10.3 ADDITIONAL ROOT-LOCUS PROPERTIES

Having presented and proved the foregoing basic rules for the construction of a root locus we shall conclude the formal presentation of the topic with a discussion of several additional features that are of particular value in control-systems work.

Intersection of branches

If two real-axis branches of a locus are approaching one another for increasing values of K, with no intervening poles or zeros of $F(s)$, they will meet at some point. At this meeting both will undergo abrupt 90° changes of direction, one moving into the upper half of the s-plane and the other into the lower, as in Figure 10.4*b*. Likewise, two complex branches may come together at some point on the real axis as K increases and then move to the right and left along the real axis for further increases in K. While the intersection of root-locus branches is not limited to pairs of branches or to intersections on the real-axis, these are the most common cases in practice. Unfortunately, determining the locations of intersection points generally requires somewhat more effort than that required by the other construction rules.

To gain insight into the conditions under which multiple branches can intersect at a single point and to determine the general features of such an intersection, we assume that s_0 is a point on the 180° locus and expand $F(s)$ in a Taylor series about s_0. Since $F(s)$ is a rational function of s it will be analytic at all points in the s-plane except at its poles, $s = p_k$, $k = 1, 2, \ldots, n$. Thus, within the circle of convergence (Appendix A)

$$F(s) = F(s_0) + F'(s_0)(s - s_0) + \tfrac{1}{2}F''(s_0)(s - s_0)^2 + \cdots \tag{1}$$

where $F'(s_0)$ denotes dF/ds evaluated at $s = s_0$. Because s_0 lies on the root locus, arg $[F(s_0)]$ must satisfy the angle criterion, which is to say that $F(s_0)$ must be real and negative. If the value of K which corresponds to the point s_0 on the root locus is denoted by K_0, it follows from the magnitude criterion that

$$F(s_0) = -\frac{1}{K_0} \tag{2}$$

For the purpose of identifying other root-locus points in the neighborhood of s_0 we consider a small circle of radius ρ centered at s_0 and seek to determine those points on the circle which satisfy the angle criterion.

First, let us assume that $F'(s_0)$ does not vanish and write the difference between the testpoint and s_0 in polar form as $s - s_0 = \rho e^{j\psi}$. Then, taking ρ small enough so that the constant term in (1) dominates the term which is linear in ρ which, in turn, dominates all higher-order terms, we are justified in writing

$$F(s) \approx -\frac{1}{K_0} + \rho F'(s_0)e^{j\psi} \tag{3}$$

In addition to depicting the s-plane relationships in Figure 10.11*a*, the complex numbers in (3) can be represented in a separate complex plane, part (*b*), referred to as the F-plane, in which we show $F(s)$ as a vector whose magnitude and angle are functions of s, namely $|F(s)|$ and arg $[F(s)]$. As required by the angle criterion, the vector representing $F(s_0)$ lies on the negative-real axis of the F-plane and remains fixed during the following discussion.

If the test point s is to be on the 180° locus, $F(s)$ must certainly be real and negative per the angle criterion. But we can see from either (3) or Figure 10.11*b* that as a consequence $\rho F'(s_0)e^{j\psi}$, which is the difference

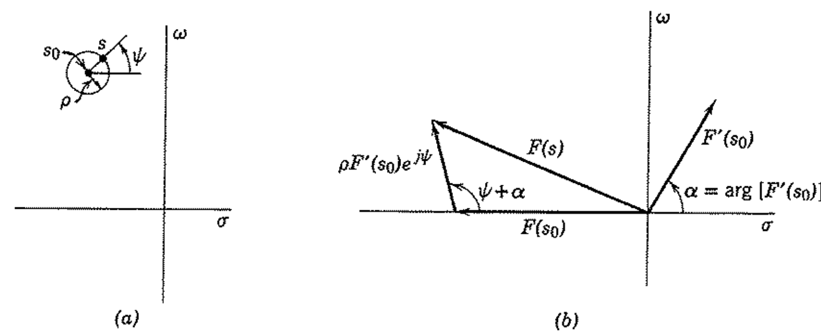

Figure 10.11. Mapping of $F(s)$ near s_0, a point on the locus. (*a*) s-plane. (*b*) F-plane.

between $F(s)$ and $F(s_0)$, must be *real* if the angle criterion is to be satisfied. Since ψ goes from 0° to 360° as the test point is moved around the circle in Figure 10.11*a* and both ρ and $F'(s_0)$ are constants, there will be only two values of ψ in this interval for which the angle criterion will be satisfied. One of these, say ψ_1, will result in $|F(s)| > |F(s_0)|$ when $\psi_1 + \arg[F'(s_0)] = 180°$, which corresponds to the branch of the root locus entering the circle at a value of $K_1 < K_0$.

The other value of ψ for which the angle criterion is satisfied will be $\psi_2 = \psi_1 + 180°$, resulting in $|F(s)| < |F(s_0)|$ and corresponding to the branch leaving the circle at a value of $K_2 > K_0$. Thus, if $F'(s)$ does not vanish at a point that is known to lie on the root locus, there is only one branch passing through that point, as in Figure 10.12*a*. In other words, $F'(s_0) = 0$ is a necessary condition for two branches to intersect at the point s_0.

To explore this point further, suppose that $F'(s_0) = 0$ but $F''(s_0) \neq 0$. Returning to (1) and placing the appropriate restriction on ρ, the Taylor-series expansion for $F(s)$ yields

$$F(s) \approx -\frac{1}{K_0} + \frac{1}{2}\rho^2 F''(s_0) e^{j2\psi} \tag{4}$$

The important difference between (3) and (4) is that the latter involves the term $e^{j2\psi}$ rather than $e^{j\psi}$. As a consequence there will be four values of ψ in the interval $0° \leq \psi < 360°$ for which the angle criterion is satisfied as the test point is moved around the circle in Figure 10.11*a*. Therefore, two branches of the locus must enter the circle and two must leave it. Furthermore, as ψ is increased from 0° to 360° the values of ψ for which the angle criterion is satisfied will be separated by 90°, with $|F(s)|$ alternating between being greater than and less than $|F(s_0)|$. Thus, the character of the root locus in the vicinity of s_0 must be as shown in Figure 10.12*b*.

Extending the above arguments to an arbitrary number of intersections, if s_0 is on the 180° locus and the first $\nu - 1$ derivatives of $F(s)$ vanish at $s = s_0$, then ν branches intersect at s_0 with entering and departing branches alternating and separated by angles of $(180/\nu)°$. The case $\nu = 3$ is shown in Figure 10.12*c*.

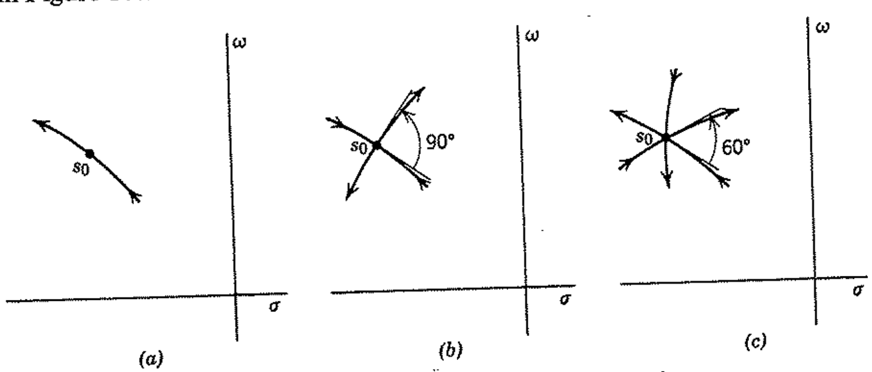

Figure 10.12. Intersecting branches. (*a*) $\nu = 1$. (*b*) $\nu = 2$. (*c*) $\nu = 3$.

The situation most often encountered in practice is the intersection of two branches, for which we have shown that

$$\left.\frac{dF}{ds}\right|_{s=s_0} = 0 \tag{5}$$

is a *necessary* condition, where s_0 is the intersection point on the root locus. However, the mere fact that $F'(s)$ vanishes for some values of s does not imply that two or more branches intersect unless the point at which $F'(s) = 0$ is on the root locus.

Finally, provided that s_0 does not coincide with a zero of $F(s)$, the condition for the intersections of loci can also be expressed as

$$\frac{d}{ds}\left(\frac{1}{F(s)}\right)_{s=s_0} = 0 \tag{6}$$

This form follows from the fact that

$$\frac{d}{ds}\left(\frac{1}{F(s)}\right) = -\frac{1}{F^2(s)}\frac{dF}{ds}$$

which vanishes for those values of s satisfying (5) provided $F(s) \neq 0$. The version given by (6) may be easier to apply than (5) in some cases.

Example 10.5

As an illustration of the manner in which branches intersect, we apply the rules given in Section 10.2 to construct the 180° locus for the system shown in Figure 10.13*a*, for which

$$F(s) = \frac{s+1}{s(s+0.5)}$$

Based upon the knowledge that the real-axis portions of the locus are the segments $-0.5 \leq s \leq 0$ and $s \leq -1$, we may infer that as K increases the two branches meet at some point $-0.5 < s_1 < 0$ and depart from the real axis for $K = K_1$ at angles of $\pm 90°$. Furthermore, the two branches must come together for some larger value of K, say $K = K_2$, at the point $s_2 < -1$.

We can solve for s_1 and s_2 by finding those solutions of (5) or (6) that lie on the locus. Differentiating $F(s)$,

$$\frac{dF}{ds} = -\frac{s^2 + 2s + 0.5}{s^2(s+0.5)^2}$$

which vanishes for

$$s_1 = -\left(1 - \frac{1}{\sqrt{2}}\right) = -0.293 \qquad \text{and} \qquad s_2 = -\left(1 + \frac{1}{\sqrt{2}}\right) = -1.707$$

Because both s_1 and s_2 lie on the 180° locus, they must be the points at which the two branches leave from and arrive at the real axis. If we had chosen to differentiate $1/F(s)$ instead, we would have obtained

$$\frac{d}{ds}\left(\frac{1}{F(s)}\right) = \frac{s^2 + 2s + 0.5}{(s+1)^2}$$

which also vanishes for the values of s_1 and s_2 found above. The complete root locus is given in Figure 10.13*b*. The combination of a pair of poles and a single zero is one that arises often, and it can be shown analytically that the branches in the upper and lower halves of the *s*-plane always comprise a circle centered at the zero of $F(s)$ – observe that the mid-point between s_1 and s_2 is indeed $s = -1$, the location of the zero (Prob. 10.18).

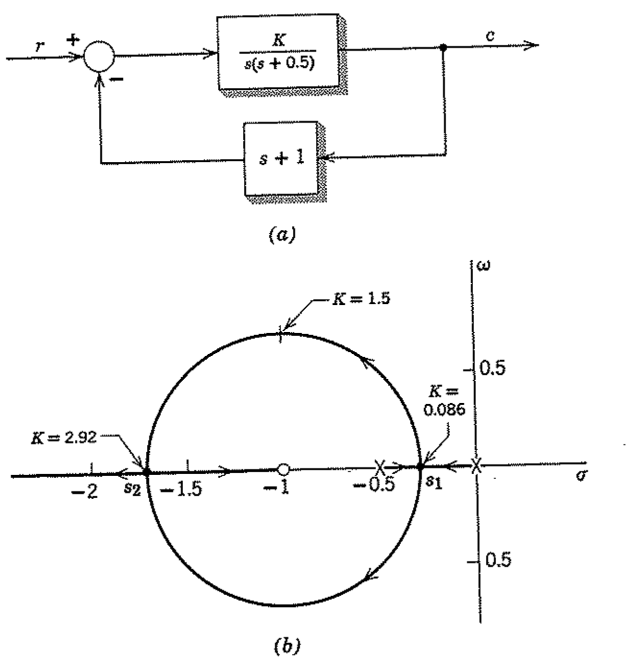

Figure 10.13. A system whose locus has intersecting branches. (*a*) Block diagram. (*b*) 180° locus.

By applying the magnitude criterion the values of K corresponding to the points s_1 and s_2 can be computed. For example, at the point s_1 at which the locus breaks away from the real axis,

$$F(s_1) = \frac{1/\sqrt{2}}{(-1 + 1/\sqrt{2})(-0.5 + 1/\sqrt{2})} = -11.64$$

so $K_1 = 1/|F(s_1)| = 0.086$.

Repeated poles and zeros of *F*(*s*)

When the basic root-locus construction rules were presented in Section 10.2, the assumption was made that all poles or zeros of $F(s)$ were distinct, i.e., not repeated. This restriction can be relaxed by making only minor adjustments in the rules.

Briefly, a pole of $F(s)$ which is repeated r times will be the starting point of r branches and should be counted r times when determining the real-axis portion of the locus, the large-gain asymptotes, and the angles of departure and arrival. The behavior of these branches in the vicinity of the repeated pole for $K \approx 0$ is governed by the preceding discussion on the intersection of branches – these happen to intersect for $K = 0$. Comparable relationships hold for a repeated zero of $F(s)$.

Example 10.6

The adjustments in the application of the root-locus construction rules necessitated by a repeated pole can be demonstrated by obtaining the locus for

$$F(s) = \frac{s+2}{s(s+1)^2}$$

By inspection, $F(s)$ has a repeated pole at $s = -1$ of multiplicity $r = 2$, along with a single pole at the origin and a zero at $s = -2$. Counting the repeated pole twice, it follows that $m = 1$ and $n = 3$; hence the locus has three branches. For $K \approx 0$ one branch emanates from $s = 0$ and two branches must start at the repeated pole. By applying the real-axis rule, we see that all points on the real axis between $s = 0$ and $s = -2$ have an odd number of poles and/or zeros to their right.

Because $n - m = 2$ and

$$\sigma_0 = \frac{[0 + 2(-1)] - [-2]}{3 - 1} = 0$$

there are two branches that approach infinity asymptotic to the imaginary axis. If the point at which the branches break away from the real axis is evaluated by solving $dF/ds = 0$, one finds that the breakaway point must satisfy $s^3+4s^2+4s+1=0$ which has as its three solutions $s_1=-0.38$, $s_2=-1$, and $s_3=-2.62$. Because s_3 does not lie on the 180° locus it need not be considered further. The fact that $F'(s)$ vanishes for $s=-1$ is a result of the fact that there is a repeated pole of $F(s)$ at that point in keeping with the discussion on intersection of branches. Thus, by the process of elimination, $s_1=-0.38$ must be the breakaway point we seek. The complete root locus is shown in Figure 10.14.

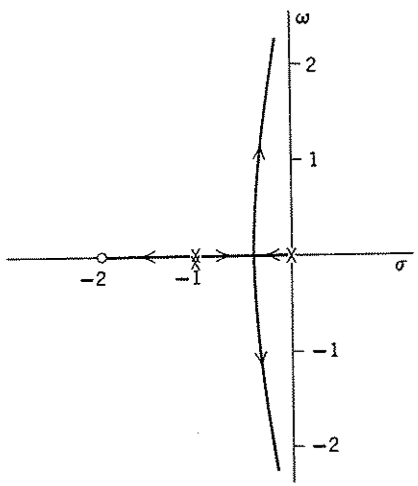

Figure 10.14. Root locus with a double pole.

Sum of the roots

When the number of open-loop poles exceeds the number of open-loop zeros by 2 or more, i.e., $m \le n-2$, the sum of the roots of the characteristic equation happens to be independent of K. This result can be derived by considering the form of the characteristic equation given in Eq. (9), Sect. 10.1, which becomes

$$s^n+a_{n-1}s^{n-1}+\cdots+a_0+K[s^m+b_{m-1}s^{m-1}+\cdots+b_0]=0$$

where the b's and a's do not depend on K. Assuming $m \le n-2$, we collect like powers of s to get

$$s^n+a_{n-1}s^{n-1}+\cdots+(a_m+K)s^m+\cdots+(a_0+Kb_0)=0 \qquad (7)$$

Appealing to the well-known theorem that the sum of the roots of a polynomial of degree n is the negative of the coefficient of the term s^{n-1}, it follows that the sum of the roots is $-a_{n-1}$, which is certainly independent of K. Furthermore, using $K=0$ shows that this sum is the sum of the poles of $F(s)$.

Referring to Figures 10.5*b* and 10.6*b* we see that in both cases the number of poles of $F(s)$ exceeds the number of zeros by at least 2. In Figure 10.5*b*, the real parts of the roots on the two complex branches must approach $+\infty$ as $K \to \infty$ at such a rate that for any given value of K the sum of the roots is equal to -1, since $p_1+p_2+p_3=-1$. Thus, for the specific value of K at which the two complex roots lie on the imaginary axis, the real root must be at $s=-1$. On the other hand, in Figure 10.6*b*, the real parts of the complex roots must remain finite as $K \to \infty$ because the real root moves to the left only as far as $-\frac{1}{2}$ as $K \to \infty$. Since the sum of the open-loop poles is -1, it follows that the real part of the asymptotes must be $-\frac{1}{4}$, which agrees with the value of σ_0 computed in Example 10.4.

The root locus in Figure 10.14 is yet another situation in which the sum of the roots is constant. On the other hand, that of Figure 10.13*b* is an example in which $m=n-1$ and the sum of the roots is not independent of K, moving to the left as K increases.

The zero-degree locus

So far, our attention has been devoted to systems that are characterized by both a minus sign at the feedback summing junction and a positive value of the root-locus gain K. However, if the feedback summing junction has a plus sign, the characteristic equation, as given by the block-diagram rules in Figure 9.8, is $1-G(s)H(s)=0$. Substituting $KF(s)$ for $G(s)H(s)$, the characteristic equation can be written as

$$F(s)=+\frac{1}{K}=\frac{1}{|K|}e^{jq\pi} \qquad (8)$$

where q is now an *even* integer. Likewise, if the feedback sign is negative but the gain K is negative, (8) is applicable, with q even. It follows that the only change to be made in the angle criterion is that in Eq. (13), Sect. 10.1, q must be even (zero is considered to be even) and the magnitude criterion is unaffected. Hence, the designation "0° locus" is used to distinguish both of these cases from the more common 180° locus.

Because their derivations did not involve the angle criterion, rules *1*, *2*, and 4 of Section 10.1 are unchanged. Rule 5 requires only the insertion of "As $K \to -\infty$" in place of "As $K \to \infty$" and rule 3 must be modified by replacing "odd" with "even"—a moment's thought will indicate that all points on the real axis will lie on either the 0° locus or the 180° locus.

In rule 6 "as $K \to -\infty$" should be inserted for "as $K \to \infty$" and Eq. (16), Sect. 10.1, should be changed to

$$\psi_\nu = \frac{\nu 360°}{n-m} \qquad \nu = 0, 1, \ldots, (n-m-1)$$

The reader may verify that there is no change needed in Eq. (15), Sect. 10.1, which is the equation for the intersection of the asymptotes. Finally, rules 7 and 8 need only be modified by restricting q to be even rather than odd.

Example 10.7

To demonstrate the application of the 0°-locus rules, the locus for the system of Example 10.5 with $K \leq 0$ will be constructed. With reference to Figure 10.13*b*, there will be two branches on the 0° locus that will be those segments of the real axis not belonging to the 180° locus: $-1 \leq \sigma \leq -0.5$ and $\sigma \geq 0$. In this case no other points lie on the 0° locus as there are no complex poles or zeros and the real roots never merge. The 0° locus is shown in Figure 10.15*a*; if the 180° portion in Figure 10.13*b* is combined with the 0° locus, Figure 10.15*b* results. The reader should note that the combination of the two portions forms a continuous locus as K goes from $-\infty$ to $+\infty$, with the change from the 0° locus to the 180° locus occurring at $K = 0$, at which point the roots coincide with the open-loop poles.

10.4 ROUTH'S CRITERION

In the initial stages of a design the situation often arises where one only needs to know whether a system is stable, rather than the precise values of all of the characteristic-equation roots. By applying *Routh's criterion* we can answer this limited question far more easily than by solving for the roots of an open-loop system directly or by drawing a root locus in the case of a feedback system. In fact, we can often use Routh's criterion as one of the steps in constructing the root locus of a feedback system. One final feature of Routh's criterion is that it may be applied to a system whose elements are expressed in literal rather than numerical form.

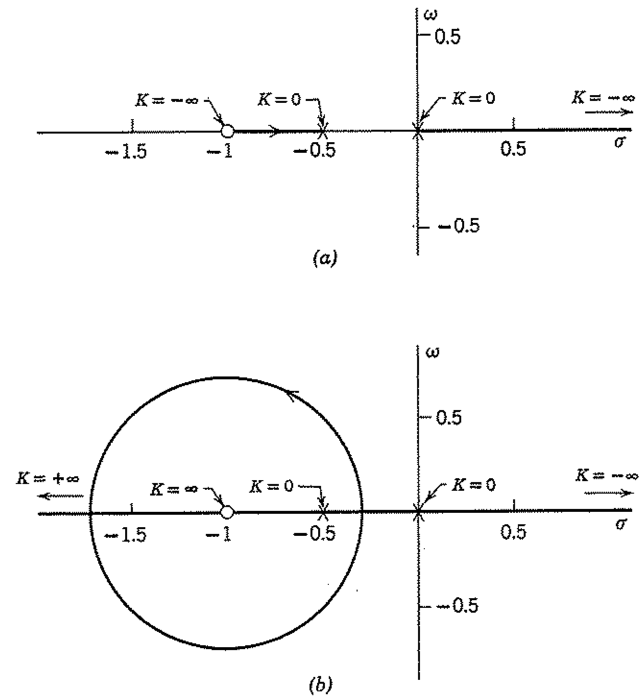

Figure 10.15. (*a*) 0° locus for the system in Figure 10.13. (*b*) Complete locus, $-\infty < K < \infty$.

It is well known that if all the roots of

$$\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0 = 0 \qquad (1)$$

are to be in the LHP it is necessary, *but not* sufficient, that all of the coefficients α_i be positive. However, we seek a necessary *and* sufficient condition in the sense that if the condition is not satisfied we shall be assured that at least one of the roots of the polynomial does not lie in the LHP. If (1) is constructed so as to be the characteristic equation, we shall have a test for asymptotic stability.

Such a test was presented by Routh in 1874 and an equivalent test was derived independently by Hurwitz in 1895; both forms are often referred to as the *Routh-Hurwitz criterion.*† Here we shall present the version given by Routh; the proof, which is rather involved, is omitted.‡ Routh's formulation states that certain quantities, $n+1$ in number and known as

†A simplification of the Routh-Hurwitz criterion which could be useful for high-order systems is the Liénard-Chipart test. See Lindorff (1965, App. A) for details.
‡See Guillemin (1956, pp. 395–409).

the *Routh series*, must all be positive if all the roots of (1) are to lie in the LHP. Furthermore, if all elements of the series are nonzero, the number of roots in the RHP is equal to the number of changes in sign encountered in going through the series in order. We shall show the manner in which the elements of the Routh series are computed and then give several results in a form particularly suited to the analysis of feedback systems.

In order to systematize the evaluation of the Routh series, the results of intermediate calculations are arranged along with the elements of the series in the *Routh array* which, in its general form, is

$$
\begin{array}{|c|cccc}
A_n & B_n & C_n & D_n & \cdots \\
A_{n-1} & B_{n-1} & C_{n-1} & D_{n-1} & \cdots \\
A_{n-2} & B_{n-2} & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & C_4 & & \\
\cdot & \cdot & 0 & & \\
A_2 & B_2 & 0 & & \\
A_1 & 0 & 0 & & \\
A_0 & 0 & 0 & &
\end{array}
$$

Routh series

The set $A_n, A_{n-1}, \ldots, A_0$ comprising the first column is the Routh series, and each element of it must be positive if all of the roots of (1) are to lie in the LHP. The B's, C's, etc., are the results of intermediate calculations and, with the exception of B_2, will be of no further use in assessing the stability of the system. There will always be $n+1$ rows in the array and the number of columns will be $(n+2)/2$ if n is even and $(n+1)/2$ if n is odd. As indicated in the partial array shown above, the column of B's will terminate with B_2, the column of C's with C_4, etc. The reason for this will be clear after the algorithm for computing the elements is given.

To start the process, the first two rows of the array are merely the coefficients of the polynomial in (1), arranged so that

$$
\begin{array}{lll}
A_n = \alpha_n & B_n = \alpha_{n-2} & C_n = \alpha_{n-4} \ldots \\
A_{n-1} = \alpha_{n-1} & B_{n-1} = \alpha_{n-3} & C_{n-1} = \alpha_{n-5} \ldots
\end{array}
$$

The filling out of the first two rows continues until α_0 is reached and it will fall in the second row if n is odd and in the first row if n is even, in which case the corresponding element in the second row is set to zero. All entries to the right of the column containing α_0 are considered to be zeros.

At this point, the entries in the third row are calculated using the elements in the first and second rows according to

$$
A_{n-2} = \frac{A_{n-1}B_n - A_n B_{n-1}}{A_{n-1}} \qquad B_{n-2} = \frac{A_{n-1}C_n - A_n C_{n-1}}{A_{n-1}} \tag{2}
$$

and so forth. The reader should be able to convince himself that the third row will have one fewer nonzero elements than the two rows above it. Once the third row is completed, the fourth and subsequent rows may be computed using the generalizations of (2):

$$
A_{i-1} = \frac{A_i B_{i+1} - A_{i+1} B_i}{A_i} \qquad B_{i-1} = \frac{A_i C_{i+1} - A_{i+1} C_i}{A_i} \tag{3}
$$

and similarly for C_{i-1}, D_{i-1}, etc., until the process ends. With the evaluation of each pair of rows the number of nonzero elements in a row is reduced by one, thus causing the process to terminate eventually. The next to last row will contain only the element A_1 and the last row only the element A_0. In fact, it is readily shown that if the sequence of operations given above is properly executed, the last nonzero element in each column of the Routh array will be α_0.

From an inspection of (3) it is obvious that the process can not proceed if $A_i = 0$ at any stage in the calculation. This is not generally a problem because we know from the statement of Routh's criterion that if one of the A_i is zero, at least one of the roots of (1) will not lie in the LHP. There are methods for continuing the calculations when this happens; see, for instance, Cannon (1967, Sect. 11.12).

Having shown how to construct the Routh series we now state the results that are of greatest utility in the study of feedback systems.

1. For all the roots of the characteristic equation to lie in the LHP, i.e., for the system to be asymptotically stable, it is both necessary and sufficient that each of the $n+1$ elements of the Routh series, $A_n, A_{n-1}, \ldots, A_0$, be positive.

2. If $A_0 = 0$ and the remaining n elements of the Routh series $A_n, A_{n-1}, \ldots, A_1$, are positive, then the characteristic equation has a single root at $s = 0$ and the remaining $n-1$ roots are in the LHP (marginal stability).

3. If $A_1 = 0$ and the remaining n elements of the Routh series,

$A_n, \ldots, A_2, A_0$, are positive, then the characteristic equation has a pair of imaginary roots at

$$s = \pm j\sqrt{\frac{A_0}{A_2}} \tag{4}$$

and the remaining $n-2$ roots are in the LHP (marginal stability).

While these rules allow us to quickly check the system's stability, they also are useful in constructing root loci and in parameter selection.

In the process of constructing the root locus for a feedback system, condition *3* can be used to locate the two points at which a pair of complex branches cross the imaginary axis if the other roots are in the LHP for the cross-over gain K_{co}. Because the A_i will be functions of the root-locus gain K, we can find K_{co} by solving the set of equations

$$A_1(K_{co}) = 0$$
$$A_i(K_{co}) > 0 \qquad i = 0, 2, \ldots, n$$

Having found K_{co}, the values of s_{co} at which the locus crosses the imaginary axis may be found by using (4) with the values of A_0 and A_2 corresponding to $K = K_{co}$:

$$s = \pm j\sqrt{\frac{A_0(K_{co})}{A_2(K_{co})}} \tag{5}$$

The above requirements also constitute a particularly useful stability test in control-system design if the characteristic equation of the closed-loop system is such that the A_i can be obtained in analytical form in terms of one or two unspecified parameters, e.g., gains, time constants. Thus, when there are two such parameters, say k_1 and k_2, the designer can construct curves in a two-dimensional *parameter space* corresponding to the solutions of the $n+1$ equations

$$A_i(k_1, k_2) = 0 \qquad i = 0, 1, \ldots, n$$

If a region in the k_1, k_2 plane exists such that each of the $A_i(k_1, k_2)$ is positive for all values of k_1 and k_2 in that region, we know that the closed-loop system can be made asymptotically stable by selecting the k_1 and k_2 corresponding to *any* point in that region (see Problem 10.26).

Example 10.8

To determine whether all the roots of

$$4s^5 + 6s^4 + 9s^3 + 2s^2 + 5s + 4 = 0$$

lie in the LHP, we begin by writing the first two rows of the Routh array directly from the polynomial. Using (3) to compute the remaining entries yields

4	9	5
6	2	4
$\frac{23}{3} = \frac{6\times 9 - 2\times 4}{6}$	$\frac{7}{3} = \frac{6\times 5 - 4\times 4}{6}$	0
$\frac{4}{23} = \left(\frac{23}{3}\times 2 - \frac{7}{3}\times 6\right)\frac{3}{23}$	$4 = \left(\frac{23}{3}\times 4 - 0\times 6\right)\frac{3}{23}$	0
$-174 = \left(\frac{4}{23}\times\frac{7}{3} - 4\times\frac{23}{3}\right)\frac{23}{4}$	0	
$4 = \frac{-174\times 4 - 0\times 4/23}{-174}$		

where the calculations leading to each entry have been included.

The Routh series, being the first column of the array, is

$$4 \quad 6 \quad 23/3 \quad 4/23 \quad -174 \quad 4$$

which contains the two sign reversals $4/23 \to -174$ and $-174 \to 4$. Thus we have ascertained that the equation has two roots that do not lie in the LHP, but we have no idea as to their specific locations—we only know that if the equation were a characteristic equation, the system would be unstable.

Example 10.9

In Example 10.3 the root locus was drawn for a feedback system that was stable only when a system parameter, designated as α, was within a finite interval. Using Routh's criterion we can readily compute the stability range for α and the points at which the complex branches of the locus cross into the RHP.

From an inspection of the block diagram in Figure 10.5*a* we can see that when the closed-loop characteristic equation is written in the form of Eq. (1), Sect. 10.1, it is

$$(s-1)(s^2+2s+5) + 5\alpha = s^3 + s^2 + 3s + 5(\alpha - 1) = 0$$

The Routh array is

$$\begin{array}{|cc} 1 & 3 \\ 1 & 5(\alpha-1) \\ 8-5\alpha & 0 \\ 5(\alpha-1) & 0 \end{array}$$

and the Routh series is

$$1 \quad 1 \quad 8-5\alpha \quad 5(\alpha-1)$$

The last two terms of the series will be positive if and only if $1 < \alpha < \frac{8}{5}$, which agrees with the results of the root-locus analysis in Example 10.3. Furthermore, if $\alpha = \frac{8}{5}$ then $A_1 = 0$ and the conditions for the existence of a pair of imaginary roots are satisfied. Using (5) with $A_0 = 3$ and $A_2 = 1$ gives $s_{co} = \pm j\sqrt{3}$, which are the points in Figure 10.5*b* at which the complex branches enter the RHP.

10.5 APPLICATION

In order to demonstrate the manner in which the analytical techniques presented in this chapter might be applied to the design of a feedback system, let us reconsider the satellite-attitude control system discussed in Section 9.3. Usually, most of the system parameters are fixed before the design of the control system begins, leaving at the discretion of the control-system designer only the form and parameter values of the control law; sometimes he is free to select or modify transducers also. For the sake of argument, we shall assume the following parameters are fixed:

Angle sensor gain: $K_\theta = 0.20$ volts/rad

Torque constant: $K_\tau = 0.10$ ft lb/volt

Moment of inertia: $I = 10.0$ slug ft^2

Hence, if we use the P+I+D control law of Eq. (12), Sect. 9.3, the amplifier gain K_a (volts/volt), the derivative gain α (seconds), and the integral gain β (seconds^{-1}) are to be selected. To complete the problem statement, certain requirements will be placed on the behavior of the closed-loop system such as specifying the locations of closed-loop poles

in the *s*-plane and bounds upon steady-state pointing errors caused by disturbance torques. Since the control laws to be discussed are not physically realizable they should be thought of as idealizations of the actual characteristics that can be obtained in practice, with their approximation by actual networks postponed until Chapter 11.

Proportional control

It was shown in Section 9.3 that the proportional control law

$$\tau_{con} = K_a K_\tau K_\theta [\theta_{ref} - \theta] \tag{1}$$

resulted in a closed-loop system whose response to any input was unsatisfactory because of the presence of an undamped oscillatory mode. Before going on to more satisfactory control laws, we can use a simple root-locus plot to verify that this will indeed be the case. Substituting the fixed parameter values and using (1), the system's block diagram becomes that shown in Figure 10.16*a*. If we restrict our interest for the moment to a determination of the roots of the closed-loop characteristic equation, the block diagram can be simplified by eliminating all inputs and outputs and by reordering any blocks within the loop – recall that we are concerned only with the properties of the product $G(s)H(s)$. Thus, for the construction of the root locus the block diagram becomes the simplified form

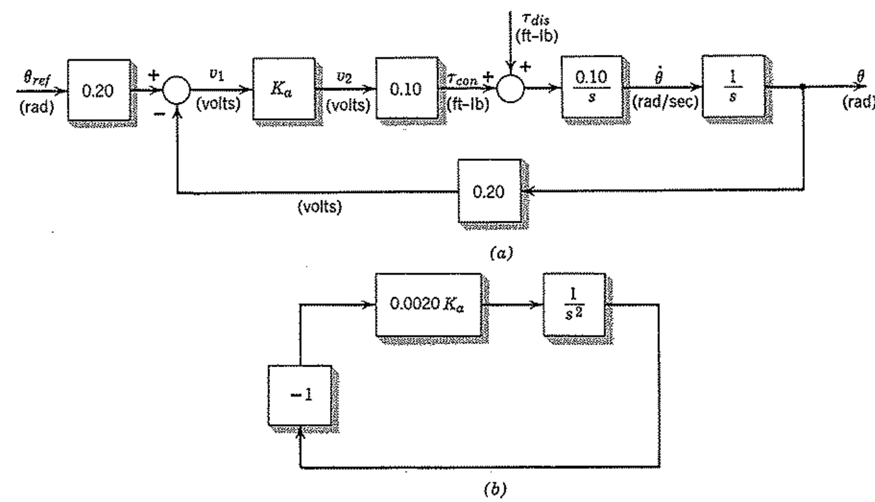

Figure 10.16. Satellite attitude system with proportional control. (*a*) Complete diagram. (*b*) Reduced loop.