1

Introduction

Classical control system design is generally a trial-and-error process in which various methods of analysis are used iteratively to determine the design parameters of an "acceptable" system. Acceptable performance is generally defined in terms of time and frequency domain criteria such as rise time, settling time, peak overshoot, gain and phase margin, and bandwidth. Radically different performance criteria must be satisfied, however, by the complex, multiple-input, multiple-output systems required to meet the demands of modern technology. For example, the design of a spacecraft attitude control system that minimizes fuel expenditure is not amenable to solution by classical methods. A new and direct approach to the synthesis of these complex systems, called optimal control theory, has been made feasible by the development of the digital computer.

The objective of optimal control theory is *to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion.* Later, we shall give a more explicit mathematical statement of "the optimal control problem," but first let us consider the matter of problem formulation.

1.1 PROBLEM FORMULATION

The axiom "A problem well put is a problem half solved" may be a slight exaggeration, but its intent is nonetheless appropriate. In this section, we shall review the important aspects of problem formulation, and introduce the notation and nomenclature to be used in the following chapters.

The formulation of an optimal control problem requires:

1. A mathematical description (or model) of the process to be controlled.
2. A statement of the physical constraints.
3. Specification of a performance criterion.

The Mathematical Model

A nontrivial part of any control problem is modeling the process. The objective is to obtain the simplest mathematical description that adequately predicts the response of the physical system to all anticipated inputs. Our discussion will be restricted to systems described by ordinary differential equations (in state variable form).† Thus, if

$$x_1(t), x_2(t), \ldots, x_n(t)$$

are the *state variables* (or simply the *states*) of the process at time t, and

$$u_1(t), u_2(t), \ldots, u_m(t)$$

are *control inputs* to the process at time t, then the system may be described by n first-order differential equations

$$\begin{aligned}
\dot{x}_1(t) &= a_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)\\
\dot{x}_2(t) &= a_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)\\
&\;\;\vdots\\
\dot{x}_n(t) &= a_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t).‡
\end{aligned} \qquad (1.1\text{-}1)$$

We shall define

$$\mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

as the *state vector* of the system, and

† The reader will find the concepts much the same for discrete systems (see [A-1]).
‡ Note that $\dot{x}_i(t)$ is in general a nonlinear time-varying function a_i of the states, the control inputs, and time.

$$\mathbf{u}(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_m(t) \end{bmatrix}$$

as the *control vector*. The state equations can then be written

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t), \qquad (1.1\text{-}1a)$$

where the definition of **a** is apparent by comparison with (1.1-1).

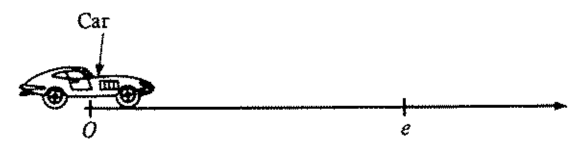

Figure 1-1 A simplified control problem

Example 1.1-1. The car shown parked in Fig. 1-1 is to be driven in a straight line away from point O. The distance of the car from O at time t is denoted by $d(t)$. To simplify the model, let us approximate the car by a unit point mass that can be accelerated by using the throttle or decelerated by using the brake. The differential equation is

$$\ddot{d}(t) = \alpha(t) + \beta(t), \qquad (1.1\text{-}2)$$

where the control α is throttle acceleration and $-\beta$ is braking deceleration. Selecting position and velocity as state variables, that is,

$$x_1(t) \triangleq d(t) \quad \text{and} \quad x_2(t) \triangleq \dot{d}(t),$$

and letting

$$u_1(t) \triangleq \alpha(t) \quad \text{and} \quad u_2(t) \triangleq \beta(t),$$

we find that the state equations become

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= u_1(t) + u_2(t),
\end{aligned} \qquad (1.1\text{-}3)$$

or, using matrix notation,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{u}(t). \qquad (1.1\text{-}3a)$$

This is the mathematical model of the process in state form.

Before we move on to the matter of physical constraints, let us consider two definitions that will be useful later. Let the system be described by Eq. (1.1-1a) for $t \in [t_0, t_f]$.†

DEFINITION 1-1

A history of control input values during the interval $[t_0, t_f]$ is denoted by $\mathbf{u}$ and is called a *control history*, or simply a *control*.

DEFINITION 1-2

A history of state values in the interval $[t_0, t_f]$ is called a *state trajectory* and is denoted by $\mathbf{x}$.

The terms "history," "curve," "function," and "trajectory" will be used interchangeably. It is most important to keep in mind the difference between a *function* and *the value of a function*. Figure 1-2 shows a single-valued function of time which is denoted by x. The *value* of the function at time t_1 is denoted by $x(t_1)$.

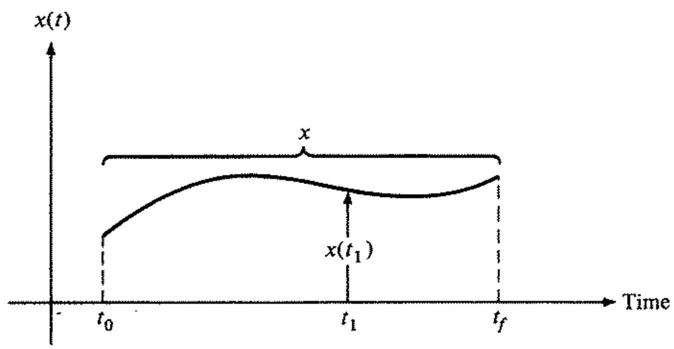

Figure 1-2 A function, x, and its value at time t_1, $x(t_1)$

Physical Constraints

After we have selected a mathematical model, the next step is to define the physical constraints on the state and control values. To illustrate some typical constraints, let us return to the automobile whose model was determined in Example 1.1-1.

Example 1.1-2. Consider the problem of driving the car in Fig. 1-1 between the points O and e. Assume that the car starts from rest and stops upon reaching point e.

† This notation means $t_0 \leq t \leq t_f$.

First let us define the state constraints. If t_0 is the time of leaving O, and t_f is the time of arrival at e, then, clearly,

$$\begin{aligned} x_1(t_0) &= 0 \\ x_1(t_f) &= e. \end{aligned} \tag{1.1-4}$$

In addition, since the automobile starts from rest and *stops* at e,

$$\begin{aligned} x_2(t_0) &= 0 \\ x_2(t_f) &= 0. \end{aligned} \tag{1.1-5}$$

In matrix notation these *boundary conditions* are

$$\mathbf{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \mathbf{x}(t_f) = \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{1.1-6}$$

If we assume that the car does not back up, then the additional constraints

$$\begin{aligned} 0 \leq x_1(t) &\leq e \\ 0 \leq x_2(t)& \end{aligned} \tag{1.1-7}$$

are also imposed.

What are the constraints on the control inputs (acceleration)? We know that the acceleration is bounded by some upper limit which depends on the capability of the engine, and that the maximum deceleration is limited by the braking system parameters. If the maximum acceleration is $M_1 > 0$, and the maximum deceleration is $M_2 > 0$, then the controls must satisfy

$$\begin{aligned} 0 \leq u_1(t) &\leq M_1 \\ -M_2 \leq u_2(t) &\leq 0. \end{aligned} \tag{1.1-8}$$

In addition, if the car starts with G gallons of gas and there are no service stations on the way, another constraint is

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]\, dt \leq G \tag{1.1-9}$$

which assumes that the rate of gas consumption is proportional to both acceleration and speed with constants of proportionality k_1 and k_2.

Now that we have an idea of typical constraints that may be encountered, let us make these concepts more precise.

DEFINITION 1-3

A control history which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an *admissible control*.

We shall denote the set of admissible controls by U, and the notation $\mathbf{u} \in U$ means that the control history $\mathbf{u}$ is admissible.

To illustrate the concept of admissibility Fig. 1-3 shows four possible acceleration histories for Example 1.1-2. $u_1^{(2)}$ and $u_1^{(4)}$ are not admissible;

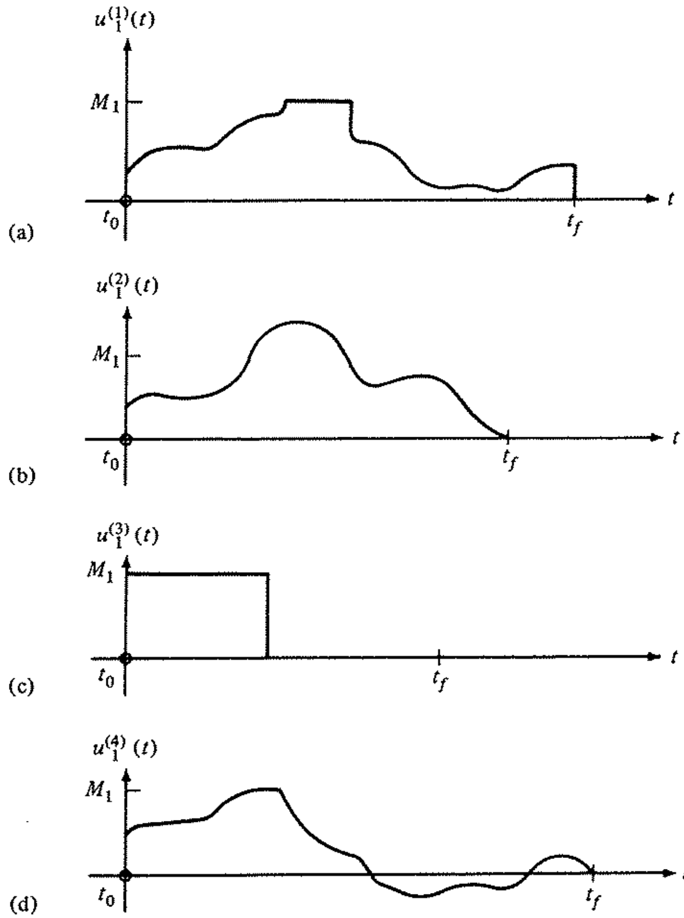

Figure 1-3 Some acceleration histories

$u_1^{(1)}$ and $u_1^{(3)}$ are admissible if they satisfy the consumed-fuel constraint of Eq. (1.1-9). In this example, the set of admissible controls U is defined by the inequalities in (1.1-8) and (1.1-9).

DEFINITION 1-4

> A state trajectory which satisfies the state variable constraints during the entire time interval $[t_0, t_f]$ is called an *admissible trajectory*.

The set of admissible state trajectories will be denoted by X, and $\mathbf{x} \in X$ means that the trajectory $\mathbf{x}$ is admissible.

In Example 1.1-2 the set of admissible state trajectories X is specified by the conditions given in Eqs. (1.1-6), (1.1-7), and (1.1-9). In general, the final state of a system will be required to lie in a specified region S of the $(n + 1)$-dimensional state-time space. We shall call S the *target set*. If the final state and the final time are fixed, then S is a point. In the automobile problem of Example 1.1-2 the target set was the line shown in Fig. 1-4(a). If the automobile had been required to arrive within three feet of e with zero terminal velocity, the target set would have been as shown in Fig. 1-4(b).

Admissibility is an important concept, because it reduces the range of values that can be assumed by the states and controls. Rather than consider all control histories and their trajectories to see which are best (according to some criterion), we investigate only those trajectories and controls that are admissible.

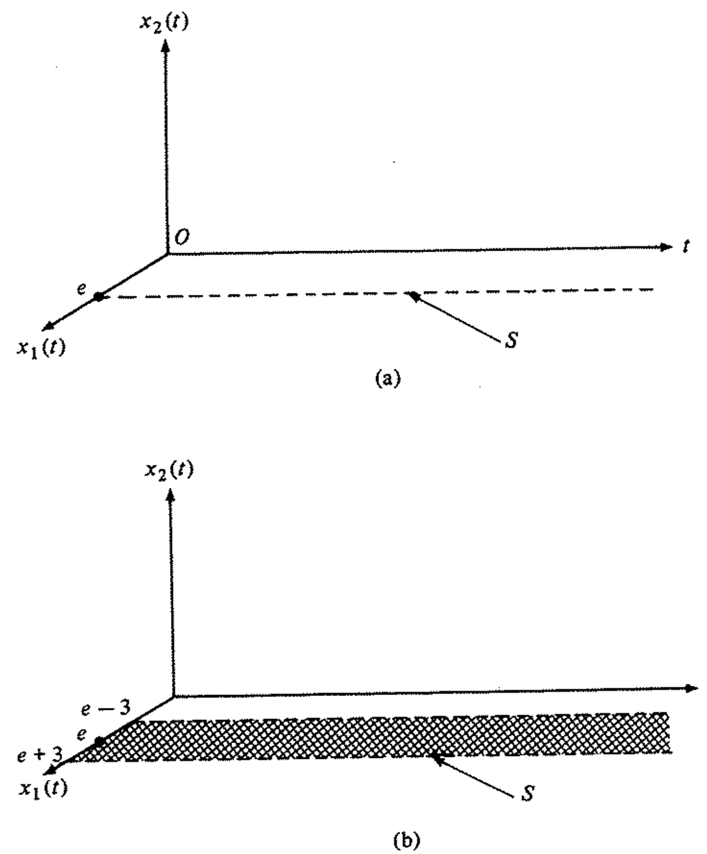

Figure 1-4 (a) The target set for Example 1.1-2. (b) The target set defined by $|x_1(t) - e| \leq 3$, $x_2(t) = 0$

The Performance Measure

In order to evaluate the performance of a system quantitatively, the designer selects a performance measure. An *optimal control* is defined as one that *minimizes* (or maximizes) the performance measure. In certain cases the problem statement may clearly indicate what to select for a performance measure, whereas in other problems the selection is a subjective matter. For example, the statement, "Transfer the system from point A to point B as quickly as possible," clearly indicates that elapsed time is the performance measure to be minimized. On the other hand, the statement, "Maintain the position and velocity of the system near zero with a small expenditure of control energy," does not instantly suggest a unique performance measure. In such problems the designer may be required to try several performance measures before selecting one which yields what he considers to be optimal performance. We shall discuss the selection of a performance measure in more detail in Chapter 2.

Example 1.1-3. Let us return to the automobile problem begun in Example 1.1-1. The state equations and physical constraints have been defined; now we turn to the selection of a performance measure. Suppose the objective is to make the car reach point e as quickly as possible; then the performance measure J is given by

$$J = t_f - t_0. \tag{1.1-10}$$

In all that follows it will be assumed that the performance of a system is evaluated by a measure of the form

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt, \tag{1.1-11}$$

where t_0 and t_f are the initial and final time; h and g are scalar functions. t_f may be specified or "free," depending on the problem statement.

Starting from the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and applying a control signal $\mathbf{u}(t)$, for $t \in [t_0, t_f]$, causes a system to follow some state trajectory; the performance measure assigns a unique real number to each trajectory of the system.

With the background material we have accumulated it is now possible to present an explicit statement of "the optimal control problem."

The Optimal Control Problem

The theory developed in the subsequent chapters is aimed at solving the following problem.

Find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.1-12}$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that *minimizes* the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt. \tag{1.1-13}$$

$\mathbf{u}^*$ is called an *optimal control* and $\mathbf{x}^*$ an *optimal trajectory*.

Several comments are in order here. First, we may not know in advance that an optimal control *exists*; that is, it may be impossible to find a control which (a) is admissible and (b) causes the system to follow an admissible trajectory. Since existence theorems are in rather short supply, we shall, in most cases, attempt to find an optimal control rather than try to prove that one exists.

Second, even if an optimal control exists, it may not be *unique*. Nonunique optimal controls may complicate computational procedures, but they do allow the possibility of choosing among several controller configurations. This is certainly helpful to the designer, because he can then consider other factors, such as cost, size, reliability, etc., which may not have been included in the performance measure.

Third, when we say that $\mathbf{u}^*$ causes the performance measure to be minimized, we mean that

$$\begin{aligned} J^* &\triangleq h(\mathbf{x}^*(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\, dt \\ &\leq h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt \end{aligned} \tag{1.1-14}$$

for all $\mathbf{u} \in U$, which make $\mathbf{x} \in X$. The above inequality states that an optimal control and its trajectory cause the performance measure to have a value smaller than (or perhaps equal to) the performance measure for *any other* admissible control and trajectory. Thus, we are seeking the *absolute* or *global minimum* of J, not merely *local minima*. Of course, one way to find the global minimum is to determine all of the local minima and then simply pick out one (or more) that yields the smallest value for the performance measure.

It may be helpful to visualize the optimization as shown in Fig. 1-5. $u^{(1)}, u^{(2)}, u^{(3)}$, and $u^{(4)}$ are "points" at which J has local, or relative, minima; $u^{(1)}$ is the "point" where J has its global, or absolute, minimum.

Finally, observe that if the objective is to maximize some measure of system performance, the theory we shall develop still applies because this

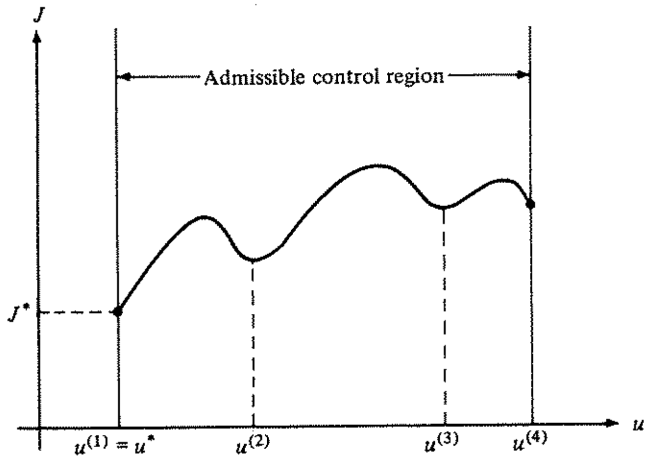

Figure 1-5 A representation of the optimization problem

is the same as minimizing the negative of this performance measure. Henceforth, we shall speak, with no lack of generality, of minimizing the performance measure.

Example 1.1-4. To illustrate a complete problem formulation, let us now summarize the results of Example 1.1-1, using the notation and definitions which have been developed.

The state equations are

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u_1(t) + u_2(t).\end{aligned} \tag{1.1-3}$$

The set of admissible states X is partially specified by the boundary conditions

$$\mathbf{x}(t_0) = \mathbf{0}, \qquad \mathbf{x}(t_f) = \begin{bmatrix} e \\ 0 \end{bmatrix}$$

and the inequalities

$$\begin{aligned}0 \le x_1(t) &\le e\\ 0 &\le x_2(t).\end{aligned} \tag{1.1-7}$$

The set of admissible controls U is partially defined by the constraints

$$\begin{aligned}0 \le u_1(t) &\le M_1\\ -M_2 \le u_2(t) &\le 0.\end{aligned} \tag{1.1-8}$$

The inequality constraint

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]\, dt \le G \tag{1.1-9}$$

completes the description of the admissible states and controls.

The solution to this problem (which is left as an exercise for the reader at the end of Chapter 5) is shown in Fig. 1-6 for the situation where $M_1 = M_2 \triangleq M$. We have also assumed that the car has enough fuel available to reach point e using the control shown.

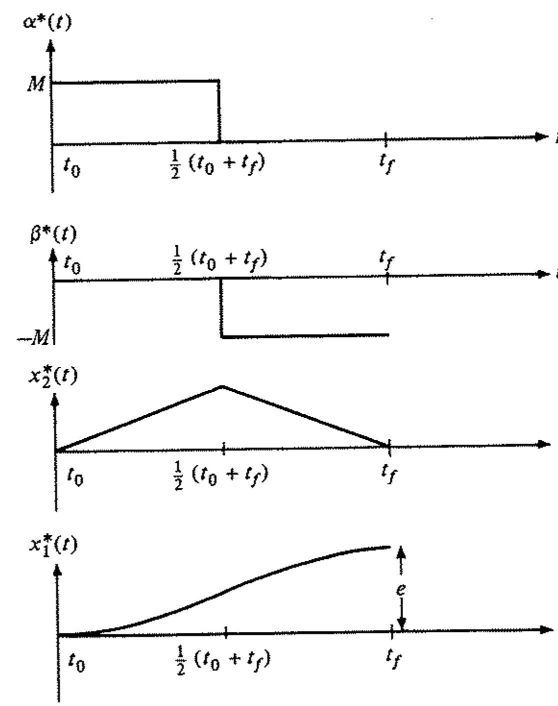

Figure 1-6 The optimal control and trajectory for the automobile problem

Example 1.1-5. Let us now consider what would happen if the preceding problem had been improperly formulated. Suppose that the control constraints had not been recognized. If we let

$$\alpha(t) + \beta(t) = e\frac{d}{dt}[\delta(t - t_0)] \tag{1.1-15}$$

where $\delta(t - t_0)$ is a unit impulse function that occurs at time t_0,† then

$$x_2(t) = e\,\delta(t - t_0) \tag{1.1-16}$$

and

$$x_1(t) = e\,\mathbb{1}(t - t_0) \tag{1.1-17}$$

[$\mathbb{1}(t - t_0)$ represents a unit step function at $t = t_0$]. Figure 1-7 shows the state trajectory which results from applying the "optimal" control in (1.1-15). Unfortunately, although the desired transfer from point O

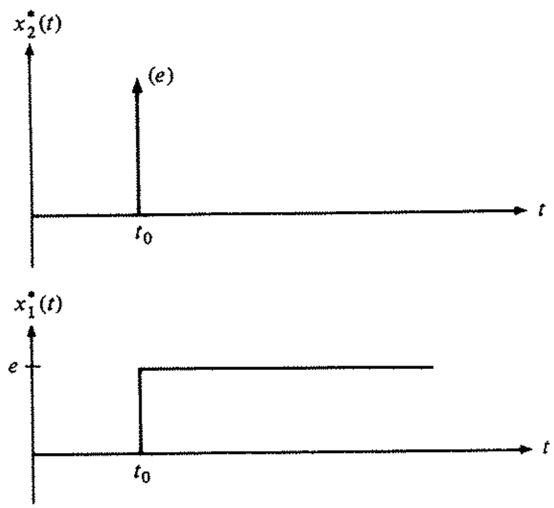

Figure 1-7 The optimal trajectory resulting from unconstrained controls

to point e is accomplished in infinitesimal time, the control required, apart from being rather unsafe, is physically impossible! Thus, we see the importance of correctly formulating problems before attempting their solution.

Form of the Optimal Control

DEFINITION 1-5

If a functional relationship of the form

$$\mathbf{u}^*(t) = \mathbf{f}(\mathbf{x}(t), t)‡ \tag{1.1-18}$$

† See reference [Z-1].

‡ Here we write $\mathbf{x}(t)$ instead of $\mathbf{x}^*(t)$ to emphasize that the control law is optimal for *all* admissible $\mathbf{x}(t)$, not just for some special state value at time t.

can be found for the optimal control at time t, then the function $\mathbf{f}$ is called the *optimal control law*, or the *optimal policy*.†

Notice that Eq. (1.1-18) implies that $\mathbf{f}$ is a rule which determines the optimal control at time t for *any* (admissible) state value at time t. For example, if

$$\mathbf{u}^*(t) = \mathbf{F}\mathbf{x}(t), \tag{1.1-19}$$

where $\mathbf{F}$ is an $m \times n$ matrix of real constants, then we would say that the optimal control law is linear, time-invariant feedback of the states.

DEFINITION 1-6

If the optimal control is determined as a function of time for a specified initial state value, that is,

$$\mathbf{u}^*(t) = \mathbf{e}(\mathbf{x}(t_0), t), \tag{1.1-20}$$

then the optimal control is said to be in *open-loop* form.

Thus the optimal open-loop control is optimal only for a *particular* initial state value, whereas, if the optimal control *law* is known, the optimal control history starting from *any* state value can be generated.

Conceptually, it is helpful to imagine the difference between an optimal control law and an open-loop optimal control as shown in Fig. 1-8; notice,

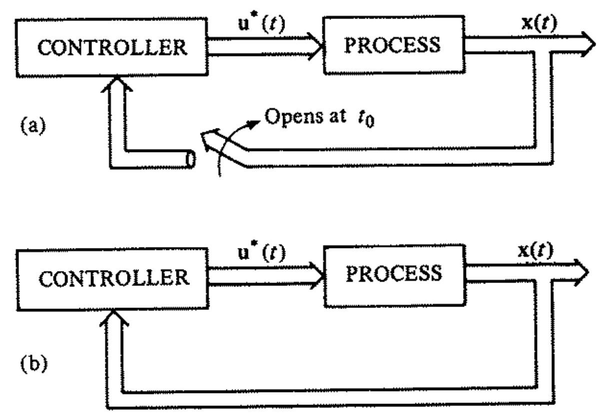

Figure 1-8 (a) Open-loop optimal control. (b) Optimal control law

however, that the mere presence of connections from the states to a controller does not, in general, guarantee an *optimal* control law.‡

† The terms *optimal feedback control*, *closed-loop optimal control*, and *optimal control strategy* are also often used.

‡ This is pursued further in reference [K-1].

Although engineers normally prefer closed-loop solutions to optimal control problems, there are cases when an open-loop control may be feasible. For example, in the radar tracking of a satellite, once the orbit is set very little can happen to cause an undesired change in the trajectory parameters. In this situation a pre-programmed control for the radar antenna might well be used.

A typical example of feedback control is in the classic servomechanism problem where the actual and desired outputs are compared and any deviation produces a control signal that attempts to reduce the discrepancy to zero.

1.2 STATE VARIABLE REPRESENTATION OF SYSTEMS

The starting point for optimal control investigations is a mathematical model in state variable form. In this section we shall summarize the results and notation to be used in the subsequent discussion. There are several excellent texts available for the reader who needs additional background material.†

Why Use State Variables?

Having the mathematical model in state variable form is convenient because

1. The differential equations are ideally suited for digital or analog solution.
2. The state form provides a unified framework for the study of nonlinear and linear systems.
3. The state variable form is invaluable in theoretical investigations.
4. The concept of state has strong physical motivation.

Definition of State of a System

When referring to the state of a system, we shall have the following definition in mind.

DEFINITION 1-7

The *state of a system* is a set of quantities $x_1(t), x_2(t), \dots, x_n(t)$

† See [D-1], [O-1], [S-1], [S-2], [T-1], [W-1], [Z-1].

which if known at $t = t_0$ are determined for $t \geq t_0$ by specifying the inputs to the system for $t \geq t_0$.

System Classification

Systems are described by the terms *linear*, *nonlinear*, *time-invariant*,† and *time-varying*. We shall classify systems according to the form of their state equations.‡ For example, if a system is *nonlinear* and *time-varying*, the state equations are written

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{1.2-1}$$

Nonlinear, time-invariant systems are represented by state equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t)). \tag{1.2-2}$$

If a system is *linear* and *time-varying* its state equations are

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{1.2-3}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $n \times n$ and $n \times m$ matrices with time-varying elements. State equations for *linear, time-invariant* systems have the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{1.2-4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices.

Output Equations

The physical quantities that can be measured are called the *outputs* and are denoted by $y_1(t), y_2(t), \dots, y_q(t)$. If the outputs are *nonlinear, time-varying* functions of the states and controls, we write the output equations

$$\mathbf{y}(t) = \mathbf{c}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{1.2-5}$$

If the output is related to the states and controls by a *linear, time-invariant* relationship, then

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{1.2-6}$$

where $\mathbf{C}$ and $\mathbf{D}$ are $q \times n$ and $q \times m$ constant matrices. A nonlinear, time-

† *Time-invariant*, *stationary*, and *fixed* will be used interchangeably.
‡ See Chapter 1 of [S-1] for an excellent discussion of system classification.

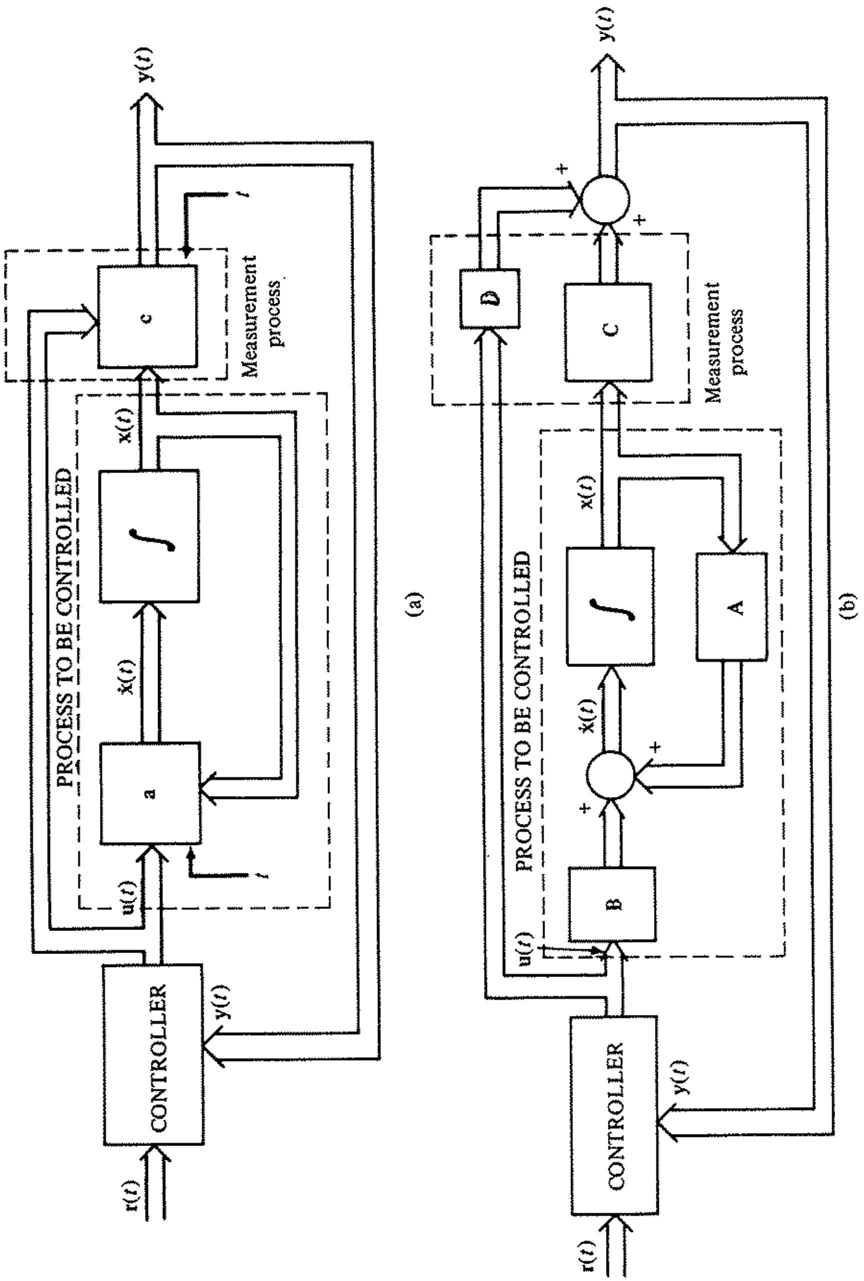

Figure 1-9 (a) Nonlinear system representation. (b) Linear system representation

varying system and a linear, time-invariant system are shown in Fig. 1-9. $\mathbf{r}(t)$, which has not been included in the state equations and represents any inputs that are not controlled, is called the *reference* or *command* input.

In our discussion of optimal control theory we shall make the simplifying assumption that the states are all available for measurement; that is, $\mathbf{y}(t) = \mathbf{x}(t)$.

Solution of the State Equations—Linear Systems

For linear systems the state equations (1.2-3) have the solution

$$\mathbf{x}(t) = \boldsymbol{\varphi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\varphi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{1.2-7}$$

where $\boldsymbol{\varphi}(t, t_0)$ is the *state transition matrix*† of the system. If the system is time-invariant as well as linear, t_0 can be set equal to 0 and the solution of the state equations is given by any of the three equivalent forms

$$\mathbf{x}(t) = \mathscr{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)\}, \tag{1.2-8a}$$

$$\mathbf{x}(t) = \mathscr{L}^{-1}\{\boldsymbol{\Phi}(s)\mathbf{x}(0) + \mathbf{H}(s)\mathbf{U}(s)\}, \tag{1.2-8b}$$

$$\mathbf{x}(t) = \epsilon^{\mathbf{A}t}\mathbf{x}(0) + \epsilon^{\mathbf{A}t}\int_{0}^{t} \epsilon^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\, d\tau, \tag{1.2-8c}$$

where $\mathbf{U}(s)$ and $\boldsymbol{\Phi}(s)$ are the Laplace transforms of $\mathbf{u}(t)$ and $\boldsymbol{\varphi}(t)$, $\mathscr{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transform of $\{\cdot\}$, and $\epsilon^{\mathbf{A}t}$ is the $n \times n$ matrix

$$\epsilon^{\mathbf{A}t} \triangleq \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots. \tag{1.2-9}$$

Equation (1.2-8a) results when the state equations (1.2-4) are Laplace transformed and solved for $\mathbf{X}(s)$. Equation (1.2-8b) can be obtained by drawing a block diagram (or signal flow graph) of the system and applying Mason's gain formula.‡ Notice that $\mathbf{H}(s)$ is the transfer function matrix. The solution in (1.2-8c) can be found by classical methods. The equivalence of these three solutions establishes the correspondences

$$\epsilon^{\mathbf{A}t} = \mathscr{L}^{-1}\{\boldsymbol{\Phi}(s)\} = \mathscr{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\} \triangleq \boldsymbol{\varphi}(t), \tag{1.2-10}$$

$$\begin{aligned}\epsilon^{\mathbf{A}t}\int_{0}^{t} \epsilon^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\, d\tau &= \mathscr{L}^{-1}\{\mathbf{H}(s)\mathbf{U}(s)\} = \mathscr{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)\} \\ &\triangleq \boldsymbol{\varphi}(t)\int_{0}^{t} \boldsymbol{\varphi}(-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau.\end{aligned} \tag{1.2-11}$$

† $\boldsymbol{\varphi}(t, t_0)$ is also called the *fundamental matrix*.
‡ See [W-1].

Properties of the State Transition Matrix

It can be verified that the state transition matrix has the properties shown in Table 1-1 for all t, t_0, t_1, and t_2.

Table 1-1 PROPERTIES OF THE LINEAR SYSTEM STATE TRANSITION MATRIX

Time-invariant systems	*Time-varying systems*
$\boldsymbol{\varphi}(0) = \mathbf{I}$	$\boldsymbol{\varphi}(t, t) = \mathbf{I}$
$\boldsymbol{\varphi}(t_2 - t_1)\boldsymbol{\varphi}(t_1 - t_0) = \boldsymbol{\varphi}(t_2 - t_0)$	$\boldsymbol{\varphi}(t_2, t_1)\boldsymbol{\varphi}(t_1, t_0) = \boldsymbol{\varphi}(t_2, t_0)$
$\boldsymbol{\varphi}^{-1}(t_2 - t_1) = \boldsymbol{\varphi}(t_1 - t_2)$	$\boldsymbol{\varphi}^{-1}(t_2, t_1) = \boldsymbol{\varphi}(t_1, t_2)$
$\frac{d}{dt}\boldsymbol{\varphi}(t) = \mathbf{A}\boldsymbol{\varphi}(t)$	$\frac{d}{dt}\boldsymbol{\varphi}(t, t_0) = \mathbf{A}(t)\boldsymbol{\varphi}(t, t_0)$

Determination of the State Transition Matrix

For systems having a constant $\mathbf{A}$ matrix, the state transition matrix, $\boldsymbol{\varphi}(t)$, can be determined by any of the following methods:

1. Inverting the matrix $[s\mathbf{I} - \mathbf{A}]$ and finding the inverse Laplace transform of each element.
2. Using Mason's gain formula to find $\boldsymbol{\Phi}(s)$ from a block diagram or signal flow graph of the system [the ijth element of the matrix $\boldsymbol{\Phi}(s)$ is given by the transmission $X_i(s)/x_j(0)$] and evaluating the inverse Laplace transform of $\boldsymbol{\Phi}(s)$.
3. Evaluating the matrix expansion

$$\epsilon^{\mathbf{A}t} \triangleq \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots\dagger \quad (1.2\text{-}9)$$

For high-order systems ($n > 4$), evaluating $\epsilon^{\mathbf{A}t}$ numerically (with the aid of a digital computer) is the most feasible of these methods.

For systems having a time-varying $\mathbf{A}$ matrix the state transition matrix can be found by numerical integration of the matrix differential equation

$$\frac{d}{dt}\boldsymbol{\varphi}(t, t_0) = \mathbf{A}(t)\boldsymbol{\varphi}(t, t_0) \quad (1.2\text{-}12)$$

with the initial condition $\boldsymbol{\varphi}(t_0, t_0) = \mathbf{I}$.

† Although a digital computer program for the evaluation of this expansion is easy to write, the running time may be excessive because of convergence properties of the series. For a discussion of more efficient numerical techniques see [O-1], p. 315ff.

Controllability and Observability†

Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (1.2\text{-}13)$$

for $t \geq t_0$ with initial state $\mathbf{x}(t_0) = \mathbf{x}_0$.

DEFINITION 1-8

> If there is a finite time $t_1 \geq t_0$ and a control $\mathbf{u}(t)$, $t \in [t_0, t_1]$, which transfers the state $\mathbf{x}_0$ to the origin at time t_1, the state $\mathbf{x}_0$ is said to be *controllable at time* t_0. If all values of $\mathbf{x}_0$ are controllable for all t_0, the system is *completely controllable*, or simply *controllable.*

Controllability is very important, because we shall consider problems in which the goal is to transfer a system from an arbitrary initial state to the origin while minimizing some performance measure; thus, controllability of the system is a necessary condition for the existence of a solution.

Kalman‡ has shown that a *linear, time-invariant* system is controllable if and only if the $n \times mn$ matrix

$$\mathbf{E} \triangleq \left[\mathbf{B} \,\vdots\, \mathbf{AB} \,\vdots\, \mathbf{A}^2\mathbf{B} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{B}\right]$$

has rank n. If there is only one control input ($m = 1$), a necessary and sufficient condition for controllability is that the $n \times n$ matrix $\mathbf{E}$ be nonsingular.

The concept of observability is defined by considering the system (1.2-13) with the control $\mathbf{u}(t) = \mathbf{0}$ for $t \geq t_0$.§

DEFINITION 1-9

> If by observing the output $\mathbf{y}(t)$ during the finite time interval $[t_0, t_1]$ the state $\mathbf{x}(t_0) = \mathbf{x}_0$ can be determined, the state $\mathbf{x}_0$ is said to be *observable at time* t_0. If all states $\mathbf{x}_0$ are observable for every t_0, the system is called *completely observable*, or simply *observable.*

Analogous to the test for controllability, it can be shown that the *linear, time-invariant* system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (1.2\text{-}14)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (1.2\text{-}15)$$

† See [K-2], [K-3].
‡ See [K-2].
§ If the system is linear and time-invariant, $\mathbf{u}$ can be any known function—see [Z-1], p. 502.

is observable if and only if the $n \times qn$ matrix

$$\mathbf{G} \triangleq \left[\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T \,\vdots\, (\mathbf{A}^T)^2\mathbf{C}^T \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^T)^{n-1}\mathbf{C}^T\right]$$

has rank n. If there is only one output ($q = 1$) $\mathbf{G}$ is an $n \times n$ matrix and a necessary and sufficient condition for observability is that $\mathbf{G}$ be nonsingular. Since we have made the simplifying assumption that all of the states can be physically measured ($\mathbf{y}(t) = \mathbf{x}(t)$), the question of observability will not arise in our subsequent discussion.

1.3 CONCLUDING REMARKS

In control system design, the ultimate objective is to obtain a controller that will cause a system to perform in a desirable manner. Usually, other factors, such as weight, volume, cost, and reliability also influence the controller design, and compromises between performance requirements and implementation considerations must be made. Classical design procedures are best suited for *linear, single-input, single-output systems with zero initial conditions.* Using simulation, mathematical analysis, or graphical methods, the designer evaluates the effects of inserting various physical devices into the system. By trial and error either an acceptable controller design is obtained, or the designer concludes that the performance requirements cannot be satisfied.

Many complex aerospace problems that are not amenable to classical techniques have been solved by using optimal control theory. However, we are forced to admit that optimal control theory does not, at the present time, constitute a generally applicable procedure for the design of simple controllers. The optimal control law, if it can be obtained, usually requires a digital computer for implementation (an important exception is the linear regulator problem discussed in Section 5.2), and *all* of the states must be available for feedback to the controller. These limitations may preclude implementation of the optimal control law; however, the theory of optimal control is still useful, because

1. Knowing the optimal control law may provide insight helpful in designing a suboptimal, but easily implemented controller.
2. The optimal control law provides a standard for evaluating proposed suboptimal designs. In other words, by knowing the optimal control law we have a quantitative measure of performance degradation caused by using a suboptimal controller.

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PROBLEMS

1-1. The tanks A and B shown in Fig. 1-P1 each have a capacity of 50 gal. Both tanks are filled at $t = 0$, tank A with 60 lb of salt dissolved in water, and

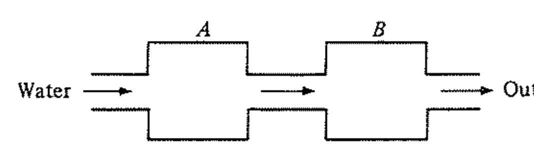

Figure 1-P1