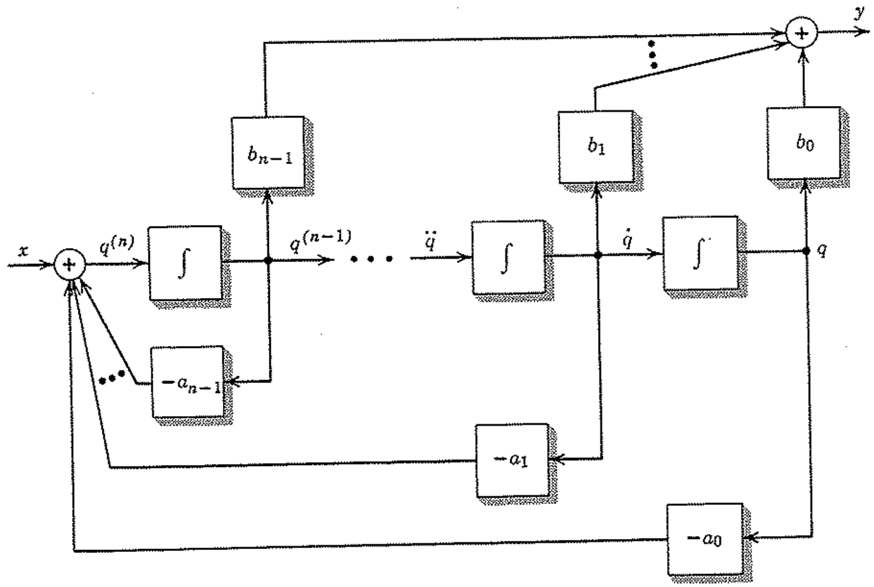

Figure 3.12. Canonic simulation of an nth-order system, $m = n - 1$.

equations instead of a single nth-order equation. For instance, similar to Example 2.6, suppose we have a second-order system described by

$$\begin{aligned} \dot{q}_1 &= A_{11}q_1 + A_{12}q_2 + B_1x \\ \dot{q}_2 &= A_{21}q_1 + A_{22}q_2 + B_2x \\ y &= C_1q_1 + C_2q_2 \end{aligned} \tag{6}$$

where the A's, B's, and C's are constants. A direct simulation of this system is given in Figure 3.13, which has the integrators in *parallel* rather than in series. The diagram nicely brings out how the two first-order subsystems are coupled to each other via the scalors A_{12} and A_{21}.

Two more items remain to be mentioned in this brief discussion. First, there is the question of initial conditions and the zero-input response. Although not included in Figure 3.7, analog-computer integrators have a special input terminal for establishing the initial output value at the time that simulation is begun, i.e., if we start the computer at time t_0, and $\dot{y}(t)$ is an integrator's input, then the corresponding output is

$$y(t) = \int_{t_0}^{t} \dot{y}(\lambda)\,d\lambda + y_0 \qquad t \geq t_0$$

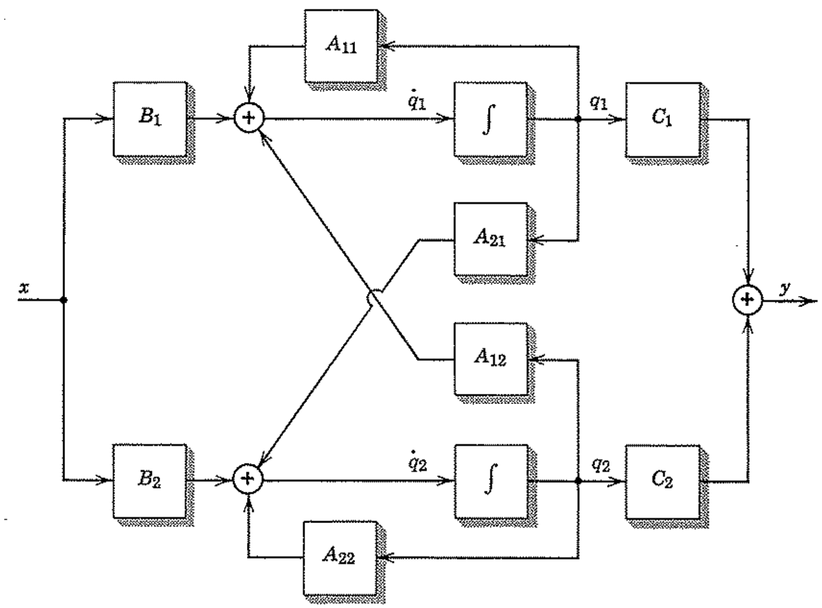

Figure 3.13.

Therefore, one must set the appropriate initial conditions on each and every variable that appears at the output of an integrator.

Second, simulation diagrams such as Figures 3.12 or 3.13 cannot be transferred directly to the analog machine. For one reason computer components have a limited range of linearity so the variables have to be *scaled in magnitude* to keep them within the linear range; for another, the rate of change of the variables may be too fast for the machine's response time or too slow for convenient observation, in which case *time scaling* is necessary. Additionally, most computer amplifiers and integrators produce a sign inversion, and amplifiers have fixed gains so that adjustable gain requires an input potentiometer. Thus, the resulting "patching" diagram looks somewhat different from the simulation diagram. Blum (1969) gives an excellent introduction to the techniques of analog computation, and is suitable for self-study.

3.5 STATE EQUATIONS

Thus far in this chapter we have dealt exclusively with single-input-output systems, and they will continue to be the major subject of later chapters. But some of the most challenging and fascinating problems of contemporary engineering are those of multi-input-output systems of rather high order. In view of the fact that space-age systems are being called on to perform many functions simultaneously, with greater accuracy and in less time, the move toward ever-increasing system complexity is an understandable and inevitable consequence. By the same token, it puts a severe burden on the engineer who designs these complicated systems with very stringent tolerances and little opportunity to experimentally check out his design as a whole.

Because of such demands, the *state-space approach* based on the concept of state variables offers a more convenient means of analysis and design than the straight input-output viewpoint. Here, we shall expand upon our earlier discussion of the state of a system, and show how state equations are related to input-output equations. As a preliminary, we recall that the state vector of an nth-order system is any signal vector $\mathbf{q}(t)$ with n components such that

$$\mathbf{q}(t) = \mathbf{S}_q[\mathbf{q}(t_0); \mathbf{x}(t)] \qquad t \geq t_0 \tag{1a}$$

and

$$\mathbf{y}(t) = \mathbf{S}[\mathbf{q}(t_0); \mathbf{x}(t)] \qquad t \geq t_0 \tag{1b}$$

which are the state and output equations, respectively.

Single-input-output systems

Consider an nth-order linear system (not necessarily fixed) with one input $x(t)$ and one output $y(t)$, whose differential equation is

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) = \sum_{k=0}^{m} b_k(t) x^{(k)}(t) \qquad m \leq n-1 \tag{2}$$

We are including the time-varying case by showing the coefficients as functions of time; if the system in question is time-invariant then the coefficients are constants.

Introducing the state variables $q_1(t), q_2(t), \ldots q_n(t)$, the system can be described by n first-order state equations of the general form

$$\begin{aligned} \dot{q}_1 &= A_{11}q_1 + A_{12}q_2 + \cdots + A_{1n}q_n + B_1 x \\ \dot{q}_2 &= A_{21}q_1 + A_{22}q_2 + \cdots + A_{2n}q_n + B_2 x \\ &\vdots \\ \dot{q}_n &= A_{n1}q_1 + A_{n2}q_2 + \cdots + A_{nn}q_n + B_n x \end{aligned} \tag{3a}$$

plus an output equation†

$$y = C_1 q_1 + C_2 q_2 + \cdots + C_n q_n \tag{3b}$$

where the new coefficients are related to the system parameters and may be time-dependent if the system is not fixed.

It is not obvious that the state variables in (3) do, in truth, have the properties required by (1). Verifying this fact would entail solving the equations to obtain the explicit forms $\mathbf{q}(t) = \mathbf{S}_q[\mathbf{q}(t_0); x(t)]$ and $y(t) = S[\mathbf{q}(t_0); x(t)]$. These solutions are not attempted here since they go beyond our intended scope. Suffice it to say that equations like (3) almost always are valid state equations.

To write (3) more compactly we use the column vector

$$\mathbf{q}(t) = [q_1(t) \quad q_2(t) \cdots q_n(t)]^T$$

and define the $n \times n$ *matrix* consisting of the coefficients $A_{ij}(t)$, namely

$$\mathbf{A}(t) \triangleq \begin{bmatrix} A_{11}(t) & A_{12}(t) & \cdots & A_{1n}(t) \\ A_{21}(t) & A_{22}(t) & \cdots & A_{2n}(t) \\ \vdots & \vdots & & \vdots \\ A_{n1}(t) & A_{n2}(t) & \cdots & A_{nn}(t) \end{bmatrix} \tag{4}$$

We also need two $n \times 1$ matrices for the B and C coefficients, say

$$\begin{aligned} \mathbf{b}(t) &\triangleq [B_1(t) \quad B_2(t) \quad \cdots \quad B_n(t)]^T \\ \mathbf{c}(t) &\triangleq [C_1(t) \quad C_2(t) \quad \cdots \quad C_n(t)]^T \end{aligned} \tag{5}$$

With these definitions, the set of state equations becomes the matrix equation

$$\dot{\mathbf{q}}(t) = \mathbf{A}(t)\mathbf{q}(t) + \mathbf{b}(t)x(t) \tag{6a}$$

while the output equation is

$$y(t) = \mathbf{c}^T(t)\mathbf{q}(t) \tag{6b}$$

where $\mathbf{c}^T(t)$, being the transpose of an $n \times 1$ column matrix, is a $1 \times n$ row matrix. If the system is fixed, then **A**, **b**, and **c** are simply constant matrices. But fixed or time-varying, (6) succinctly constitutes a state-space model of the system.

†For $m > n-1$ or some selections of the state variables, the output equation also must include $x(t)$.

Having thus disposed of notational liabilities, one may still ask if the state-space description is worth the effort. Indeed, from a quick comparison of (2) and (6), it might appear that the latter has little to offer in the way of advantages, and has several seeming disadvantages: Where, for instance, do the state variables come from? And why bother with n simultaneous equations plus an output equation when the same information is contained in one nth-order equation?

Taking these questions one at a time, we have already shown that state variables often arise quite naturally in the course of modeling a system from its physical description. In fact, it usually takes additional labor to get a single higher-order equation in terms of just the input and output. But, as previously mentioned, the state vector is not unique; any $\mathbf{q}(t)$ satisfying (1) qualifies as a state vector. This means that we have considerable latitude in choosing the state variables, and that they need not even correspond to physical variables in the system – although that is perhaps the most appealing choice.

Emphasizing this point, suppose all we have is the differential equation (2) instead of a physical description of the system. Several systematic procedures exist for obtaining state equations therefrom, one being based on the canonic simulation, Figure 3.12; that diagram is repeated here as Figure 3.14. Since the input to each integrator is the first derivative of its output, we can immediately write down a set of state equations by taking the state variables as those *integrator outputs*, i.e.,

$$\dot{q}_1 = q_2 \quad \dot{q}_2 = q_3 \quad \cdots \quad \dot{q}_{n-1} = q_n$$

$$\dot{q}_n = -a_0 q_1 - a_1 q_2 - \cdots - a_{n-1} q_n + x$$

Also by inspection of the diagram, the output equation is

$$y = b_0 q_1 + b_1 q_2 + \cdots + b_{n-1} q_n.$$

Thus, for this choice of state variables, the coefficient matrices directly display the coefficients of the differential equation:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 \cdots & -a_{n-1} \end{bmatrix} \tag{7a}$$

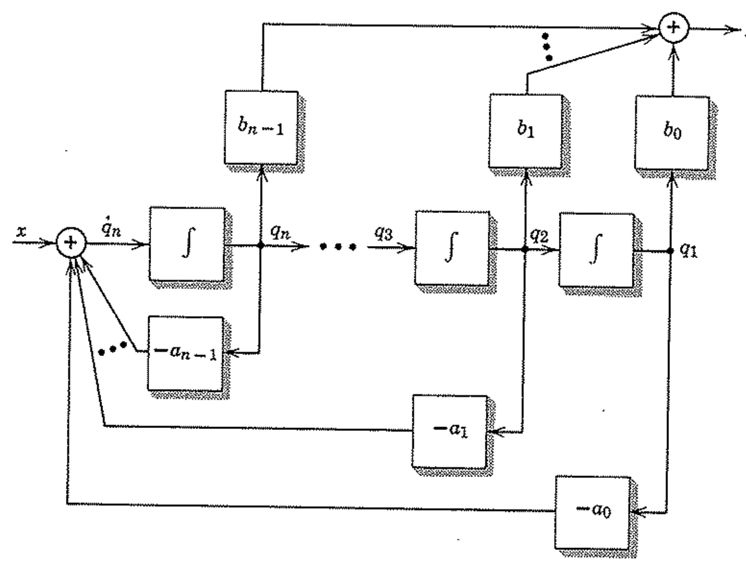

Figure 3.14.

$$\mathbf{b} = [0 \quad 0 \cdots 0 \quad 1]^T \tag{7b}$$

$$\mathbf{c} = [b_0 \quad b_1 \cdots b_{n-1}]^T \tag{7c}$$

which, when inserted in (6), constitutes the *canonic* state-variable model of an SIO system.

Granted that the state equations may be easier to formulate than the corresponding nth-order equation, this in itself is only a minor benefit. A major benefit of the state-space approach is the conceptual clarity and notational convenience when dealing with MIO systems, a subject we shall turn to momentarily. Another important benefit is the knowledge one can gain about what is going on *inside* the system. This aspect is illustrated by the following example.

Example 3.3

Suppose a physical system corresponds to the simulation diagrammed in Figure 3.15. With the state variables as indicated

$$\dot{q}_1 = q_1 + x \qquad \dot{q}_2 = q_1 - 3q_2$$

and

$$y = -\tfrac{1}{4} q_1 + q_2$$

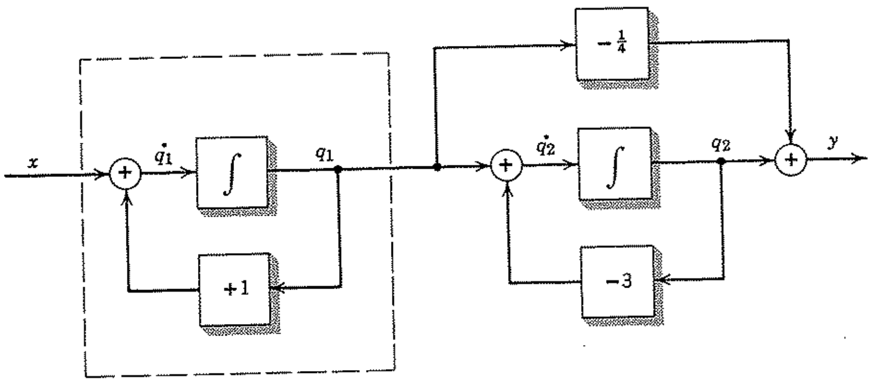

Figure 3.15.

Thus, upon eliminating q_1 and q_2, the input-output relation is

$$\dot{y}+3y=-\tfrac{1}{4}x$$

Judging only from the latter, one would conclude that the system is *first-order* and *asymptotically stable* — as follows from the characteristic equation $s+3=0$. But the simulation has two integrators, implying a *second-order* system. Moreover, the first state equation can be recast as $\dot{q}_1-q_1=x$, which corresponds to an RHP root, namely at $s=+1$; thus, the q_1 mode grows as e^{+t} and the subsystem inside the dashed lines is, by itself, *unstable*.

These seemingly contradicting conclusions are perfectly correct. The contradiction is resolved with further study of the diagram, for such a study will reveal that the unstable behavior is cancelled at the output summing junction and does not appear in the output.† (The reader should verify this for himself by taking $q_1=e^t$ and solving for y.) Consequently, insofar as input-output is concerned, the total system acts as if it were first-order and stable.

Nonetheless, the internal variable q_1 may very well increase without bound while x and y remain finite. Furthermore, a slight change in any of the parameters voids the cancellation effect and the total system becomes unstable. Clearly, the engineer ought to be aware of these potentially grave circumstances. Equally clear, the necessary information may be missing from the input-output equation and contained only in the state equations.

†We therefore say that q_1 is an *unobservable* state. See Zadeh and Desoer (1963, Chap. 11).

Multi-input-output systems

Consider an nth-order linear system with p input signals and r output signals. To formulate the state equations we observe that, paralleling (3a), every element of $\dot{\mathbf{q}}(t)$ will be a linear combination of all the state variables plus the p input signals, i.e.,

$$\dot{q}_i=A_{i1}q_1+\cdots+A_{in}q_n+B_{i1}x_1+\cdots+B_{ip}x_p$$

Similarly, there are r output equations of the form

$$y_i=C_{i1}q_1+\cdots+C_{in}q_n$$

Expressing these in matrix notation we define an $n\times p$ matrix for the B coefficients

$$\mathbf{B}(t)\triangleq\begin{bmatrix}B_{11}(t) & B_{12}(t) & \cdots & B_{1p}(t)\\ \vdots & \vdots & & \vdots\\ B_{n1}(t) & B_{n2}(t) & \cdots & B_{np}(t)\end{bmatrix}\qquad(8a)$$

and an $r\times n$ matrix for the C coefficients

$$\mathbf{C}(t)\triangleq\begin{bmatrix}C_{11}(t) & C_{12}(t) & \cdots & C_{1n}(t)\\ \vdots & \vdots & & \vdots\\ C_{r1}(t) & C_{r2}(t) & \cdots & C_{rn}(t)\end{bmatrix}\qquad(8b)$$

which correspond to $\mathbf{b}(t)$ and $\mathbf{c}^T(t)$ in the SIO case. Then, with $\mathbf{x}(t)$ and $\mathbf{y}(t)$ as the input and output signal vectors, we have the state-space matrix equations

$$\dot{\mathbf{q}}(t)=\mathbf{A}(t)\mathbf{q}(t)+\mathbf{B}(t)\mathbf{x}(t)\qquad(9a)$$

$$\mathbf{y}(t)=\mathbf{C}(t)\mathbf{q}(t)\qquad(9b)$$

where $\mathbf{q}(t)$ is the state vector and $\mathbf{A}(t)$ is as previously defined by (4). Corresponding to these equations, one can draw a *matrix block diagram* of the system, Figure 3.16, where double flow lines mean vector signals and boldface symbols are matrix operations.

Probably the most striking feature of (9) or Figure 3.16 is that it is not appreciably more complicated than a *first-order SIO* system. The addition of multiple inputs or outputs changes the *degree* of the problem but not the basic solution methods, all other factors being equal. In witness of this assertion, let us outline how one could tackle the zero-input response of a linear, time-invariant MIO system. The details and the complete response will be discussed in Chapter 8.

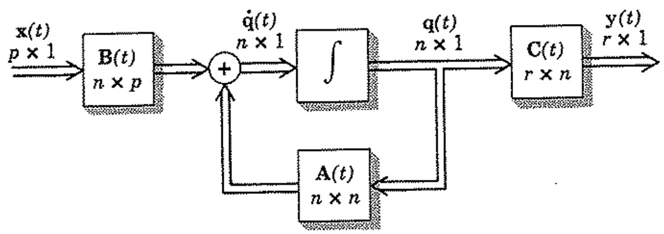

Figure 3.16. Matrix block diagram for an MIO system.

Setting $\mathbf{x}(t)=\mathbf{0}$, $\mathbf{A}(t)=\mathbf{A}$ and $\mathbf{C}(t)=\mathbf{C}$ in (9) gives

$$\mathbf{y}_{zi}(t)=\mathbf{C}\mathbf{q}(t) \tag{10a}$$

where $\mathbf{q}(t)$ must satisfy the homogeneous state equation

$$\dot{\mathbf{q}}(t)-\mathbf{A}\mathbf{q}(t)=\mathbf{0} \tag{10b}$$

subject to the initial condition $\mathbf{q}(t_0)$. We begin, as before, by assuming an exponential trial solution for (10b), say

$$\mathbf{q}(t)=[G_1e^{st} \quad G_2e^{st}\cdots G_ne^{st}]^T$$

in which the G's are constants. It then follows that $\dot{\mathbf{q}}(t)=s\mathbf{q}(t)$ and, hence, (10b) becomes

$$(s\mathbf{I}-\mathbf{A})\mathbf{q}(t)=\mathbf{0} \tag{11}$$

where $\mathbf{I}$, the $n\times n$ *identity matrix*, has been introduced to permit factoring.

Discarding the trivial case $\mathbf{q}(t)=\mathbf{0}$, matrix theory says that (11) requires the *determinant* of $(s\mathbf{I}-\mathbf{A})$ to vanish, i.e.,

$$|s\mathbf{I}-\mathbf{A}|=0 \tag{12}$$

and it can be shown that $|s\mathbf{I}-\mathbf{A}|$ is an nth-order polynomial in s having n roots, $s=p_1, p_2,\ldots p_n$. Accordingly, if the roots are distinct, our trial solution must be modified such that each element of $\mathbf{q}(t)$ takes the form

$$q_i(t)=G_{i1}e^{p_1t}+G_{i2}e^{p_2t}+\cdots+G_{in}e^{p_nt}$$

Then, in principle, the G_{ij} can be evaluated from the initial conditions and $\mathbf{q}(t)$ inserted into (10a) to yield $\mathbf{y}_{zi}(t)$. However, that is not the real point of our investigation, since a better method for proceeding from (12) will be given in Chapter 8.

The important point to notice here is that (12) plays the role of the

characteristic equation, i.e., from it one can find the values of s such that the trial solution satisfies the differential equation. Thus, we shall *define* $|s\mathbf{I}-\mathbf{A}|$ as the *characteristic polynomial* in the MIO case. Having done that, this valuable concept and its interpretations are generalized to MIO systems, and SIO systems become a subclass thereof.

Example 3.4

As a simple check on this last point, consider a third-order SIO system described by $\dddot{y}+a_2\ddot{y}+a_1\dot{y}+a_0y=F_x$. Earlier we said its characteristic polynomial is $s^3+a_2s^2+a_1s+a_0$, which we now compare with the generalized definition $|s\mathbf{I}-\mathbf{A}|$.

Taking the canonic state-variable model, the $\mathbf{A}$ matrix is given by (7a) as

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}$$

Hence,

$$|s\mathbf{I}-\mathbf{A}|=\left|\begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\right|$$

$$=\begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ a_0 & a_1 & s+a_2\end{vmatrix}=s^3+a_2s^2+a_1s+a_0$$

so we have perfect agreement. Of course, the choice of $\mathbf{A}$ is not sacred since the state vector is not unique; the interesting implication is that, for a given system, $|s\mathbf{I}-\mathbf{A}|$ will be the same with *any* valid $\mathbf{A}$.

Advantages of the state-space approach

To summarize, the state-space approach has at least three major advantages compared to straight input-output analysis: conceptual clarity, greater information about the system itself, and computational convenience.

From the conceptual viewpoint, the state equations provide a mathematical model of great generality that is readily extended to include time-varying, nonlinear, and MIO systems. Similarly, the matrix notation is a compact vehicle for analytic manipulation, particularly when the powerful techniques of linear algebra are brought to bear on the problem. As a matter of fact, without these features the main results of modern systems theory would have been quite difficult to obtain.

When the state variables are chosen as directly related to internal variables of the system, it is clear that we will get more information about the system itself than we do just from input-output considerations, and sometimes that additional information is critical. But regardless of how the state variables are selected, the state equations further enhance our intuitive grasp of the problem.

Finally, in the case of very complex systems – e.g., large n, nonlinearities, many inputs or outputs, etc. – computer-aided solution is almost always a necessity. For this purpose, the state equations are far better suited to analog simulation or digital computation than an nth-order differential equation, whether or not one is also interested in the state variables themselves.

Having thus expounded upon the merits of state-space analysis, the reader may find it puzzling that the method is not fully detailed in later chapters. There are two reasons for our stopping short on the subject. First, solving the state equations requires a knowledge of matrix theory above the level assumed in this book. Second, the real payoff of the state-space approach comes when one is faced with highly complex systems – whereas we shall have our hands full just investigating rather simple systems with a reasonable degree of thoroughness. However, from time to time we shall invoke the state concept as an aid to interpretation and understanding.

Problems

3.1 Verify that Eq. (13), Sect. 3.1, gives the specified values for y and $\dot{y}$ at $t = t_0$.

3.2 Show that Eq. (15), Sect. 3.1, is a solution of Eq. (7) when the roots are repeated, and confirm that Eq. (16) yields the specified initial values.

3.3 Suppose the roots of a second-order system are nearly, but not exactly equal, so $p_1 = p + \epsilon$ and $p_2 = p - \epsilon$ where $|\epsilon|^2 << |p|^2$. Show that

$$y(t) \approx (B_1 + B_2 \epsilon t) e^{pt}$$

3.4 The characteristic equation of a certain system is $(s+1)(s^2+9) = 0$. Solve for $y_{zi}(t)$, $t \geq 0$, if $y_0 = 10$ and $\dot{y}_0 = \ddot{y}_0 = 0$ at $t = 0$. *Answer*: $9e^{-t} + \cos 3t + 3 \sin 3t$.

3.5 Consider a third-order system having $s^3 + s^2 + \frac{5}{4}s = 0$. Find and sketch the zero-input response when $\dot{y}_0 = 2$ and $y_0 = \ddot{y}_0 = 0$.

3.6 Make a sketch similar to Figure 3.3 for the mode functions corresponding to a repeated complex root.

3.7 Consider a fourth-order system having $(s^2 + s + 2)^2 = 0$. Plot the roots in the s-plane and sketch a typical zero-input response.

3.8 Express $|A|$ and arg $[A]$ in Eq. (5), Sect. 3.2, in terms of the initial conditions $y(t_0)$ and $\dot{y}(t_0)$.

3.9 Classify the stability of the zero-input response for each of the following characteristic polynomials: (a) $s^2 + 3s$; (b) $(s^2 + 3s)^2$; (c) $s^2 + 7s + 10$; (d) $s^2 + 2s + 5$; (e) $s^2 - 2s + 5$.

3.10 For a second-order system with repeated roots, $p_1 = p_2 = p$, assume the trial solution $y(t) = \alpha_1(t) e^{pt} + \alpha_2(t) t e^{pt}$ to derive the complete response

$$y(t) = \int_{t_0}^{t} F_x(\lambda)(t-\lambda) e^{p(t-\lambda)} d\lambda + [\alpha_1(t_0) e^{pt} + \alpha_2(t_0) t e^{pt}]$$

3.11 Rewrite $y_{zs}(t)$ in Example 3.2 in terms of σ and ω when $p_1, p_2 = \sigma \pm j\omega$. Simplify your result so that it contains no imaginary quantities.

3.12 Using Eq. (16), Sect. 3.3, obtain an integral expression for $y_{zs}(t)$ given that $\dddot{y} + 5\ddot{y} + 11\dot{y} + 15y = \ddot{x} + 15x$. *Hint*: one of the roots is -3.

3.13 Suppose the RC circuit in Example 2.4 has $y(t) = y_0$ at $t = 0$ and

$$x(t) = \begin{cases} -y_0 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

where $T >> RC$. Find and sketch $y(t)$, identifying the transient and steady-state components.

3.14 Draw a simulation diagram for the system in Problem 3.12.

3.15 Draw a simulation diagram for Eqs. (17) and (18), Sect. 2.3.

3.16 Find the differential equation relating x and y from the simulation diagram of Figure P3.1. *Answer*: $\dddot{y} + 3\ddot{y} + y = 2\ddot{x} + \dot{x} + 5x$.

3.17 Devise a simulation diagram for the circuit in Figure P3.2 such that each circuit element is represented by one and only one scalor. (These are called *isolated-parameter* simulations, and have obvious advantages for experi-