consist of the three exponential terms e^{-t}, e^{-2t}, and e^{-3t}, the coefficients of which are found from a partial-fraction expansion of $Y_{zs}(s)$. Performing the calculations, we find that for $t > 0$

$$y_{zs}(t) = -\tfrac{1}{2}e^{-t} + 3e^{-2t} - \tfrac{5}{2}e^{-3t}$$

which can be added to the previously calculated zero-input response to obtain the complete response $y(t)$.

8.4 STATE-VARIABLE EQUATIONS

Although the state-variable equations and the structure of the zero-input response of multi-input-output (MIO) systems were derived in Chapter 3, no attempt was made to solve these equations in any systematic fashion. However, it is possible to use Laplace transforms, coupled with the results of our study of single-input-output (SIO) systems, to provide insight into both the methods of solution and some of the general properties of such systems. Bear in mind that a definitive study of the relevant theory requires an understanding of linear algebra, e.g., eigenvectors, functions of a matrix, etc., none of which will be attempted here. Furthermore, in practice, calculations are usually carried out on a digital computer because the use of such methods is warranted only when studying systems of at least moderate complexity.

Transfer-function matrix

The Laplace transform of a vector function of time can be defined to be the vector whose elements are the transforms of the elements of the time-function vector.† For example, the n-vector $\mathbf{q}(t)$ and its transform, the vector $\mathbf{Q}(s) \triangleq \mathcal{L}[\mathbf{q}(t)]$, form the transform pair $\mathbf{q}(t) \leftrightarrow \mathbf{Q}(s)$, which is a shorthand notation for the relationship

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \leftrightarrow \begin{bmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \\ Q_n(s) \end{bmatrix} \tag{1}$$

†Boldface capital letters will be used to denote the Laplace transforms of both vectors and matrices. The argument associated with the transform will distinguish it from a constant matrix.

These vector transforms will be applied to the state-variable equations of a fixed MIO system, namely the *state equation*

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{x}(t) \tag{2}$$

and the *output equation*

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) \tag{3}$$

where $\mathbf{q}$ is the $n \times 1$ state vector, $\mathbf{x}$ is the $p \times 1$ input vector, $\mathbf{y}$ is the $r \times 1$ output vector, and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are constant matrices of appropriate dimension.

Using Eq. (8), Sect. 8.1, with (1) it follows that $\mathcal{L}[\dot{\mathbf{q}}(t)] = s\mathbf{Q}(s) - \mathbf{q}(0)$, where $\mathbf{q}(0)$ is the initial state of the system and is also an $n \times 1$ vector. Therefore, transforming each term in (2) yields the following *algebraic* equation for the transform of the state-vector:

$$s\mathbf{Q}(s) - \mathbf{q}(0) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}\mathbf{X}(s) \tag{4}$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)]$. The transform of the zero-state response vector $\mathbf{q}_{zs}(t)$ can be found by setting $\mathbf{q}(0) = \mathbf{0}$ in (4) and collecting the terms involving $\mathbf{Q}_{zs}(s) \triangleq \mathcal{L}[\mathbf{q}_{zs}(t)]$, giving

$$s\mathbf{Q}_{zs}(s) - \mathbf{A}\mathbf{Q}_{zs}(s) = \mathbf{B}\mathbf{X}(s)$$

Introducing the $n \times n$ identity matrix $\mathbf{I}$ in order to combine the terms involving $\mathbf{Q}_{zs}$, we have

$$(s\mathbf{I} - \mathbf{A})\mathbf{Q}_{zs}(s) = \mathbf{B}\mathbf{X}(s) \tag{5}$$

which, when premultiplied by the inverse† of $(s\mathbf{I} - \mathbf{A})$, results in

$$\mathbf{Q}_{zs}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{X}(s) \tag{6}$$

The transform of the zero-state response of the output vector $\mathbf{y}_{zs}(t)$ is obtained by transforming the output equation (3) and substituting (6) to yield

$$\mathbf{Y}_{zs}(s) = \mathbf{C}\overbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{X}(s)}^{\mathbf{Q}_{zs}(s)} \tag{7}$$

Recalling that the scalar transfer function $H(s)$ was defined so as to satisfy $Y_{zs}(s) = H(s)X(s)$ for SIO systems, it is reasonable to define the

†The temptation to divide matrices rather than use the inverse operator must be resisted.

transfer-function matrix as

$$\mathbf{H}(s) \triangleq \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \tag{8}$$

which has r rows and p columns and depends only upon the system matrices **A**, **B**, and **C**. Having made this definition, (7) becomes

$$\mathbf{Y}_{zs}(s) = \mathbf{H}(s)\mathbf{X}(s) \tag{9}$$

Characteristic values and poles

The inverse of the matrix $(s\mathbf{I}-\mathbf{A})$ used in forming $\mathbf{H}(s)$ in (8) can be written in terms of its adjoint matrix – not to be confused with the "adjoint system" of optimal-control theory – and its determinant as

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\text{Adj}\,[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|} \tag{10}$$

The denominator $|s\mathbf{I}-\mathbf{A}|$ will be recognized as the system's characteristic polynomial, first introduced in Section 3.5 and its roots are known as the characteristic values (eigenvalues) of the matrix **A**. Because $(s\mathbf{I}-\mathbf{A})$ has the rather special form

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s-a_{nn} \end{bmatrix} \tag{11}$$

it follows that Adj $[s\mathbf{I}-\mathbf{A}]$ will be an $n\times n$ matrix, each element of which is a polynomial in s of degree $n-1$ or less. Hence, the n^2 elements of $(s\mathbf{I}-\mathbf{A})^{-1}$ will be rational functions of s, each having the nth-degree polynomial $|s\mathbf{I}-\mathbf{A}|$ as its denominator.

Because the matrices **B** and **C** do not depend upon the variable s, the $r\times p$ elements of the transfer-function matrix $\mathbf{H}(s)$ will also be functions of s, each having the polynomial $|s\mathbf{I}-\mathbf{A}|$ as its denominator – at least before any factors that are common to both numerator and denominator are canceled. To be more explicit, substitution of (10) into (8) gives

$$\mathbf{H}(s) = \frac{\mathbf{C}\,\text{Adj}\,[s\mathbf{I}-\mathbf{A}]\mathbf{B}}{|s\mathbf{I}-\mathbf{A}|} \tag{12}$$

where the element $H_{ij}(s)$ is the transfer function $Y_i(s)/X_j(s)$.

Hence, all input-output transfer functions associated with the system

will have as their poles the characteristic values of **A** and these characteristic values will be the p_k in the exponential terms $e^{p_k t}$ comprising the system's response modes. If the numerator of a particular $H_{ij}(s)$ should have a root that coincides with one of the characteristic values of **A**, then the corresponding mode will not appear in the response of $y_i(t)$ to the impulsive input $x_j(t) = \delta(t)$. This fundamental relationship is just one manifestation of the fact that once the dynamics of a fixed linear system are put into state-variable form, the very powerful analytical methods of linear algebra and computational capabilities of the digital computer can be brought to bear on the problems of analysis and synthesis.

State-transition matrix

Having extended the transfer-function notion to MIO systems, we shall consider the zero-input response of the state vector, denoted by $\mathbf{q}_{zi}(t)$, which results from the initial state $\mathbf{q}(0)$. Returning to (4) and setting $\mathbf{X}(s) = \mathbf{0}$ we obtain, after a slight rearrangement of terms,

$$(s\mathbf{I}-\mathbf{A})\mathbf{Q}_{zi}(s) = \mathbf{q}(0) \tag{13}$$

where $\mathbf{Q}_{zi}(s) \triangleq \mathcal{L}[\mathbf{q}_{zi}(t)]$. As before, $\mathbf{Q}_{zi}(s)$ may be solved for by using the inverse of $(s\mathbf{I}-\mathbf{A})$, yielding

$$\mathbf{Q}_{zi}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{q}(0) \tag{14}$$

Taking the inverse Laplace transform of both sides of (14), the zero-input response vector must be expressable in the form

$$\mathbf{q}_{zi}(t) = \mathbf{F}(t)\mathbf{q}(0) \qquad t \geq 0 \tag{15}$$

where

$$\mathbf{F}(t) \triangleq \mathcal{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}] = \mathcal{L}^{-1}\left[\frac{\text{Adj}\,[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|}\right] \tag{16}$$

In words, (15) says that in the absence of any inputs the state at $t = 0$ is transformed into the state at any time $t \geq 0$ according to the matrix $\mathbf{F}(t)$ which, from (16), depends strictly on **A** and t.

Actually, (15) is part of a more general relationship in which the state at *any* time t (positive or negative) can be related to the state at any other time t_0 by

$$\mathbf{q}_{zi}(t) = \mathbf{\Phi}(t-t_0)\mathbf{q}(t_0) \qquad -\infty < t < \infty \tag{17}$$

provided that all inputs are zero during the interval between t_0 and t. The matrix function $\Phi(t)$, which provides the connection between the states at any two times, is known as the *state-transition matrix* and is equal to the exponential function $e^{\mathbf{A}t}$ which is *defined* by the infinite series

$$e^{\mathbf{A}t} \triangleq \mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2!}+\mathbf{A}^3\frac{t^3}{3!}+\cdots \tag{18}$$

The fact that

$$\Phi(t)=e^{\mathbf{A}t} \tag{19}$$

can be proved by showing that the infinite series given in (18) satisfies the matrix differential equation $\dot{\Phi}(t)=\mathbf{A}\Phi(t)$ with the initial condition $\Phi(0)=\mathbf{I}$, which implies that $\Phi(t)\mathbf{q}(0)$ must satisfy (2) with $\mathbf{x}(t)=\mathbf{0}$. [See Zadeh and Desoer (1963, Chap. 5) for the proof and for the other numerous properties of the exponential function.] The relevant point here is that the matrix $\mathbf{F}(t)$ in (15) must be the state-transition matrix $\Phi(t)$ over the interval $t \geq 0$.

Example 8.5

For illustrative purposes, we shall find the transfer function and state-transition matrix of the second-order system discussed in Example 3.3, although it is a bit like using a cannon to shoot a humming bird. Rewriting the state and output equations in matrix form, we have

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}}_{\mathbf{A}}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}}x(t)$$

$$y=\underbrace{\begin{bmatrix} -\frac{1}{4} & 1 \end{bmatrix}}_{\mathbf{C}}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

from which the matrices **A**, **B**, and **C** are readily identified.

In order to find the transfer-function matrix $\mathbf{H}(s)$ – in this case a scalar – we write

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s-1 & 0 \\ -1 & s+3 \end{bmatrix}$$

from which

$$|s\mathbf{I}-\mathbf{A}|=(s-1)(s+3)$$

and

$$\text{Adj}\,[s\mathbf{I}-\mathbf{A}]=\begin{bmatrix} s+3 & 0 \\ 1 & s-1 \end{bmatrix}$$

Substituting into (8) and using (10), the scalar transfer function is

$$\mathbf{H}(s)=\frac{\begin{bmatrix} -\frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} s+3 & 0 \\ 1 & s-1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(s-1)(s+3)}$$

$$=\frac{-\frac{1}{4}(s-1)}{(s-1)(s+3)}=-\frac{1}{4}\frac{1}{(s+3)}$$

which is in agreement with the calculations in Example 3.3. As before, we note that $\mathbf{H}(s)$ has a zero which coincides with its pole at $s=1$, thereby eliminating the mode e^t from the impulse response of the output, although it is present in the state variables q_1 and q_2.

As for the state-transition matrix, we can use (16) to get an analytical result; or, if a computer is handy and numerical values of $\Phi(t)$ computed at discrete points in time are satisfactory, (18) can be used. Pursuing the former,

$$\mathbf{F}(t)=\mathcal{L}^{-1}\left[\frac{\begin{bmatrix} s+3 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)(s+3)}\right]=\mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)(s+3)} & \frac{1}{s+3} \end{bmatrix}$$

Inverting the transform matrix term by term and noting that although $\mathbf{F}(t)$ is defined only for $t \geq 0$, $\Phi(t)=\mathbf{F}(t)$ over this interval but is defined for $-\infty < t < \infty$, it follows that

$$\Phi(t)=\begin{bmatrix} e^t & 0 \\ \frac{1}{4}(e^t-e^{-3t}) & e^{-3t} \end{bmatrix}$$

Having obtained the state-transition matrix, the zero-input response of $\mathbf{q}(t)$ can be written in terms of the initial state from (15) as

$$q_1(t)=q_1(0)e^t$$

$$q_2(t)=\tfrac{1}{4}(e^t-e^{-3t})q_1(0)+e^{-3t}q_2(0)$$

Now, employing the output equation (3),

$$y_{zi}(t)=-\tfrac{1}{4}q_1(t)+q_2(t)=[-\tfrac{1}{4}q_1(0)+q_2(0)]e^{-3t}$$

Complete response

A time-domain expression can be found for $\mathbf{q}_{zs}(t)$ by observing that the matrix $(s\mathbf{I}-\mathbf{A})^{-1}$ and the state-transition matrix form a Laplace-transform pair, i.e.,

$$\mathbf{\Phi}(t) \leftrightarrow (s\mathbf{I}-\mathbf{A})^{-1}$$

Thus the frequency-domain multiplication in (6) becomes a time-domain convolution of the matrices $\mathbf{\Phi}(t)$ and $\mathbf{Bx}(t)$, which can be written as

$$\mathbf{q}_{zs}(t) = \int_0^t \mathbf{\Phi}(t-\lambda)\mathbf{Bx}(\lambda)\,d\lambda \tag{20}$$

where the three matrices must appear in the sequence shown. Although the notation used in (20) may appear to be rather overpowering at first glance, the equation merely says that any element of the vector $\mathbf{q}_{zs}(t)$ is the integral of the corresponding row of the matrix product $\mathbf{\Phi}(t-\lambda)\mathbf{Bx}(\lambda)$.

When $\mathbf{q}_{zs}(t)$ is added to the zero-input response of the state vector as given by (15), and the output equation (3) is included, we have the complete response of both the state and output vectors:

$$\mathbf{q}(t) = \mathbf{\Phi}(t)\mathbf{q}(0) + \int_0^t \mathbf{\Phi}(t-\lambda)\mathbf{Bx}(\lambda)\,d\lambda \tag{21}$$

$$\mathbf{y}(t) = \mathbf{Cq}(t) \tag{22}$$

Example 8.6

To find the complete response of the system discussed in Example 8.5 to a unit step with an arbitrary initial state, we let $\mathbf{x}(t)$ be the scalar $u(t)$. Substituting the appropriate matrices from the previous example into (21), the state vector is

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \overbrace{\begin{bmatrix} e^t & 0 \\ \frac{1}{4}(e^t - e^{-3t}) & e^{-3t} \end{bmatrix}}^{\mathbf{\Phi}(t)} \begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix}$$

$$+ \int_0^t \overbrace{\begin{bmatrix} e^{(t-\lambda)} & 0 \\ \frac{1}{4}(e^{(t-\lambda)} - e^{-3(t-\lambda)}) & e^{-3(t-\lambda)} \end{bmatrix}}^{\mathbf{\Phi}(t-\lambda)} \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\mathbf{B}} u(\lambda)\,d\lambda$$

In component form, the first row reduces to

$$q_1(t) = q_1(0)e^t + e^t \int_0^t e^{-\lambda}d\lambda = e^t[q_1(0) + 1 - e^{-t}]$$

and $q_2(t)$ may be found in a similar manner, the detailed calculations being omitted. Substituting q_1 and q_2 into the output equation, the complete response is found to be

$$y(t) = \underbrace{[-\tfrac{1}{4}q_1(0) + q_2(0)]e^{-3t}}_{y_{zi}(t)} + \underbrace{\tfrac{1}{12}(e^{-3t} - 1)}_{y_{zs}(t)}$$

where the zero-input and zero-state components are readily identified. We notice that whenever the output variable is computed, the mode e^t never appears, in keeping with the fact that that particular mode is unobservable.

Problems

8.1 Derive the following transform relationships from Section 8.1: (a) Superposition, Eq. (5); (b) Scale change, Eq. (6); (c) Integration, Eq. (7).

8.2 Derive the transforms of the causal functions $\sin \omega t$ and $\cos \omega t$ as given by Eqs. (17) and (18), Sect. 8.1.

8.3 Prove the initial-value theorem.

8.4 (a) Derive the transform relationships

$$e^{at}v(t) \leftrightarrow V(s-a)$$

$$tv(t) \leftrightarrow -\frac{dV(s)}{ds}$$

(b) Derive the transform pairs

$$t \sin \omega t \leftrightarrow \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$t^2 e^{-\alpha t} \leftrightarrow \frac{2}{(s+\alpha)^3}$$

8.5 Use Laplace transforms to derive the following expressions for the zero-state response of a second-order system: (a) Eq. (9b), Sect. 3.3; (b) Eq. (11), Sect. 3.3. In both cases, take $t_0 = 0$.