eq. (10.101), where **D** is known.
2. Minimize J by determining the minimum of Eq. (10.100).

Example 10.11. Consider the control system shown in Fig. 10.23 in signal-flow graph form. The state variables are identified as x_1 and x_2. The performance of this

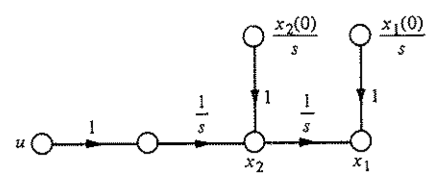


Fig. 10.23. The signal-flow graph of the control system of Example 10.10.

system is quite unsatisfactory since an undamped response results for a step input or disturbance signal. The vector differential equation of this system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{10.102}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We will choose a feedback control system so that

$$u(t) = -k_1 x_1 - k_2 x_2, \tag{10.103}$$

and therefore the control signal is a linear function of the two state variables. The sign of the feedback is negative in order to provide negative feedback. Then Eq. (10.102) becomes

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -k_1 x_1 - k_2 x_2, \end{aligned} \tag{10.104}$$

or, in matrix form, we have

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{D}\mathbf{x} \\ &= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}. \end{aligned} \tag{10.105}$$

We note that x_1 would represent the position of a position-control system and the transfer function of the system would be $G(s) = 1/Ms^2$, where $M = 1$ and the friction is negligible. In any case, in order to avoid needless algebraic manipulation, we will let $k_1 = 1$ and determine a suitable value for k_2 so that the performance index is minimized. Then, writing Eq. (10.101), we have

$$\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D} = -\mathbf{I},$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -k_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{10.106}$$

Completing the matrix multiplication and addition, we have

$$\begin{aligned} -p_{12} - p_{12} &= -1, \\ p_{11} - k_2 p_{12} - p_{22} &= 0, \\ p_{12} - k_2 p_{22} + p_{12} - k_2 p_{22} &= -1. \end{aligned} \tag{10.107}$$

Then, solving these simultaneous equations, we obtain

$$p_{12} = \frac{1}{2}, \quad p_{22} = \frac{1}{k_2}, \quad p_{11} = \frac{k_2^2 + 2}{2k_2}.$$

The integral performance index is then

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0), \tag{10.108}$$

and we shall consider the case where each state is initially displaced one unit from equilibrium so that $\mathbf{x}^T(0) = [1, 1]$. Therefore Eq. (10.108) becomes

$$\begin{aligned} J &= [1, 1]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [1, 1]\begin{bmatrix} (p_{11} + p_{12}) \\ (p_{12} + p_{22}) \end{bmatrix} \\ &= (p_{11} + p_{12}) + (p_{12} + p_{22}) = p_{11} + 2p_{12} + p_{22}. \end{aligned} \tag{10.109}$$

Substituting the values of the elements of $\mathbf{P}$, we have

$$\begin{aligned} J &= \frac{k_2^2 + 2}{2k_2} + 1 + \frac{1}{k_2} \\ &= \frac{k_2^2 + 2k_2 + 4}{2k_2}. \end{aligned} \tag{10.110}$$

In order to minimize as a function of k_2, we take the derivative with respect to k_2 and set it equal to zero as follows:

$$\frac{\partial J}{\partial k_2} = \frac{2k_2(2k_2 + 2) - 2(k_2^2 + 2k_2 + 4)}{(2k_2)^2} = 0, \tag{10.111}$$

and therefore $k_2^2 = 4$ and $k_2 = 2$ when J is a minimum. The minimum value of J is obtained by substituting $k_2 = 2$ into Eq. (10.110) and thus we obtain

$$J_{\min} = 3.$$

The system matrix $\mathbf{D}$, obtained for the compensated system, is then

$$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{10.112}$$

The characteristic equation of the compensated system is therefore

$$\begin{aligned} \det[\lambda\mathbf{I} - \mathbf{D}] &= \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix} \\ &= \lambda^2 + 2\lambda + 1. \end{aligned} \tag{10.113}$$

Since this is a second-order system, we note that the characteristic equation is of the form $(s^2 + 2\zeta\omega_n s + \omega_n^2)$, and therefore the damping ratio of the compensated system is $\zeta = 1.0$. This compensated system is considered to be an optimum system in that the compensated system results in a minimum value for the performance index. Of course, we recognize that this system is only optimum for the specific set of initial conditions that were assumed. The compensated system is shown in Fig.

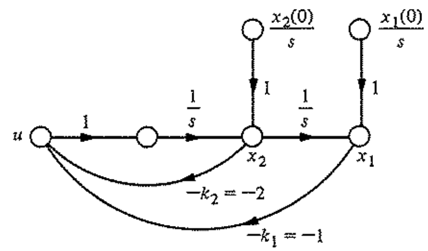


Fig. 10.24. The compensated control system of Example 10.10.

10.24. A curve of the performance index as a function of k_2 is shown in Fig. 10.25. It is clear that this system is not very sensitive to changes in k_2 and will maintain a near minimum performance index if the k_2 is altered some percentage. We define the sensitivity of an optimum system as

$$S_k^{\text{opt}} = \frac{\Delta J/J}{\Delta k/k}, \tag{10.114}$$

where k is the design parameter. Then, for this example, we have $k = k_2$ and therefore

$$S_{k_2}^{\text{opt}} \simeq \frac{0.08/3}{0.5/2} = 0.107. \tag{10.115}$$

Example 10.12. Now let us reconsider the system of the previous example where both the feedback gains, k_1 and k_2, are unspecified. In order to simplify the algebra, without any loss in insight into the problem, let us set $k_1 = k_2 = k$. The reader may prove that if k_1 and k_2 are unspecified then $k_1 = k_2$ when the minimum of the

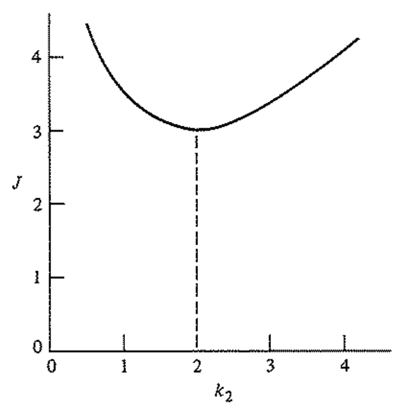


Fig. 10.25. The performance index versus the parameter k_2.

performance index (Eq. 10.100) is obtained. Then for the system of the previous example, Eq. (10.105) becomes

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{D}\mathbf{x}\\ &= \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix}\mathbf{x}.\end{aligned} \tag{10.116}$$

In order to determine the **P** matrix, we utilize Eq. (10.101), which is

$$\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D} = -\mathbf{I}. \tag{10.117}$$

Solving the set of simultaneous equations resulting from Eq. (10.117), we find that

$$p_{12} = \frac{1}{2k}, \quad p_{22} = \frac{(k+1)}{2k^2}, \quad p_{11} = \frac{(1+2k)}{2k}.$$

Let us consider the case where the system is initially displaced one unit from equilibrium so that $\mathbf{x}^T(0) = [1, 0]$. Then the performance index (Eq. 10.100) becomes

$$J = \int_0^\infty \mathbf{x}^T\mathbf{x}\, dt = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = p_{11}. \tag{10.118}$$

Thus the performance index to be minimized is

$$J = p_{11} = \frac{(1+2k)}{2k} = 1 + \frac{1}{2k}. \tag{10.119}$$

Clearly, the minimum value of J is obtained when k approaches infinity; the result is $J_{\min} = 1$. A plot of J versus k is shown in Fig. 10.26. This plot illustrates that the

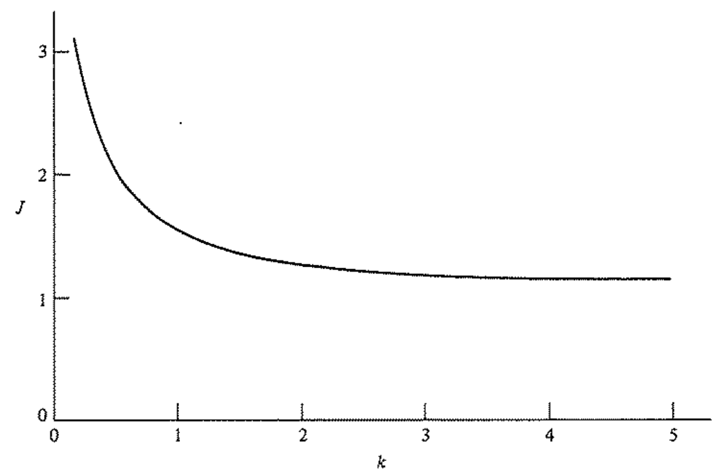


Fig. 10.26. Performance index versus the feedback gain k for Example 10.11.

performance index approaches a minimum asymptotically as k approaches an infinite value. Now we recognize that in providing a very large gain k, we cause the feedback signal

$$u(t) = -k(x_1(t) + x_2(t))$$

to be very large. However, we are restricted to realizable magnitudes of the control signal $u(t)$. Therefore we must introduce a *constraint* on $u(t)$ so that the gain k is not made too large. Then, for example, if we establish a constraint on $u(t)$ so that

$$|u(t)| \leq 50, \tag{10.120}$$

we require that the maximum acceptable value of k in this case

$$k_{\max} = \frac{|u|_{\max}}{x_1(0)} = 50. \tag{10.121}$$

Then the minimum value of J is

$$\begin{aligned} J_{\min} &= 1 + \frac{1}{2k_{\max}} \\ &= 1.01, \end{aligned} \tag{10.122}$$

which is sufficiently close to the absolute minimum of J in order to satisfy our requirements.

Upon examination of the performance index (Eq. 10.95), we recognize that the reason the magnitude of the control signal is not accounted for in the original calculations is that $u(t)$ is not included within the expression for the performance index. However, there are many cases where we are concerned with the expenditure of the control signal energy. For example, in a space vehicle attitude control system, $[u(t)]^2$ represents the expenditure of jet fuel energy and must be restricted in order to conserve the fuel energy for long periods of flight. In order to account for the expenditure of the energy of the control signal, we will utilize the performance index

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{I}\mathbf{x} + \lambda\mathbf{u}^T\mathbf{u})dt, \tag{10.123}$$

where λ is a scalar weighting factor and $\mathbf{I}$ = identity matrix. The weighting factor λ will be chosen so that the relative importance of the state variable performance is contrasted with the importance of the expenditure of the system energy resource which is represented by $\mathbf{u}^T\mathbf{u}$. As in the previous paragraphs we will represent the state variable feedback by the matrix equation

$$\mathbf{u} = \mathbf{H}\mathbf{x} \tag{10.124}$$

and the system with this state variable feedback as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ &= \mathbf{D}\mathbf{x}. \end{aligned} \tag{10.125}$$

Now, substituting Eq. (10.124) into Eq. (10.123), we have

$$\begin{aligned} J &= \int_0^\infty (\mathbf{x}^T\mathbf{I}\mathbf{x} + \lambda(\mathbf{H}\mathbf{x})^T(\mathbf{H}\mathbf{x}))\, dt \\ &= \int_0^\infty [\mathbf{x}^T(\mathbf{I} + \lambda\mathbf{H}^T\mathbf{H})\mathbf{x}]\, dt \\ &= \int_0^\infty \mathbf{x}^T\mathbf{Q}\mathbf{x}\, dt, \end{aligned} \tag{10.126}$$

where $\mathbf{Q} = (\mathbf{I} + \lambda\mathbf{H}^T\mathbf{H})$ is an $n \times n$ matrix. Following the development of Eqs. (10.95) through (10.99), we postulate the existence of an exact differential so that

$$\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) = -\mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{10.127}$$

Then in this case we require that

$$\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D} = -\mathbf{Q}, \tag{10.128}$$

and thus we have as before (Eq. 10.99)

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0). \tag{10.129}$$

Now the design steps are exactly as for Eqs. (10.100) and (10.101) with the exception that the left side of Eq. (10.128) equals $-\mathbf{Q}$ instead of $-\mathbf{I}$. Of course, if $\lambda = 0$, Eq. (10.128) reduces to Eq. (10.101). Now, let us reconsider the previous example when λ is other than zero and account for the expenditure of control signal energy.

Example 10.13. Let us reconsider the system of the previous example which is shown in Fig. 10.23. For this system we use a state variable feedback so that

$$\begin{aligned} \mathbf{u} &= \mathbf{H}\mathbf{x} \\ &= \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\mathbf{I}\mathbf{x}. \end{aligned} \tag{10.130}$$

Therefore, the matrix $\mathbf{Q}$ is then

$$\begin{aligned} \mathbf{Q} &= (\mathbf{I} + \lambda\mathbf{H}^T\mathbf{H}) \\ &= (\mathbf{I} + \lambda k^2\mathbf{I}) \\ &= (1 + \lambda k^2)\mathbf{I}. \end{aligned} \tag{10.131}$$

As in the previous example we will let $\mathbf{x}^T(0) = [1, 0]$ so that $J = p_{11}$. We evaluate p_{11} from Eq. (10.128) as

$$\begin{aligned} \mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D} &= -\mathbf{Q} \\ &= -(1 + \lambda k^2)\mathbf{I}. \end{aligned} \tag{10.132}$$

Thus we find that

$$J = p_{11} = (1 + \lambda k^2)\left(1 + \frac{1}{2k}\right), \tag{10.133}$$

and we note that the right-hand side of Eq. (10.133) reduces to Eq. (10.119) when $\lambda = 0$. Now the minimum of J is found by taking the derivative of J, which is

$$\frac{dJ}{dk} = 2\lambda k + \frac{\lambda}{2} - \frac{1}{2k^2} = \frac{4\lambda k^3 + \lambda k^2 - 1}{2k^2} = 0. \qquad (10.134)$$

Therefore, the minimum of the performance index occurs when $k = k_{min}$, where k_{min} is the solution of Eq. (10.134).

A simple method of solution for Eq. (10.134) is the Newton-Raphson method illustrated in Section 5.4. Let us complete this example for the case where the control energy and the state variables squared are equally important so that $\lambda = 1$. Then Eq. (10.134) becomes $4k^3 + k^2 - 1 = 0$, and using the Newton-Raphson method, we find that $k_{min} = 0.555$. The value of the performance index J obtained with k_{min} is considerably greater than that of the previous example, since the expenditure of energy is equally weighted as a cost. The plot of J versus k for this case is shown in Fig. 10.27. Also the plot of J versus k for Example 10.11 is shown for comparison on Fig. 10.27. It has become clear from this and the previous examples that the actual minimum obtained depends upon the initial conditions, the definition of the performance index, and the value of the scalar factor λ.

The design of several parameters may be accomplished in a similar manner to that illustrated in the examples. Also, the design procedure can be carried out for higher-order systems. However, one must then consider the use of a digital computer to determine the solution of Eq. (10.101) in order to obtain the **P** matrix. Also, the computer would provide a suitable approach for evaluating the minimum value of J for the several parameters. The newly emerging field of adaptive and optimal

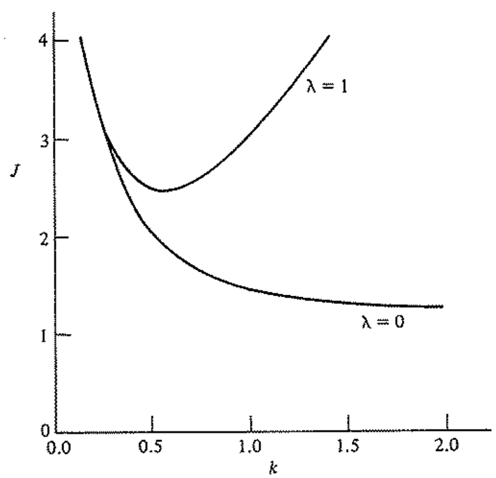


Fig. 10.27. Performance index versus the feedback gain k for Example 10.12.

control systems is based on the formulation of the time-domain equations and the determination of an optimum feedback control signal $\mathbf{u}(t)$ [5, 6, 10]. The design of control systems using time-domain methods will continue to develop in the future and will provide the control engineer with many interesting challenges and opportunities.

10.11 STATE-VARIABLE FEEDBACK

In the previous section we considered the use of state-variable feedback in achieving optimization of a performance index. In this section we will use *state-variable feedback* in order to achieve the desired pole location of the closed-loop transfer function $T(s)$. The approach is based on the feedback of all the state variables, and therefore

$$\mathbf{u} = \mathbf{H}\mathbf{x}. \qquad (10.135)$$

When using this state-variable feedback, the roots of the characteristic equation are placed where the transient performance meets the desired response.

As an example of state-variable feedback, consider the feedback system shown in Fig. 10.28. This position control uses a field controlled motor, and the transfer function was obtained in Section 2.5 as

$$G(s) = \frac{K}{s(s + f/J)(s + R_f/L_f)}, \qquad (10.136)$$

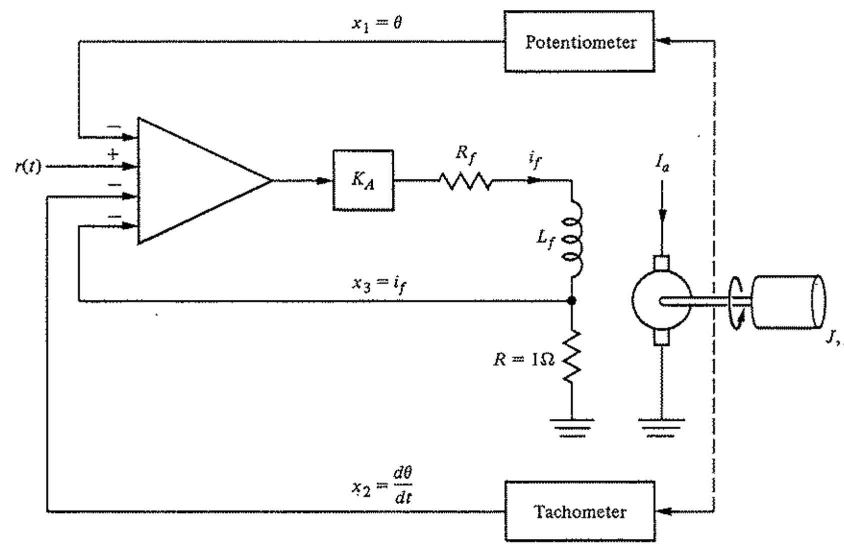


Fig. 10.28. A position control system with state-variable feedback.

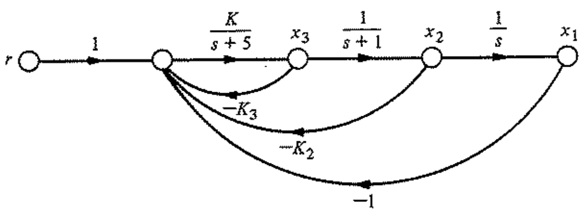


Fig. 10.29. The signal flow graph of the state-variable feedback system.

where $K = K_a K_m / J L_f$. For our purposes we will assume that $f/J = 1$ and $R_f/L_f = 5$. As shown in Fig. 10.28, the system has feedback of the three state variables: position, velocity, and field current. We will assume that the feedback constant for the position is equal to 1, as shown in Fig. 10.29, which provides a signal-flow graph representation of the system. Without state-variable feedback of x_2 and x_3, we set $K_3 = K_2 = 0$ and we have

$$G(s) = \frac{K}{s(s+1)(s+5)}. \tag{10.137}$$

This system will become unstable when $K \geq 30$. However, with variable feedback of all the state variables we can assure that the system is stable and set the transient performance of the system to a desired performance.

In general, the state-variable feedback signal-flow graph can be converted to the block diagram form shown in Fig. 10.30. The transfer function $G(s)$ remains unaffected (as in Eq. 10.137) and the $H(s)$ accounts for the state variable feedback. Therefore,

$$H(s) = K_3\left[s^2 + \left(\frac{K_3 + K_2}{K_3}\right)s + \frac{1}{K_3}\right] \tag{10.138}$$

and

$$G(s)H(s) = \frac{M[s^2 + Qs + (1/K_3)]}{s(s+1)(s+5)}, \tag{10.139}$$

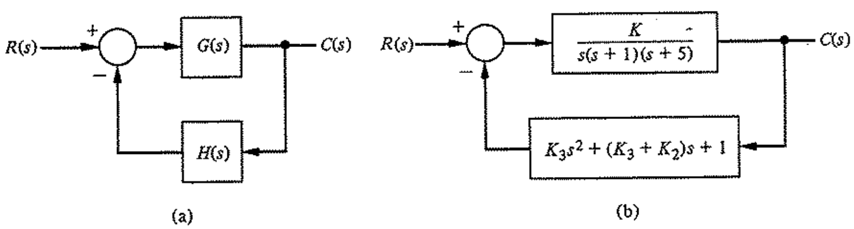


Fig. 10.30. An equivalent block diagram representation of the state-variable feedback system.

where $M = KK_3$ and $Q = (K_3 + K_2)/K_3$. Since K_3 and K_2 may be set independently, the designer may select the location of the zeros of $G(s)H(s)$.

As an illustration, let us choose the zeros of $GH(s)$ so that they cancel the real poles of $G(s)$. We set the numerator polynomial

$$\begin{aligned} H(s) &= K_3\left(s^2 + Qs + \frac{1}{K_3}\right) \\ &= K_3(s+1)(s+5). \end{aligned} \tag{10.140}$$

This requires $K_3 = 1/5$ and $Q = 6$, which sets $K_2 = 1$. Then

$$GH(s) = \frac{M(s+1)(s+5)}{s(s+1)(s+5)}, \tag{10.141}$$

where $M = KK_3$. The closed-loop transfer function is then

$$\frac{C(s)}{R(s)} = T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{(s+1)(s+5)(s+M)}. \tag{10.142}$$

Therefore, while we could choose $M = 10$ which would ensure the stability of the system, the closed-loop response of the system will be dictated by the poles at $s = -1$ and $s = -5$. Therefore we will usually choose the zeros of $GH(s)$ in order to achieve closed-loop roots in a desirable location in the left-hand plane and assure system stability.

Example 10.14. Let us again consider the system of Fig. 10.30(b) and set the zeros of $GH(s)$ at $s = -4 + j2$ and $s = -4 - j2$. Then the numerator of $GH(s)$ will be

$$\begin{aligned} H(s) &= K_3\left(s^2 + Qs + \frac{1}{K_3}\right) \\ &= K_3(s + 4 + j2)(s + 4 - j2) \\ &= K_3(s^2 + 8s + 20). \end{aligned} \tag{10.143}$$

Therefore $K_3 = 1/20$ and $Q = 8$ resulting in $K_2 = 7/20$. The resulting root locus for

$$G(s)H(s) = \frac{M(s^2 + 8s + 20)}{s(s+1)(s+5)} \tag{10.144}$$

is shown in Fig. 10.31. The system is stable for all values of gain $M = KK_3$. For $M = 10$ the complex roots have $\zeta = 0.73$, so that we might expect an overshoot for a step input of approximately 5%. The settling time will be approximately 1 second. The closed-loop transfer function is

$$\begin{aligned} \frac{C(s)}{R(s)} = T(s) &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{200}{(s + 3.45 + j3.2)(s + 3.45 - j3.2)(s + 9.1)}. \end{aligned} \tag{10.145}$$

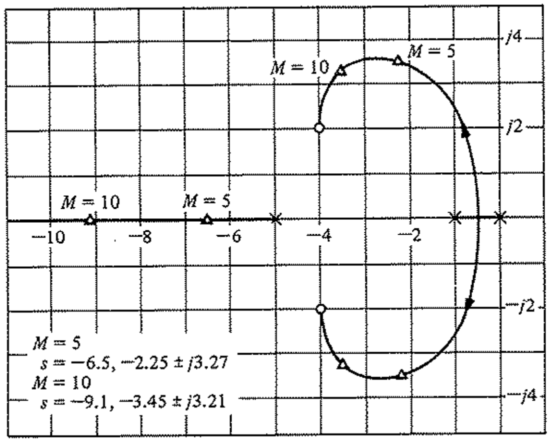


Fig. 10.31. The compensated system root locus.

An alternative approach is to set the closed-loop roots of $1 + G(s)H(S) = 0$ at desired locations and then solve for the gain values of K, K_3 and K_2 that are required. For example, if we desire closed-loop roots at $s = -10$, $s = -5 + j$ and $s = -5-j$ we have the characteristic equation

$$q(s) = (s + 10)(s^2 + 10s + 26) = s^3 + 20s^2 + 126s + 260 = 0. \quad (10.146)$$

Since

$$1 + G(s)H(s) = s(s + 1)(s + 5) + M\left(s^2 + Qs + \frac{1}{K_3}\right) = 0, \quad (10.147)$$

we equate Eq. (10.146) and Eq. (10.147), obtaining $M = 14$, $Q = 121$, $K_3 = 14/260$ and $K_2 = 6.462$.

In many cases the state variables are available and we can use state variable feedback to obtain a stable, well-compensated system.

10.12 SUMMARY

In this chapter we have considered several alternative approaches to the design and compensation of feedback control systems. In the first two sections, we discussed the concepts of design and compensation and noted the several design cases which we completed in the preceding chapters. Then, the possibility of introducing cascade compensation networks within the feedback loops of control systems was examined. The cascade compensation networks are useful for altering the shape of the root locus or frequency response of a system. The phase-lead network and the

phase-lag network were considered in detail as candidates for system compensators. Then, system compensation was studied by using a phase-lead s-plane network on the Bode diagram and the root locus s-plane diagram successively. System compensation using integration networks and phase-lag networks was also considered on the Bode diagram and the s-plane. We noted that the phase-lead compensator increases the phase margin of the system and thus provides additional stability. When the design specifications include an error constant, the design of a phase lead network is more readily accomplished on the Bode diagram. Alternatively, when an error constant is not specified, but the settling time and overshoot for a step input are specified, the design of a phase-lead network is more readily carried

Table 10.1 A Summary of the Characteristics of Phase-Lead and Phase-Lag Compensation Networks

	Compensation	
	Phase-lead	Phase-lag
Approach	Addition of phase-lead angle near the crossover frequency or to yield the desired dominant roots in the s-plane	Addition of phase-lag to yield an increased error constant while maintaining the desired dominant roots in the s-plane or phase margin on the Bode diagram
Results	1. Increases system bandwidth 2. Increases gain at higher frequencies	1. Decreases system bandwidth
Advantages	1. Yields desired response 2. Faster dynamic response	1. Suppresses high frequency noise 2. Reduces the steady-state error
Disadvantages	1. Requires additional amplifier gain 2. Increases bandwidth and thus susceptibility to noise 3. May require large values of components for the RC network	1. Slows down transient response 2. May require large values of components for the RC network
Applications	1. When fast transient response is desired	1. When error constants are specified
Not applicable	1. When phase decreases rapidly near the crossover frequency	1. When no low frequency range exists where the phase is equal to the desired phase margin

out on the s-plane. When large error constants are specified for a feedback system, it is usually easier to compensate the system by using integration (phase-lag) networks. We also noted that the phase-lead compensation increases the system bandwidth, while the phase-lag compensation decreases the system bandwidth. The bandwidth often may be an important factor when noise is present at the input and generated within the system. Also we noted that a satisfactory system is obtained when the asymptotic course for magnitude of the compensated system crosses the 0 db line with a slope of −6 db/octave. The characteristics of the phase-lead and phase-lag compensation networks are summarized in Table 10.1. Also, the design of control systems in the time domain was briefly examined. Specifically, the optimum design of a system using state-variable feedback and an integral performance index was considered. Finally, the s-plane design of systems utilizing state-variable feedback was examined.

PROBLEMS

10.1. The design of the Lunar Excursion Module (LEM) is an interesting control problem [6]. The Apollo 11 lunar landing vehicle is shown in Fig. P10.1(a). The attitude control system for the lunar vehicle is shown in Fig. P10.1(b). The vehicle damping is negligible and the attitude is controlled by gas jets. The torque, as a first approximation, will be considered to be proportional to the signal $V(s)$ so that $T(s) = K_2 V(s)$. The loop gain may be selected by the designer in order to provide a suitable damping. A damping ratio of $\zeta = 0.5$ with a settling time less than 2 sec is required. Using a lead-network compensation, select the necessary compensator $G_c(s)$ by using (a) frequency response techniques, and (b) root locus methods.

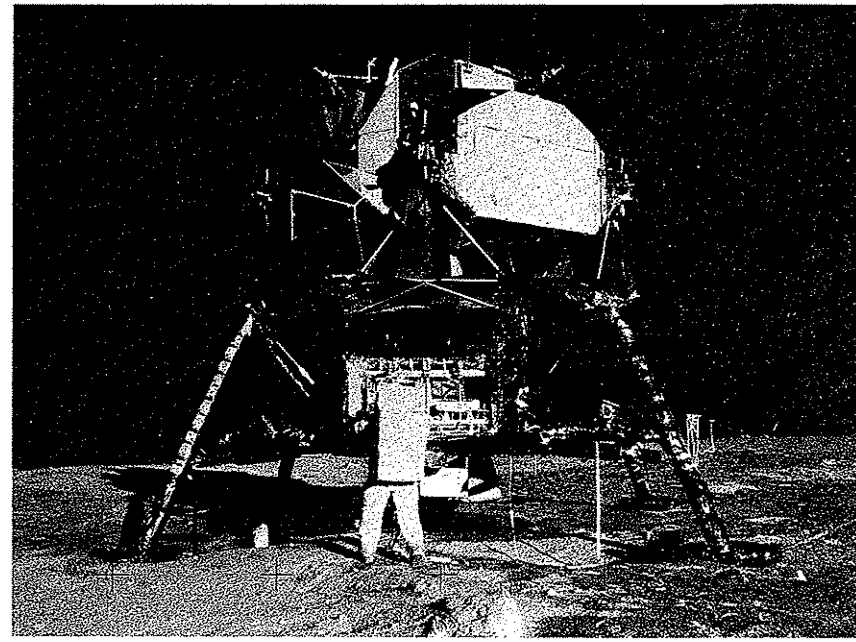
10.2. A magnetic tape-recorder transport for modern computers requires a high-accuracy, rapid-response control system. The requirements for a specific transport are as follows: (1) the tape must stop or start in 3 msec; (2) it must be possible to read 45,000 characters per second. This system was discussed in Problem 6.11. It is desired to set $J = 5 \times 10^{-3}$, and K_a is set on the basis of the maximum error allowable for a velocity input. In this case, it is desired to maintain a steady-state speed error of less than 2%. However, it is not possible to use a tachometer in this case and thus $K_2 = 0$. In order to provide a suitable performance, a compensator $G_c(s)$ is inserted in cascade between the photocell transducer and the amplifier. Select a compensator $G_c(s)$ so that the overshoot of the system for a step input is less than 30%.

10.3. A simplified version of the attitude rate control for the F-94 or X-15 type aircraft is shown in Fig. P10.3. When the vehicle is flying at four times the speed of sound (Mach 4) at an altitude of 100,000 feet, the parameters are

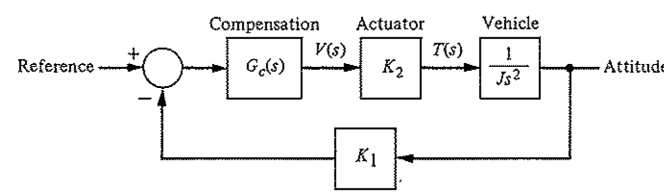
$$\frac{1}{\tau_a} = 0.04, \qquad K_1 = 0.02, \qquad \zeta\omega_a = 0.04, \qquad \omega_a = 2.$$

Design a compensation network so that the complex poles have approximately a $\zeta = 0.707$ and $\omega_n = 3$.

10.4. Magnetic particle clutches are useful actuator devices for high power requirements, since they can typically provide a 200-watt mechanical power output. The particle clutches



(a)



(b)

Fig. P10.1. Apollo 11 lunar excursion module viewed from the command ship. Inside the LEM were astronauts Neil Armstrong and Edwin Aldrin, Jr. The LEM landed on the moon on July 20, 1969. (Photo courtesy of NASA Manned Spacecraft Center.)

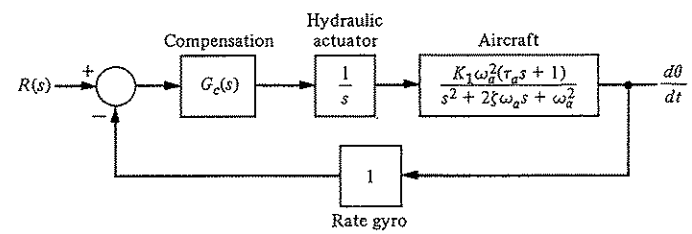


Figure P10.3