5

The Variational Approach to Optimal Control Problems

In this chapter we shall apply variational methods to optimal control problems. We shall first derive necessary conditions for optimal control assuming that the admissible controls are not bounded. These necessary conditions are then employed to find the optimal control law for the important linear regulator problem. Next, Pontryagin's minimum principle is introduced heuristically as a generalization of the fundamental theorem of the calculus of variations, and problems with bounded control and state variables are discussed. The three concluding sections of the chapter are devoted to time-optimal problems, minimum control-effort systems, and problems involving singular intervals.

5.1 NECESSARY CONDITIONS FOR OPTIMAL CONTROL

Let us now employ the techniques introduced in Chapter 4 to determine necessary conditions for optimal control. As stated in Chapter 1, the problem is to find an admissible control $\mathbf{u}^*$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{5.1-1}$$

to follow an admissible trajectory $\mathbf{x}^*$ that minimizes the performance measure

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\,dt.\dagger \tag{5.1-2}$$

We shall initially assume that the admissible state and control regions are not bounded, and that the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and the initial time t_0 are specified. As usual, $\mathbf{x}$ is the $n \times 1$ state vector and $\mathbf{u}$ is the $m \times 1$ vector of control inputs.

In the terminology of Chapter 4, we have a problem involving $n + m$ functions which must satisfy the n differential equation constraints (5.1-1). The m control inputs are the independent functions.

The only difference between Eq. (5.1-2) and the functionals considered in Chapter 4 is the term involving the final states and final time. However, assuming that h is a differentiable function, we can write

$$h(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[h(\mathbf{x}(t), t)]\,dt + h(\mathbf{x}(t_0), t_0), \tag{5.1-3}$$

so that the performance measure can be expressed as

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt + h(\mathbf{x}(t_0), t_0). \tag{5.1-4}$$

Since $\mathbf{x}(t_0)$ and t_0 are fixed, the minimization does not affect the $h(\mathbf{x}(t_0), t_0)$ term, so we need consider only the functional

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt. \tag{5.1-5}$$

Using the chain rule of differentiation, we find that this becomes

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \right\} dt. \tag{5.1-6}$$

To include the differential equation constraints, we form the augmented functional

$$\begin{aligned} J_a(\mathbf{u}) = \int_{t_0}^{t_f} \bigg\{ & g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \\ & + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] \bigg\} dt \end{aligned} \tag{5.1-7}$$

by introducing the Lagrange multipliers $p_1(t), \ldots, p_n(t)$. Let us define

† In general, the functional J depends on $\mathbf{x}(t_0)$, t_0, $\mathbf{x}$, $\mathbf{u}$, the target set S, and t_f. However, here it is assumed that $\mathbf{x}(t_0)$ and t_0 are specified; hence, $\mathbf{x}$ is determined by $\mathbf{u}$ and we write $J(\mathbf{u})$—the dependence of J on S and t_f will not be explicitly indicated.

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t)$$

so that

$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} \{g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t)\}\, dt. \tag{5.1-8}$$

We shall assume that the end points at $t = t_f$ can be specified or free. To determine the variation of J_a, we introduce the variations $\delta\mathbf{x}$, $\delta\dot{\mathbf{x}}$, $\delta\mathbf{u}$, $\delta\mathbf{p}$, and δt_f. From *Problem 4a* in the preceding chapter this gives [see Eq. (4.3-16)], on an extremal,

$$\begin{aligned}\delta J_a(\mathbf{u}^*) = 0 = &\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f \\ &+ \Big[g_a(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\ &- \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\Big]\delta t_f \\ &+ \int_{t_0}^{t_f}\Big\{\Big[\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \\ &- \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T\Big]\delta\mathbf{x}(t) \\ &+ \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t) \\ &+ \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{p}(t)\Big\}\, dt.\end{aligned} \tag{5.1-9}$$

Notice that the above result is obtained because $\dot{\mathbf{u}}(t)$ and $\dot{\mathbf{p}}(t)$ do not appear in g_a.

Next, let us consider only those terms inside the integral which involve the function h; these terms contain

$$\frac{\partial}{\partial \mathbf{x}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right]^T \dot{\mathbf{x}}^*(t) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t), t)\right] - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\mathbf{x}}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right]^T \dot{\mathbf{x}}^*(t)\right]\right\}. \tag{5.1-10}$$

Writing out the indicated partial derivatives gives

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial \mathbf{x}}(\mathbf{x}^*(t), t)\right] - \frac{d}{dt}\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right], \tag{5.1-11}$$

or, if we apply the chain rule to the last term,

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial \mathbf{x}}(\mathbf{x}^*(t), t)\right] - \left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t) - \left[\frac{\partial^2 h}{\partial \mathbf{x} \partial t}(\mathbf{x}^*(t), t)\right]. \tag{5.1-12}$$

If it is assumed that the second partial derivatives are continuous, the order of differentiation can be interchanged, and these terms add to zero. In the integral term we have, then,

$$\begin{aligned}\int_{t_0}^{t_f}\Big\{&\Big[\left[\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T + \mathbf{p}^{*T}(t)\left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right] \\ &- \frac{d}{dt}[-\mathbf{p}^{*T}(t)]\Big]\delta\mathbf{x}(t) + \Big[\left[\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T \\ &+ \mathbf{p}^{*T}(t)\left[\frac{\partial \mathbf{a}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]\Big]\delta\mathbf{u}(t) + \Big[[\mathbf{a}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)]^T\Big]\delta\mathbf{p}(t)\Big\}\, dt.\end{aligned} \tag{5.1-13}$$

This integral must vanish on an extremal regardless of the boundary conditions. We first observe that the constraints

$$\dot{\mathbf{x}}^*(t) = \mathbf{a}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \tag{5.1-14a}$$

must be satisfied by an extremal so that the coefficient of $\delta\mathbf{p}(t)$ is zero. The Lagrange multipliers are arbitrary, so let us select them to make the coefficient of $\delta\mathbf{x}(t)$ equal to zero, that is,

$$\dot{\mathbf{p}}^*(t) = -\left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T \mathbf{p}^*(t) - \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t). \tag{5.1-14b}$$

We shall henceforth call (5.1-14b) the *costate equations* and $\mathbf{p}(t)$ the *costate*.

The remaining variation $\delta\mathbf{u}(t)$ is independent, so its coefficient must be zero; thus,

$$\mathbf{0} = \frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left[\frac{\partial \mathbf{a}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T \mathbf{p}^*(t). \tag{5.1-14c}$$

Equations (5.1-14) are important equations; we shall be using them throughout the remainder of this chapter. We shall find that even when the admissible controls are bounded, only Eq. (5.1-14c) is modified.

There are still the terms outside the integral to deal with; since the variation must be zero, we have

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \Big[g(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) + \mathbf{p}^{*T}(t_f)[\mathbf{a}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f)]\Big]\delta t_f = 0. \tag{5.1-15}$$

In writing (5.1-15), we have used the fact that $\dot{\mathbf{x}}^*(t_f) = \mathbf{a}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f)$. Equation (5.1-15) admits a variety of situations, which we shall discuss shortly.

Equations (5.1-14) are the necessary conditions we set out to determine. Notice that these necessary conditions consist of a set of $2n$, *first-order* differential equations—the state and costate equations (5.1-14a) and (5.1-14b)—and a set of m algebraic relations—(5.1-14c)—which must be satisfied throughout the interval $[t_0, t_f]$. The solution of the state and costate equations will contain $2n$ constants of integration. To evaluate these constants we use the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$ and an additional set of n or $(n+1)$ relationships —depending on whether or not t_f is specified—from Eq. (5.1-15). Notice that, as expected, we are again confronted by a two-point boundary-value problem.

In the following we shall find it convenient to use the function $\mathscr{H}$, called the *Hamiltonian*, defined as

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)]. \quad (5.1\text{-}16)$$

Using this notation, we can write the necessary conditions (5.1-14) through (5.1-15) as follows:

$$\dot{\mathbf{x}}^*(t) = \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \quad (5.1\text{-}17a)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \qquad \text{for all } t \in [t_0, t_f] \quad (5.1\text{-}17b)$$

$$\mathbf{0} = \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \quad (5.1\text{-}17c)$$

and

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]\delta t_f = 0. \quad (5.1\text{-}18)$$

Let us now consider the boundary conditions that may occur.

Boundary Conditions

In a particular problem either g or h may be missing; in this case, we simply strike out the terms involving the missing function. To determine the boundary conditions is a matter of making the appropriate substitutions in Eq. (5.1-18). In all cases it will be assumed that we have the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$.

Problems with Fixed Final Time. If the final time t_f is specified, $\mathbf{x}(t_f)$ may be specified, free, or required to lie on some surface in the state space.

CASE I. *Final state specified.* Since $\mathbf{x}(t_f)$ and t_f are specified, we substitute $\delta\mathbf{x}_f = \mathbf{0}$ and $\delta t_f = 0$ in (5.1-18). The required n equations are

$$\mathbf{x}^*(t_f) = \mathbf{x}_f. \quad (5.1\text{-}19)$$

CASE II. *Final state free.* We substitute $\delta t_f = 0$ in Eq. (5.1-18); since $\delta\mathbf{x}_f$ is arbitrary, the n equations

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}\dagger \quad (5.1\text{-}20)$$

must be satisfied.

CASE III. *Final state lying on the surface defined by* $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$. Since this is a new situation, let us consider an introductory example. Suppose that the final state of a second-order system is required to lie on the circle

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0 \quad (5.1\text{-}21)$$

shown in Fig. 5-1. Notice that admissible changes in $\mathbf{x}(t_f)$ are (to first-order) tangent to the circle at the point $(\mathbf{x}^*(t_f), t_f)$. The tangent line is normal to the gradient vector

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) = \begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix} \quad (5.1\text{-}22)$$

at the point $(\mathbf{x}^*(t_f), t_f)$. Thus, $\delta\mathbf{x}(t_f)$ must be normal to the gradient (5.1-22), so that

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T \delta\mathbf{x}(t_f) = 2[x_1^*(t_f) - 3]\,\delta x_1(t_f) + 2[x_2^*(t_f) - 4]\,\delta x_2(t_f) = 0. \quad (5.1\text{-}23)$$

† Since the final time is fixed, h will not depend on t_f.

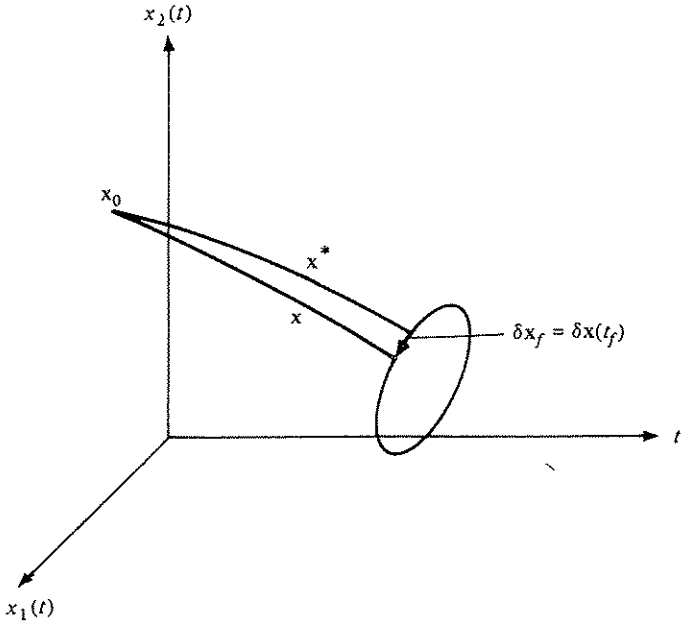


Figure 5-1 An extremal and a comparison curve that terminate on the curve $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$ at the specified final time, t_f

Solving for $\delta x_2(t_f)$ gives

$$\delta x_2(t_f) = \frac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]} \delta x_1(t_f), \tag{5.1-24}$$

which, when substituted in Eq. (5.1-18), gives

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \begin{bmatrix} 1 \\ \dfrac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]} \end{bmatrix} = 0 \tag{5.1-25}$$

since $\delta t_f = 0$ and $\delta x_1(t_f)$ is arbitrary. The second required equation at the final time is

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \tag{5.1-26}$$

In the general situation there are n state variables and $1 \leq k \leq n - 1$ relationships that the states must satisfy at $t = t_f$. In this case we write

$$\mathbf{m}(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t)) \end{bmatrix} = \mathbf{0}, \tag{5.1-27}$$

and each component of $\mathbf{m}$ represents a hypersurface in the n-dimensional state space. Thus, the final state lies on the intersection of these k hypersurfaces, and $\delta\mathbf{x}(t_f)$ is tangent to each of the hypersurfaces at the point $(\mathbf{x}^*(t_f), t_f)$. This means that $\delta\mathbf{x}(t_f)$ is normal to each of the gradient vectors

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \ldots, \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \tag{5.1-28}$$

which are assumed to be linearly independent. From Eq. (5.1-18) we have, since $\delta t_f = 0$,

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}(t_f) \triangleq \mathbf{v}^T \delta\mathbf{x}(t_f) = 0. \tag{5.1-29}$$

It can be shown that this equation is satisfied if and only if the vector $\mathbf{v}$ is a linear combination of the gradient vectors in Eq. (5.1-28), that is,

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]. \tag{5.1-30}$$

To determine the $2n$ constants of integration in the solution of the state-costate equations, and $d_1, \ldots, d_k$, we have the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$, the n equations (5.1-30), and the k equations

$$\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}. \tag{5.1-31}$$

Let us show that Eqs. (5.1-30) and (5.1-31) lead to the results obtained in our introductory example. The constraining relation is

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0. \tag{5.1-21}$$

From Eq. (5.1-30) we obtain the two equations

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d\begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix}, \tag{5.1-32}$$

and (5.1-31) gives

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \qquad (5.1\text{-}33)$$

By solving the second of Eqs. (5.1-32) for d and substituting this into the first equation of (5.1-32), Eq. (5.1-25) is obtained.

Problems with Free Final Time. If the final time is free, there are several situations that may occur.

CASE I. *Final state fixed.* The appropriate substitution in Eq. (5.1-18) is $\delta\mathbf{x}_f = \mathbf{0}$. δt_f is arbitrary, so the $(2n + 1)$st relationship is

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0. \qquad (5.1\text{-}34)$$

CASE II. *Final state free.* $\delta\mathbf{x}_f$ and δt_f are arbitrary and independent; therefore, their coefficients must be zero; that is,

$$\mathbf{p}^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \qquad (n \text{ equations}) \qquad (5.1\text{-}35)$$

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0 \qquad (1 \text{ equation}). \qquad (5.1\text{-}36)$$

Notice that if $h = 0$

$$\mathbf{p}^*(t_f) = \mathbf{0} \qquad (5.1\text{-}37)$$

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) = 0. \qquad (5.1\text{-}38)$$

CASE III. $\mathbf{x}(t_f)$ *lies on the moving point* $\boldsymbol{\theta}(t)$. Here $\delta\mathbf{x}_f$ and δt_f are related by

$$\delta\mathbf{x}_f \doteq \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f;$$

making this substitution in Eq. (5.1-18) yields the equation

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \times \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0. \qquad (5.1\text{-}39)$$

This gives one equation; the remaining n required relationships are

$$\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f).$$

CASE IV. *Final state lying on the surface defined by* $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$. As an example of this type of end point constraint, suppose that the final state is required to lie on the curve

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0. \qquad (5.1\text{-}40)$$

Since the final time is free, the admissible end points lie on the cylindrical surface shown in Fig. 5-2. Notice that

1. To first-order, the change in $\mathbf{x}(t_f)$ must be in the plane tangent to the cylindrical surface at the point $(\mathbf{x}^*(t_f), t_f)$.
2. The change in $\mathbf{x}(t_f)$ is independent of δt_f.

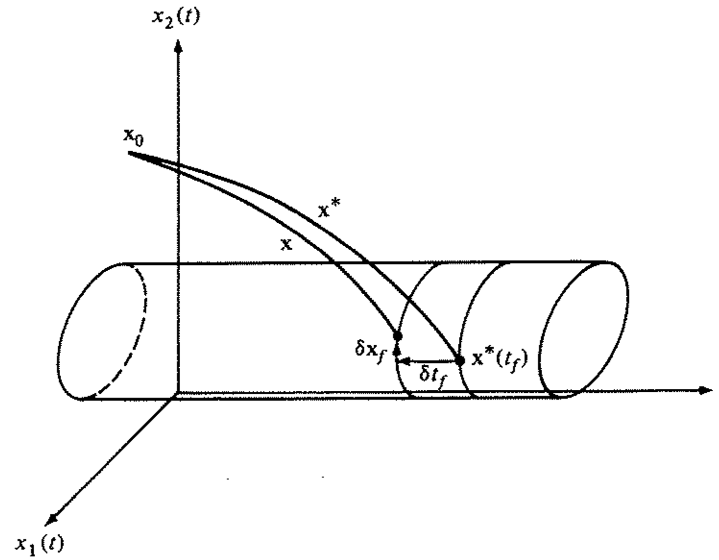


Figure 5-2 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$

Since $\delta\mathbf{x}_f$ is independent of δt_f, the coefficient of δt_f must be zero, and

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0. \qquad (5.1\text{-}41)$$

The plane that is tangent to the cylinder at the point $(\mathbf{x}^*(t_f), t_f)$ is described by its normal vector or gradient; that is, every vector in the plane is normal to the vector

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) = \begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix}. \qquad (5.1\text{-}42)$$

This means that

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T \delta\mathbf{x}_f = 2[x_1^*(t_f) - 3]\,\delta x_{1_f} + 2[x_2^*(t_f) - 4]\,\delta x_{2_f} = 0. \quad (5.1\text{-}43)$$

Solving for δx_{2_f} gives

$$\delta x_{2_f} = \frac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]}\,\delta x_{1_f}. \quad (5.1\text{-}44)$$

Substituting this for δx_{2_f} in Eq. (5.1-18) gives

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \begin{bmatrix} 1 \\ \dfrac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]} \end{bmatrix} \delta x_{1_f} = 0. \quad (5.1\text{-}45)$$

Since δx_{1_f} is arbitrary, its coefficient must be zero. Equations (5.1-41) and (5.1-45) give two relationships; the third is the constraint

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \quad (5.1\text{-}46)$$

In the general situation we have n state variables, and there may be $1 \le k \le n - 1$ relationships that the states are required to satisfy at the terminal time. In this case we write

$$\mathbf{m}(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t)) \end{bmatrix} = \mathbf{0}, \quad (5.1\text{-}47)$$

and each component of $\mathbf{m}$ describes a hypersurface in the n-dimensional state space. This means that the final state lies on the intersection of the hypersurfaces defined by $\mathbf{m}$, and that $\delta\mathbf{x}_f$ is (to first order) tangent to each of the hypersurfaces at the point $(\mathbf{x}^*(t_f), t_f)$. Thus, $\delta\mathbf{x}_f$ is normal to each of the gradient vectors

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \ldots, \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \quad (5.1\text{-}48)$$

which we assume to be linearly independent. It is left as an exercise for the reader to show that the reasoning used in Case III with *fixed* final time also applies in the present situation and leads to the $(2n + k + 1)$ equations

$$\begin{aligned}
&\mathbf{x}^*(t_0) = \mathbf{x}_0 \\
&\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] \\
&\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0} \\
&\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0
\end{aligned} \quad (5.1\text{-}49)$$

involving the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f. It is also easily shown that Eqs. (5.1-49) give Eqs. (5.1-41), (5.1-45), and (5.1-46) in the preceding example.

CASE V. *Final state lying on the moving surface defined by* $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$. Suppose that the final state must lie on the surface

$$m(\mathbf{x}(t), t) = [x_1(t) - 3]^2 + [x_2(t) - 4 - t]^2 - 4 = 0 \quad (5.1\text{-}50)$$

shown in Fig. 5-3. Notice that δt_f does influence the admissible values of $\delta\mathbf{x}_f$; that is, to remain on the surface $m(\mathbf{x}(t), t) = 0$ the value of $\delta\mathbf{x}_f$ depends on δt_f. The vector with components $\delta x_{1_f}, \delta x_{2_f}, \delta t_f$ must be contained in a plane tangent to the surface at the point $(\mathbf{x}^*(t_f), t_f)$. This means that the normal to this tangent plane is the vector

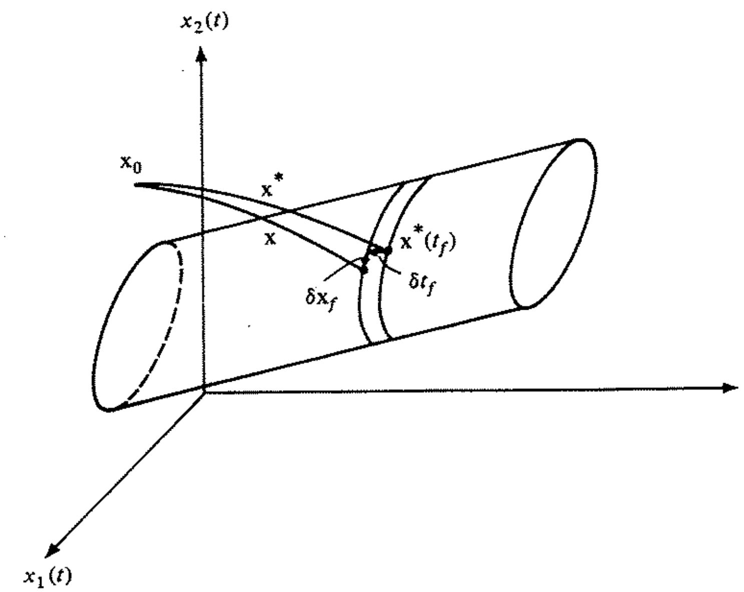


Figure 5-3 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4 - t]^2 - 4 = 0$

$$\begin{bmatrix} \dfrac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f), t_f) \\ \dfrac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f), t_f) \\ \dfrac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} \triangleq \begin{bmatrix} \dfrac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \dfrac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} \tag{5.1-51}$$

in the three-dimensional space. Thus, admissible variations must be normal to the vector (5.1-51), so

$$\left[\frac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f), t_f)\right] \delta x_{1_f} + \left[\frac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f), t_f)\right] \delta x_{2_f} + \left[\frac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0. \tag{5.1-52}$$

For the surface specified we have

$$2[x_1^*(t_f) - 3]\,\delta x_{1_f} + 2[x_2^*(t_f) - 4 - t_f]\,\delta x_{2_f} - 2[x_2^*(t_f) - 4 - t_f]\,\delta t_f = 0. \tag{5.1-53}$$

Solving for δt_f gives

$$\delta t_f = \frac{[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4 - t_f]}\,\delta x_{1_f} + \delta x_{2_f}. \tag{5.1-54}$$

Substituting in Eq. (5.1-18) and collecting terms, we obtain

$$\begin{aligned} &\Big[\frac{\partial h}{\partial x_1}(\mathbf{x}^*(t_f), t_f) - p_1^*(t_f) + \Big[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\ &\quad + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\Big]\Big[\frac{x_1^*(t_f) - 3}{x_2^*(t_f) - 4 - t_f}\Big]\Big]\,\delta x_{1_f} \\ &\quad + \Big[\frac{\partial h}{\partial x_2}(\mathbf{x}^*(t_f), t_f) - p_2^*(t_f) + \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\ &\quad + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\Big]\,\delta x_{2_f} = 0. \end{aligned} \tag{5.1-55}$$

Since there is one constraint involving the three variables $(\delta x_{1_f}, \delta x_{2_f}, \delta t_f)$, δx_{1_f} and δx_{2_f} can be varied independently; therefore, the coefficients of δx_{1_f} and δx_{2_f} must be zero. This gives two equations; the third equation is

$$m(\mathbf{x}^*(t_f), t_f) = 0. \tag{5.1-56}$$

In general, we may have $1 \le k \le n$ relationships

$$\mathbf{m}(\mathbf{x}(t), t) = \begin{bmatrix} m_1(\mathbf{x}(t), t) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t), t) \end{bmatrix} = \mathbf{0}, \tag{5.1-57}$$

which must be satisfied by the $(n + 1)$ variables $\mathbf{x}(t_f)$ and t_f. Reasoning as in the situation where $\mathbf{m}$ is not dependent on time, we deduce that the admissible values of the $(n + 1)$ vector

$$\begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix}$$

are normal to each of the gradient vectors

$$\begin{bmatrix} \dfrac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \dfrac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \ldots, \begin{bmatrix} \dfrac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \dfrac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \tag{5.1-58}$$

which are assumed to be linearly independent. Writing Eq. (5.1-18) as

$$\begin{bmatrix} \dfrac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) \\ \hline \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \dfrac{\partial h}{\partial t}(\mathbf{x}^*(t_f)\ t_f) \end{bmatrix}^T \begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix} = 0 \triangleq \mathbf{v}^T \begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix} \tag{5.1-59}$$

and again using the result that $\mathbf{v}$ must be a linear combination of the gradient vectors in (5.1-58), we obtain

$$\mathbf{v} = d_1 \begin{bmatrix} \dfrac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \dfrac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} + \cdots + d_k \begin{bmatrix} \dfrac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \dfrac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \tag{5.1-60}$$

or

$$\begin{aligned} \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) &= d_1\Big[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\Big] \\ &\quad + \cdots + d_k\Big[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\Big] \end{aligned}$$

and

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = d_1\left[\frac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]. \quad (5.1\text{-}61)$$

Equations (5.1-61), the k equations

$$\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}, \quad (5.1\text{-}62)$$

and the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$ comprise a set of $(2n + k + 1)$ equations in the $2n$ constants of integration, the variables $d_1, d_2, \ldots, d_k$, and t_f. It is left as an exercise for the reader to verify that (5.1-62) and (5.1-61) yield Eqs. (5.1-55) and (5.1-56).

The boundary conditions which we have discussed are summarized in Table 5-1. Of course, mixed situations can arise, but these can be handled by returning to Eq. (5.1-18) and applying the ideas introduced in the preceding discussion.

Although the boundary condition relationships may look foreboding, setting up the equations is not difficult; obtaining solutions is another matter. This should not surprise us, however, for we already suspect that numerical techniques are required to solve most problems of practical interest. Let us now illustrate the determination of the boundary-condition equations by considering several examples.

Example 5.1-1. The system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_2(t) + u(t)\end{aligned} \quad (5.1\text{-}63)$$

is to be controlled so that its control effort is conserved; that is, the performance measure

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}u^2(t)\,dt \quad (5.1\text{-}64)$$

is to be minimized. The admissible states and controls are not bounded. Find necessary conditions that must be satisfied for optimal control.

The first step is to form the Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t). \quad (5.1\text{-}65)$$

From Eqs. (5.1-17b) and (5.1-17c) necessary conditions for optimality are

$$\begin{aligned}\dot{p}_1^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_1} = 0\\ \dot{p}_2^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t),\end{aligned} \quad (5.1\text{-}66)$$

and

$$0 = \frac{\partial\mathscr{H}}{\partial u} = u^*(t) + p_2^*(t). \quad (5.1\text{-}67)$$

If Eq. (5.1-67) is solved for $u^*(t)$ and substituted into the state equations (5.1-63), we have

$$\begin{aligned}\dot{x}_1^*(t) &= x_2^*(t)\\ \dot{x}_2^*(t) &= -x_2^*(t) - p_2^*(t).\end{aligned} \quad (5.1\text{-}68)$$

Equations (5.1-68) and (5.1-66)—the state and costate equations—are a set of $2n$ linear first-order, homogeneous, constant-coefficient differential equations. Solving these equations gives

$$\begin{aligned}x_1^*(t) &= c_1 + c_2[1 - \epsilon^{-t}] + c_3[-t - \tfrac{1}{2}\epsilon^{-t} + \tfrac{1}{2}\epsilon^{t}] + c_4[1 - \tfrac{1}{2}\epsilon^{-t} - \tfrac{1}{2}\epsilon^{t}]\\ x_2^*(t) &= c_2\epsilon^{-t} + c_3[-1 + \tfrac{1}{2}\epsilon^{-t} + \tfrac{1}{2}\epsilon^{t}] + c_4[\tfrac{1}{2}\epsilon^{-t} - \tfrac{1}{2}\epsilon^{t}]\\ p_1^*(t) &= c_3\\ p_2^*(t) &= c_3[1 - \epsilon^{t}] + c_4\epsilon^{t}.\end{aligned} \quad (5.1\text{-}69)$$

Now let us consider several possible sets of boundary conditions.

a. Suppose $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}(2) = [5 \quad 2]^T$. From $\mathbf{x}(0) = \mathbf{0}$ we obtain $c_1 = c_2 = 0$; the remaining two equations to be solved are

$$\begin{aligned}5 &= c_3[-2 - \tfrac{1}{2}\epsilon^{-2} + \tfrac{1}{2}\epsilon^{2}] + c_4[1 - \tfrac{1}{2}\epsilon^{-2} - \tfrac{1}{2}\epsilon^{2}]\\ 2 &= c_3[-1 + \tfrac{1}{2}\epsilon^{-2} + \tfrac{1}{2}\epsilon^{2}] + c_4[\tfrac{1}{2}\epsilon^{-2} - \tfrac{1}{2}\epsilon^{2}].\end{aligned} \quad (5.1\text{-}70)$$

Solving these linear algebraic equations gives $c_3 = -7.289$ and $c_4 = -6.103$, so the optimal trajectory is

$$\begin{aligned}x_1^*(t) &= 7.289t - 6.103 + 6.696\epsilon^{-t} - 0.593\epsilon^{t}\\ x_2^*(t) &= 7.289 - 6.696\epsilon^{-t} - 0.593\epsilon^{t}.\end{aligned} \quad (5.1\text{-}71)$$

b. Let $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}(2)$ be unspecified; consider the performance measure

$$J(u) = \tfrac{1}{2}[x_1(2) - 5]^2 + \tfrac{1}{2}[x_2(2) - 2]^2 + \tfrac{1}{2}\int_0^2 u^2(t)\,dt. \quad (5.1\text{-}72)$$

Table 5-1 SUMMARY OF BOUNDARY CONDITIONS IN OPTIMAL CONTROL PROBLEMS

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	6. $\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta\mathbf{x}_f = \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $+ \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	7. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f
	8. $\mathbf{x}(t_f)$ on the moving surface $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$ $\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $= \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f.

The modified performance measure affects only the boundary conditions at $t = 2$. From entry 2 of Table 5-1 we have

$$p_1^*(2) = x_1^*(2) - 5$$
$$p_2^*(2) = x_2^*(2) - 2. \tag{5.1-73}$$

c_1 and c_2 are again zero because $\mathbf{x}^*(0) = \mathbf{0}$. Putting $t = 2$ in Eq. (5.1-69) and substituting in (5.1-73), we obtain the linear algebraic equations

$$\begin{bmatrix} 0.627 & -2.762 \\ 9.151 & -11.016 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \tag{5.1-74}$$

Solving these equations, we find that $c_3 = -2.697$, $c_4 = -2.422$; hence,

$$x_1^*(t) = 2.697t - 2.422 + 2.560\epsilon^{-t} - 0.137\epsilon^{t}$$
$$x_2^*(t) = 2.697 - 2.560\epsilon^{-t} - 0.137\epsilon^{t}. \tag{5.1-75}$$

c. Next, suppose that the system is to be transferred from $\mathbf{x}(0) = \mathbf{0}$ to the line

$$x_1(t) + 5x_2(t) = 15 \tag{5.1-76}$$

while the original performance measure (5.1-64) is minimized. As before, the solution of the state and costate equations is given by Eq. (5.1-69), and $c_1 = c_2 = 0$. The boundary conditions at $t = 2$ are, from entry 3 of Table 5-1,

$$x_1^*(2) + 5x_2^*(2) = 15$$
$$-p_1^*(2) = d \tag{5.1-77}$$
$$-p_2^*(2) = 5d.$$

Eliminating d and substituting $t = 2$ in (5.1-69), we obtain the equations

$$\begin{bmatrix} 15.437 & -20.897 \\ 11.389 & -7.389 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix}, \tag{5.1-78}$$

which have the solution $c_3 = -0.894$, $c_4 = -1.379$. The optimal trajectory is then

$$x_1^*(t) = 0.894t - 1.379 + 1.136\epsilon^{-t} + 0.242\epsilon^{t}$$
$$x_2^*(t) = 0.894 - 1.136\epsilon^{-t} + 0.242\epsilon^{t}. \tag{5.1-79}$$

Example 5.1-2. The space vehicle shown in Fig. 5-4 is in the gravity field of the moon. Assume that the motion is planar, that aerodynamic forces are negligible, and that the thrust magnitude T is constant. The control

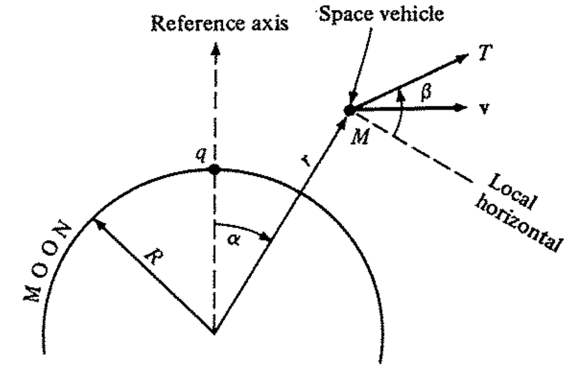


Figure 5-4 A space vehicle in the gravity field of the moon

variable is the thrust direction $\beta(t)$, which is measured from the local horizontal. To simplify the state equations, we shall approximate the vehicle as a particle of mass M. The gravitational force exerted on the vehicle is $F_g(t) = Mg_0R^2/r^2(t)$; g_0 is the gravitational constant at the surface of the moon, R is the radius of the moon, and r is the distance of the spacecraft from the center of the moon. The instantaneous velocity of the vehicle is the vector $\mathbf{v}$, and α is the angular displacement from the reference axis. Selecting $x_1 \triangleq r$, $x_2 \triangleq \alpha$, $x_3 \triangleq \dot{r}$, and $x_4 \triangleq r\dot{\alpha}$ as the states of the system, letting $u \triangleq \beta$, and neglecting the change in mass resulting from fuel consumption, we find that the state equations are

$$\dot{x}_1(t) = x_3(t)$$
$$\dot{x}_2(t) = \frac{x_4(t)}{x_1(t)}$$
$$\dot{x}_3(t) = \frac{x_4^2(t)}{x_1(t)} - \frac{g_0R^2}{x_1^2(t)} + \left[\frac{T}{M}\right]\sin u(t) \tag{5.1-80}$$
$$\dot{x}_4(t) = -\frac{x_3(t)x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right]\cos u(t).$$

Notice that these differential equations are nonlinear in both the states and the control variable. Let us consider several possible missions for the space vehicle.

Mission a. Suppose that the spacecraft is to be launched from the point q on the reference axis at $t = 0$ into a circular orbit of altitude D, as shown in Fig. 5-5(a), in minimum time. $\alpha(t_f)$ is unspecified, and the vehicle starts from rest; thus, the initial conditions are $\mathbf{x}(0) = [R \quad 0 \quad 0 \quad 0]^T$.

From the performance measure

$$J(u) = \int_0^{t_f} dt \tag{5.1-81}$$

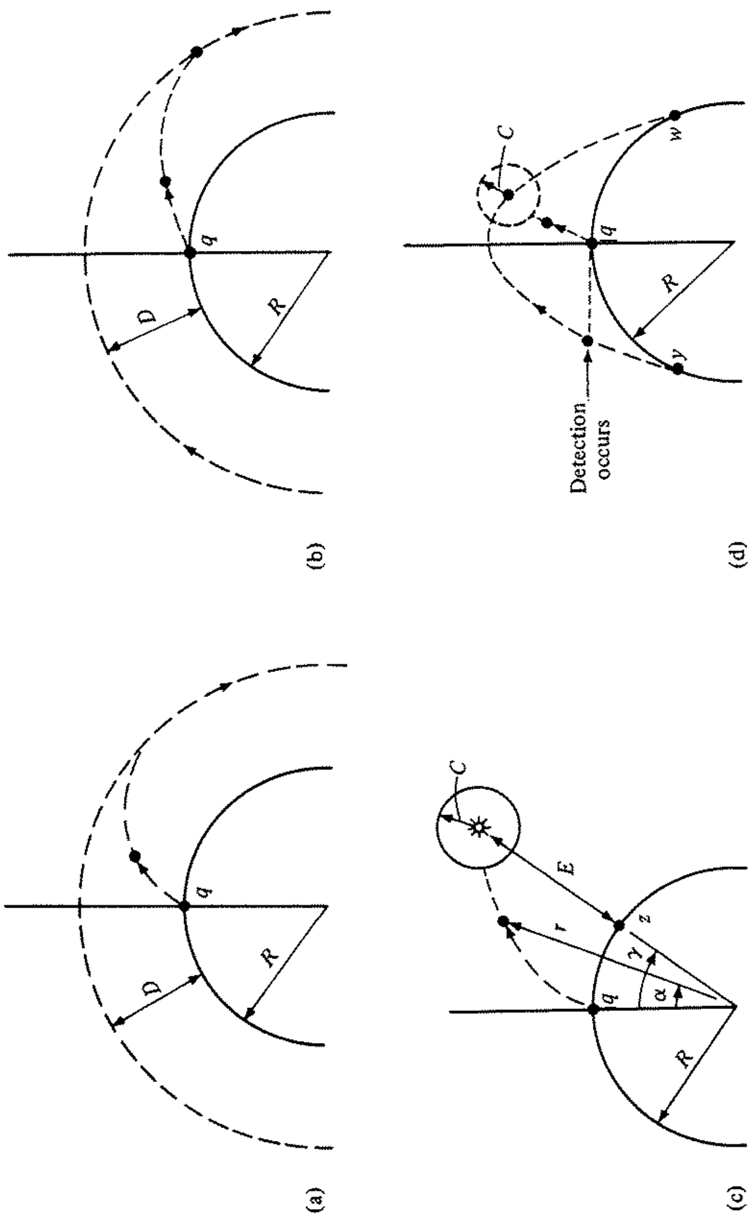


Figure 5-5 (a) Orbit injection. (b) Rendezvous. (c) Reconnaissance of synchronous satellite. (d) Reconnaissance of approaching spacecraft

and the state equations, the Hamiltonian is

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) &= 1 + p_1(t)x_3(t) + \frac{p_2(t)x_4(t)}{x_1(t)} \\ &\quad + p_3(t)\left[\frac{x_4^2(t)}{x_1(t)} - \frac{g_0R^2}{x_1^2(t)} + \left[\frac{T}{M}\right]\sin u(t)\right] \\ &\quad + p_4(t)\left[\frac{-x_3(t)x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right]\cos u(t)\right].^{\dagger}\end{aligned} \tag{5.1-82}$$

The costate equations are, from (5.1-17b),

$$\begin{aligned}\dot{p}_1^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_1} = \frac{p_2^*(t)x_4^*(t)}{x_1^{*2}(t)} + p_3^*(t)\left[\frac{x_4^{*2}(t)}{x_1^{*2}(t)} - \frac{2g_0R^2}{x_1^{*3}(t)}\right] - \frac{p_4^*(t)x_3^*(t)x_4^*(t)}{x_1^{*2}(t)} \\ \dot{p}_2^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_2} = 0 \\ \dot{p}_3^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_3} = -p_1^*(t) + \frac{p_4^*(t)x_4^*(t)}{x_1^*(t)} \\ \dot{p}_4^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_4} = -\frac{p_2^*(t)}{x_1^*(t)} - \frac{2p_3^*(t)x_4^*(t)}{x_1^*(t)} + \frac{p_4^*(t)x_3^*(t)}{x_1^*(t)}.\end{aligned} \tag{5.1-83}$$

The state equations

$$\dot{\mathbf{x}}^*(t) = \mathbf{a}(\mathbf{x}^*(t), u^*(t)) \tag{5.1-84}$$

must be satisfied by an optimal trajectory, and Eq. (5.1-17c) gives the algebraic relationship

$$0 = \frac{\partial\mathscr{H}}{\partial u} = \left[\frac{T}{M}\right][p_3^*(t)\cos u^*(t) - p_4^*(t)\sin u^*(t)]. \tag{5.1-85}$$

Solving Eq. (5.1-85) for $u^*(t)$ gives

$$u^*(t) = \tan^{-1}\frac{p_3^*(t)}{p_4^*(t)}, \tag{5.1-86}$$

or, equivalently,

$$\sin u^*(t) = \frac{p_3^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}} \tag{5.1-87a}$$

$$\cos u^*(t) = \frac{p_4^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}}. \tag{5.1-87b}$$

† Notice that $\mathscr{H}$ is not explicitly dependent on time; hence, the argument t is omitted.

By substituting (5.1-87a) and (5.1-87b) in the state equations, $u^*(t)$ can be eliminated; unfortunately, as is often the case, the resulting $2n$ first-order differential equations are nonlinear.

Next, let us determine the boundary conditions at the final time. There will be five relationships to be satisfied at $t = t_f$; hence, the initial and final boundary conditions will give nine equations involving the eight constants of integration and t_f. From the problem statement we know that $x_1^*(t_f)$ must equal $R + D$. In addition, to have a circular orbit, the centrifugal force must be exactly balanced by the gravitational force; therefore, $M[r^*(t)\dot{\alpha}^*(t)]^2/r^*(t) = Mg_0R^2/r^{*2}(t)$ for $t \geq t_f$. Evaluating this expression at $t = t_f$ and using the specified value of $x_1^*(t_f)$, we obtain $x_4^*(t_f) = \sqrt{g_0R^2/[R + D]}$. The radial velocity must be zero at $t = t_f$, so $x_3^*(t_f) = 0$. The final time is not related to the unspecified final state value $x_2^*(t_f)$, so in Eq. (5.1-18) the coefficients of δt_f and δx_{2_f} must both be zero. To summarize, the required boundary condition relationships are

$$\begin{aligned} x_1^*(t_f) &= R + D \\ p_2^*(t_f) &= 0 \\ x_3^*(t_f) &= 0 \\ x_4^*(t_f) &= \sqrt{\frac{g_0R^2}{[R + D]}} \\ \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) &= 0 \end{aligned} \tag{5.1-88}$$

In writing the last equation it has been assumed that $u^*(t)$ has been eliminated from the Hamiltonian by using Eqs. (5.1-87).

Mission b. In this mission, shown in Fig. 5-5(b), the space vehicle is to be launched from point q and is to rendezvous with another spacecraft that is in a fixed circular orbit D miles above the moon with a period of two hours. At $t = 0$ both spacecraft are on the reference axis. The rendezvous is to be accomplished in minimum time.

Only the boundary conditions are changed from Mission a. The final state values of the controlled vehicle must lie on the moving point

$$\boldsymbol{\theta}(t) = \begin{bmatrix} R + D \\ \underset{2\pi}{\text{modulo}}(\pi t) \\ 0 \\ \pi[R + D] \end{bmatrix}.$$

$\underset{2\pi}{\text{Modulo}}\,(\pi t)$ means that after each revolution 2π radians are subtracted from the angular displacement of the spacecraft. Only the final value of x_2 depends on t, so we have

$$\begin{aligned} \delta x_{2_f} &= \left[\frac{d\theta_2}{dt}(t_f)\right]\delta t_f \\ &= \pi\,\delta t_f. \end{aligned} \tag{5.1-89}$$

Thus, from Eq. (5.1-18), or entry 6 of Table 5-1,

$$-\pi p_2^*(t_f) + \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 0, \tag{5.1-90}$$

since $h = 0$. The remaining four boundary relationships are

$$\mathbf{x}^*(t_f) = \begin{bmatrix} R + D \\ \underset{2\pi}{\text{modulo}}(\pi t_f) \\ 0 \\ \pi[R + D] \end{bmatrix} = \boldsymbol{\theta}(t_f). \tag{5.1-91}$$

Mission c. A satellite is in synchronous orbit E miles above the point z shown in Fig. 5-5(c). It is desired to investigate this satellite with a spacecraft as quickly as possible. The spacecraft transmits television pictures to the lunar base upon arriving at a distance of C miles from the satellite.

Again, the state and costate equations and Eq. (5.1-85) remain unchanged. For this mission, however, the final states must lie on the curve given by

$$\begin{aligned} m(\mathbf{x}(t)) = {} & [r(t)\cos\alpha(t) - [R + E]\cos\gamma]^2 \\ & + [r(t)\sin\alpha(t) - [R + E]\sin\gamma]^2 - C^2 = 0. \end{aligned} \tag{5.1-92}$$

Since the curve $m(\mathbf{x}(t))$ does not depend explicitly on t, we have from entry 7 of Table 5-1 (putting $h = 0$),

$$-\mathbf{p}^*(t_f) = d\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right].$$

Performing the gradient operation, and simplifying, we obtain

$$\begin{aligned} -\mathbf{p}^*(t_f) &= d\begin{bmatrix} 2r^*(t_f) - 2[R + E]\cos(\alpha^*(t_f) - \gamma) \\ 2r^*(t_f)[R + E]\sin(\alpha^*(t_f) - \gamma) \\ 0 \\ 0 \end{bmatrix} \\ &= 2d\begin{bmatrix} x_1^*(t_f) - [R + E]\cos(x_2^*(t_f) - \gamma) \\ x_1^*(t_f)[R + E]\sin(x_2^*(t_f) - \gamma) \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{5.1-93}$$

where d is an unknown variable.

Thus, $p_3^*(t_f) = 0$ and $p_4^*(t_f) = 0$. The other boundary condition equations are

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f)\cos x_2^*(t_f) - [R + E]\cos\gamma]^2 + [x_1^*(t_f)\sin x_2^*(t_f) - [R + E]\sin\gamma]^2 - C^2 = 0, \quad (5.1\text{-}94)$$

and

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 0. \quad (5.1\text{-}95)$$

Equations (5.1-93) through (5.1-95) and $\mathbf{x}^*(0) = [R \quad 0 \quad 0 \quad 0]^T$ give a total of ten equations involving the eight constants of integration, the variable d, and t_f.

Mission d. A lunar-based radar operator detects an approaching spacecraft at $t = 0$ in the position shown in Fig. 5-5(d), and at this time a reconnaissance spacecraft is dispatched from point q. The reconnaissance vehicle is to close to a distance of C miles of the approaching spacecraft as quickly as possible, and relay television pictures to the lunar base. From the radar data the position history of the approaching spacecraft is

$$m(\mathbf{x}(t), t) = [r(t)\cos\alpha(t) - 2.78Rt + 6.95Rt^2 - R]^2 + [r(t)\sin\alpha(t) - 1.85Rt + 0.32R]^2 - C^2 = 0. \quad (5.1\text{-}96)$$

It is to be assumed that this position history will not change.
From Table 5-1, entry 8, we have

$$-\mathbf{p}^*(t_f) = d\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]. \quad (5.1\text{-}97)$$

Performing the gradient operation and simplifying, we obtain

$$\begin{aligned} -p_1^*(t_f) &= 2d[x_1^*(t_f) + R\{[-2.78t_f + 6.95t_f^2 - 1]\cos x_2^*(t_f) \\ &\quad + [-1.85t_f + 0.32]\sin x_2^*(t_f)\}] \\ -p_2^*(t_f) &= -2d[Rx_1^*(t_f)\{[-2.78t_f + 6.95t_f^2 - 1]\sin x_2^*(t_f) \\ &\quad + [1.85t_f - 0.32]\cos x_2^*(t_f)\}] \\ -p_3^*(t_f) &= 0 \\ -p_4^*(t_f) &= 0. \end{aligned} \quad (5.1\text{-}98)$$

In addition, the specified constraint

$$[x_1^*(t_f)\cos x_2^*(t_f) - 2.78Rt_f + 6.95Rt_f^2 - R]^2 + [x_1^*(t_f)\sin x_2^*(t_f) - 1.85Rt_f + 0.32R]^2 - C^2 = 0, \quad (5.1\text{-}99)$$

must be satisfied and, from Table 5-1,

$$\begin{aligned}\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 2\,dR\{&[-2.78 + 13.9t_f][x_1^*(t_f)\cos x_2^*(t_f) \\ &- 2.78Rt_f + 6.95Rt_f^2 - R] - 1.85[x_1^*(t_f)\sin x_2^*(t_f) \\ &- 1.85Rt_f + 0.32R]\}. \end{aligned}\quad (5.1\text{-}100)$$

With the specified initial conditions, we have ten equations in ten unknowns.

5.2 LINEAR REGULATOR PROBLEMS

In this section we shall consider an important class of optimal control problems—linear regulator systems. We shall show that for linear regulator problems the optimal control law can be found as a linear time-varying function of the system states. Under certain conditions, which we shall discuss, the optimal control law becomes time-invariant. The results presented here are primarily due to R. E. Kalman.†

The plant is described by the linear state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad (5.2\text{-}1)$$

which may have time-varying coefficients. The performance measure to be minimized is

$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{H}\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}[\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]\,dt; \quad (5.2\text{-}2)$$

the final time t_f is fixed, **H** and **Q** are real symmetric positive semi-definite matrices, and **R** is a real symmetric positive definite matrix. It is assumed that the states and controls are not bounded, and $\mathbf{x}(t_f)$ is free. We attach the following physical interpretation to this performance measure: It is desired to maintain the state vector close to the origin without an excessive expenditure of control effort.

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \tfrac{1}{2}\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \tfrac{1}{2}\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t) + \mathbf{p}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{B}(t)\mathbf{u}(t), \quad (5.2\text{-}3)$$

and necessary conditions for optimality are

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t) \quad (5.2\text{-}4)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial\mathscr{H}}{\partial\mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t) \quad (5.2\text{-}5)$$

† See references [K-5], [K-6], and [K-7].

$$0 = \frac{\partial \mathscr{H}}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t). \tag{5.2-6}$$

Equation (5.2-6) can be solved for $\mathbf{u}^*(t)$ to give

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t); \tag{5.2-7}$$

the existence of $\mathbf{R}^{-1}$ is assured, since $\mathbf{R}$ is a positive definite matrix. Substituting (5.2-7) into (5.2-4) yields

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t); \tag{5.2-8}$$

thus, we have the set of $2n$ linear homogeneous differential equations

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \hline \dot{\mathbf{p}}^*(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A}(t) & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) \\ \hline -\mathbf{Q}(t) & -\mathbf{A}^T(t) \end{array}\right] \begin{bmatrix} \mathbf{x}^*(t) \\ \hline \mathbf{p}^*(t) \end{bmatrix}. \tag{5.2-9}$$

The solution to these equations has the form

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ \hline \mathbf{p}^*(t_f) \end{bmatrix} = \boldsymbol{\varphi}(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ \hline \mathbf{p}^*(t) \end{bmatrix}, \tag{5.2-10}$$

where $\boldsymbol{\varphi}$ is the transition matrix of the system (5.2-9). Partitioning the transition matrix, we have

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ \hline \mathbf{p}^*(t_f) \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{\varphi}_{11}(t_f, t) & \boldsymbol{\varphi}_{12}(t_f, t) \\ \hline \boldsymbol{\varphi}_{21}(t_f, t) & \boldsymbol{\varphi}_{22}(t_f, t) \end{array}\right] \begin{bmatrix} \mathbf{x}^*(t) \\ \hline \mathbf{p}^*(t) \end{bmatrix}, \tag{5.2-10a}$$

where $\boldsymbol{\varphi}_{11}, \boldsymbol{\varphi}_{12}, \boldsymbol{\varphi}_{21}$, and $\boldsymbol{\varphi}_{22}$ are $n \times n$ matrices.

From the boundary-condition equations—entry 2 of Table 5-1—we find that

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f). \tag{5.2-11}$$

Substituting this for $\mathbf{p}^*(t_f)$ in (5.2-10a) gives

$$\begin{aligned} \mathbf{x}^*(t_f) &= \boldsymbol{\varphi}_{11}(t_f, t)\mathbf{x}^*(t) + \boldsymbol{\varphi}_{12}(t_f, t)\mathbf{p}^*(t) \\ \mathbf{H}\mathbf{x}^*(t_f) &= \boldsymbol{\varphi}_{21}(t_f, t)\mathbf{x}^*(t) + \boldsymbol{\varphi}_{22}(t_f, t)\mathbf{p}^*(t). \end{aligned} \tag{5.2-12}$$

Substituting the upper equation into the lower, we obtain

$$\begin{aligned} \mathbf{H}\boldsymbol{\varphi}_{11}(t_f, t)\mathbf{x}^*(t) + \mathbf{H}\boldsymbol{\varphi}_{12}(t_f, t)\mathbf{p}^*(t) &= \boldsymbol{\varphi}_{21}(t_f, t)\mathbf{x}^*(t) \\ &+ \boldsymbol{\varphi}_{22}(t_f, t)\mathbf{p}^*(t), \end{aligned} \tag{5.2-13}$$

which, when solved for $\mathbf{p}^*(t)$, yields

$$\mathbf{p}^*(t) = [\boldsymbol{\varphi}_{22}(t_f, t) - \mathbf{H}\boldsymbol{\varphi}_{12}(t_f, t)]^{-1}[\mathbf{H}\boldsymbol{\varphi}_{11}(t_f, t) - \boldsymbol{\varphi}_{21}(t_f, t)]\mathbf{x}^*(t). \tag{5.2-14}$$

Kalman [K-7] has shown that the required inverse exists for all $t \in [t_0, t_f]$. Equation (5.2-14) can also be written as

$$\mathbf{p}^*(t) \triangleq \mathbf{K}(t)\mathbf{x}^*(t), \tag{5.2-15}$$

which means that $\mathbf{p}^*(t)$ is a linear function of the states of the system; $\mathbf{K}$ is an $n \times n$ matrix. Actually, $\mathbf{K}$ depends on t_f also, but t_f is specified.

Substituting in (5.2-7), we obtain

$$\boxed{\begin{aligned} \mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) \\ &\triangleq \mathbf{F}(t)\mathbf{x}(t),^{\dagger} \end{aligned}} \tag{5.2-16}$$

which indicates that the optimal control law is a linear, albeit time-varying, combination of the system states. Notice that even if the plant is fixed, the feedback gain matrix $\mathbf{F}$ is time-varying.‡ In addition, measurements of all of the state variables must be available to implement the optimal control law. Figure 5-6 shows the plant and its optimal controller.

To determine the feedback gain matrix $\mathbf{F}$, we need the transition matrix for the system given in (5.2-9). If all of the matrices involved ($\mathbf{A}, \mathbf{B}, \mathbf{R}, \mathbf{Q}$) are time-invariant, the required transition matrix can be found by evaluating the inverse Laplace transform of the matrix

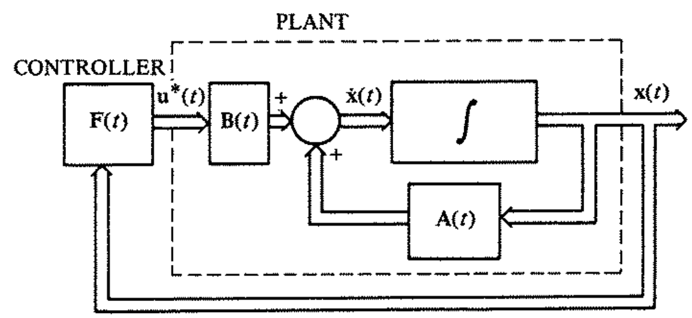


Figure 5-6 Plant and optimal feedback controller for linear regulator problems

† Here we drop the * notation because the optimal control law applies for all $\mathbf{x}(t)$.
‡ In certain cases it may be possible to implement a nonlinear, but time-invariant, optimal control law—see [J-1].

$$\left\{ s\mathbf{I} - \left[\begin{array}{c|c} \mathbf{A} & -\mathbf{BR}^{-1}\mathbf{B}^T \\ \hline -\mathbf{Q} & -\mathbf{A}^T \end{array}\right]\right\}^{-1},$$

and substituting $(t_f - t)$ for t. Unfortunately, when the order of the system is large this becomes a tedious and time-consuming task. If any of the matrices in (5.2-9) is time-varying, we must generally resort to a numerical procedure for evaluating $\varphi(t_f, t)$.

There is an alternative approach, however; it can be shown (see Problem 5-9) that the matrix $\mathbf{K}$ satisfies the matrix differential equation

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t), \quad (5.2\text{-}17)$$

with the boundary condition $\mathbf{K}(t_f) = \mathbf{H}$.

This matrix differential equation is of the Riccati type; in fact, we shall call (5.2-17) the *Riccati equation*.† Since $\mathbf{K}$ is an $n \times n$ matrix, Eq. (5.2-17) is a system of n^2 first-order differential equations. Actually, it can be shown (see Problem 5-9), that $\mathbf{K}$ is symmetric; hence, not n^2, but $n(n+1)/2$ first-order differential equations must be solved. These equations can be integrated numerically by using a digital computer. The integration is started at $t = t_f$ and proceeds backward in time to $t = t_0$; $\mathbf{K}(t)$ is stored, and the feedback gain matrix is determined from Eq. (5.2-16).

Let us illustrate these concepts with the following examples.

Example 5.2-1. Find the optimal control law for the system

$$\dot{x}(t) = ax(t) + u(t) \quad (5.2\text{-}18)$$

to minimize the performance measure

$$J(u) = \tfrac{1}{2}Hx^2(T) + \int_0^T \tfrac{1}{4}u^2(t)\,dt. \quad (5.2\text{-}19)$$

The admissible state and control values are unconstrained, the final time T is specified, $H > 0$, and $x(T)$ is free.

Equation (5.2-9) gives

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} = \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}, \quad (5.2\text{-}20)$$

† The Riccati equation is also derived in Section 3.12, where the Hamilton-Jacobi-Bellman equation is used.

which has the transition matrix

$$\varphi(t) = \left[\begin{array}{c|c} \epsilon^{at} & \frac{1}{a}\epsilon^{-at} - \frac{1}{a}\epsilon^{at} \\ \hline 0 & \epsilon^{-at} \end{array}\right];$$

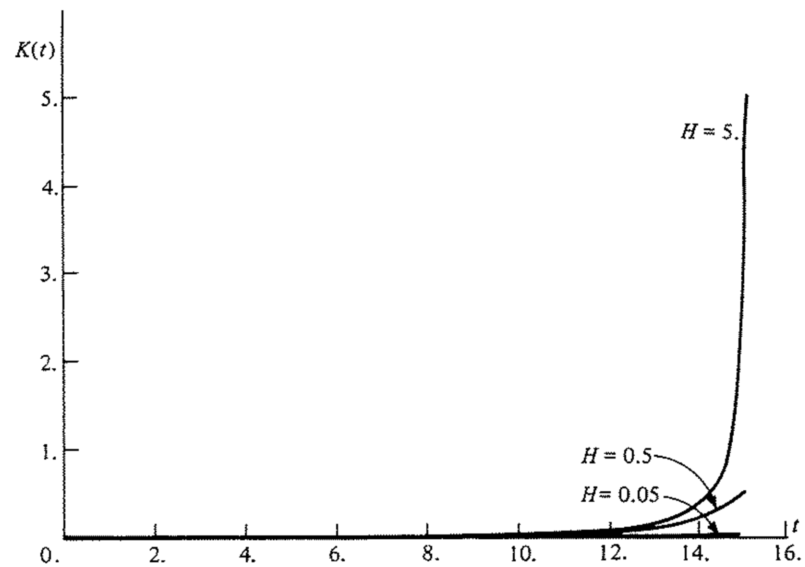
thus, from Eqs. (5.2-14) and (5.2-15) we have

$$K(t) = \left[\epsilon^{-a(T-t)} - \frac{H}{a}[\epsilon^{-a(T-t)} - \epsilon^{a(T-t)}]\right]^{-1}[H\epsilon^{a(T-t)}], \quad (5.2\text{-}21)$$

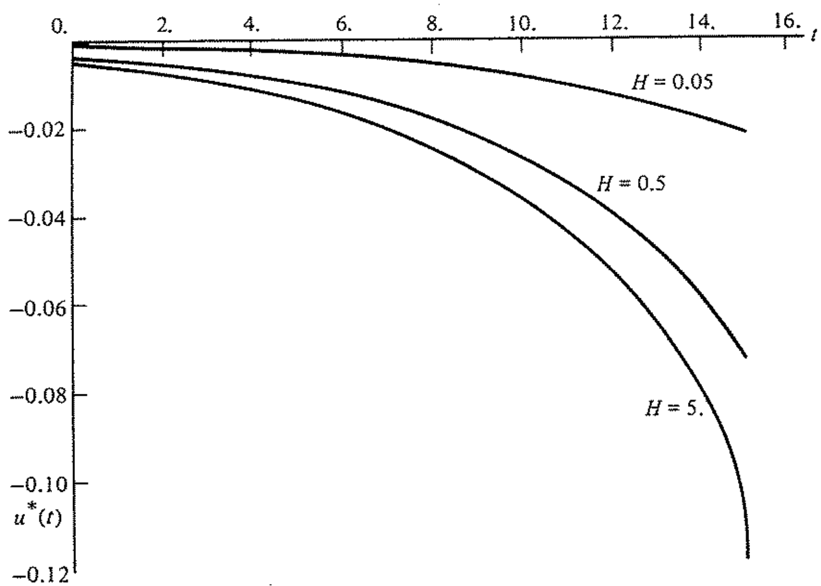
and from Eq. (5.2-16) the optimal control law is

$$u^*(t) = -2K(t)x(t) \quad (5.2\text{-}22)$$

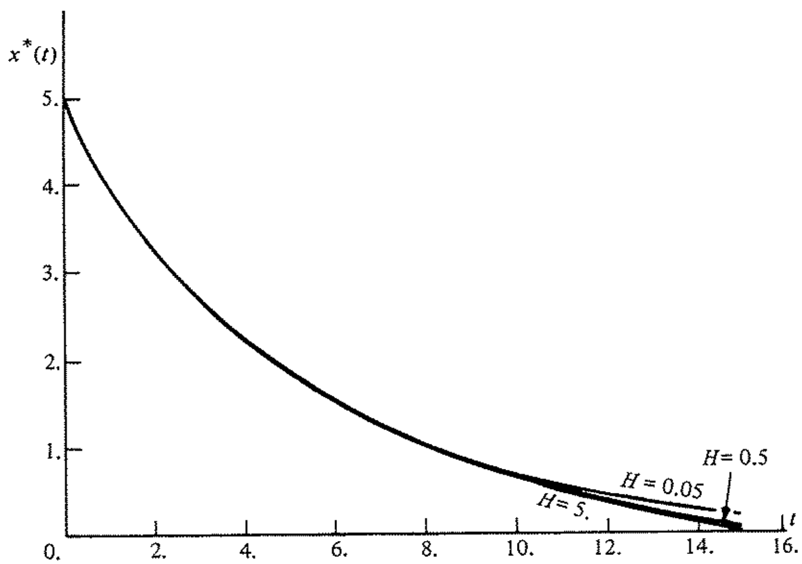
Figure 5-7(a) shows $K(t)$ as a function of time for $a = -0.2$ and $T = 15$, with $H = 5.0$, 0.5, and 0.05. The corresponding control histories and state trajectories for $x(0) = 5.0$ are shown in Fig. 5-7(b), (c). Notice that the state trajectories are almost identical and that the control signals are small in all three cases. These qualitative observations can be explained physically by noting that with $a = -0.2$ the plant is stable and tends toward zero—the desired state—even if no control is applied. Observe in Fig. 5-7(b) that the larger the value of H, the larger the control signal required. This occurs because a larger H indicates that it is



(a)



(b)

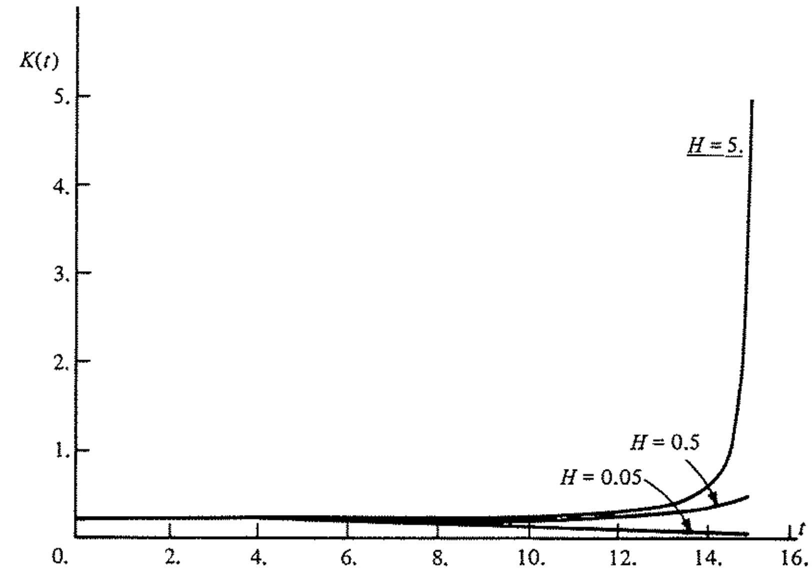


(c)

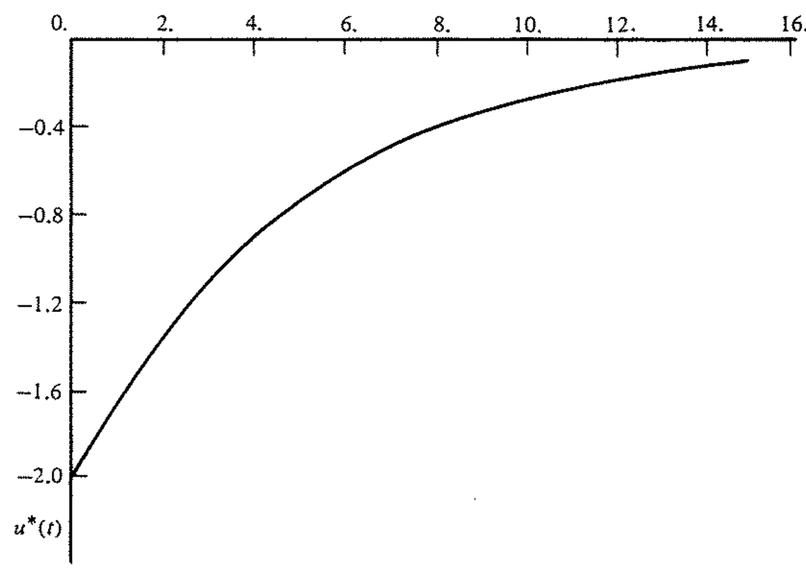
Figure 5-7 (a) Solution of the Riccati equation for $a = -0.2$, $H = 5, 0.5, 0.05$. (b) The optimal control histories for $a = -0.2$, $H = 5, 0.5, 0.05$. (c) The optimal trajectories for $a = -0.2$, $H = 5, 0.5$, and 0.05.

desired to be closer to $x(15) = 0$ than with a smaller H—even if more control effort is required.

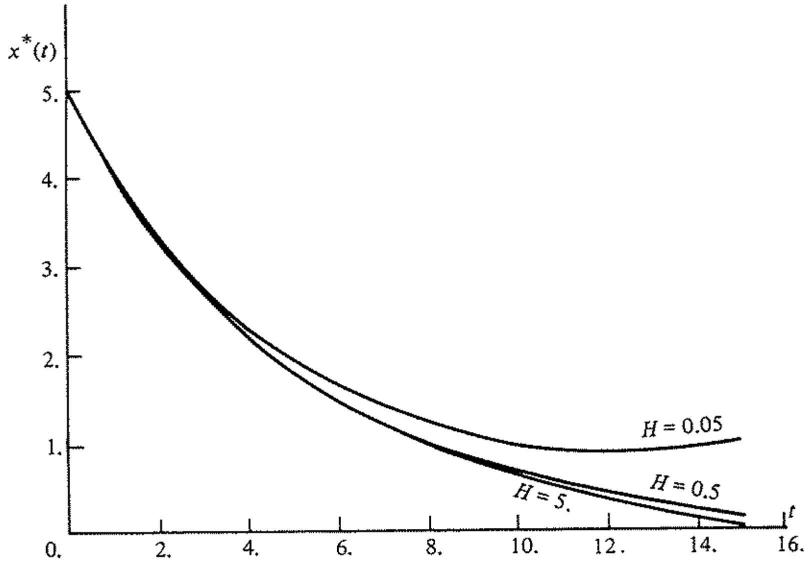
If $a = 0.2$ the results are as shown in Fig. 5-8. Notice that the control signals (which are essentially identical with one another) are much larger than when $a = -0.2$. This is expected, because the plant with $a = 0.2$ is unstable.



(a)



(b)



(c)

Figure 5-8 (a) Solution of the Riccati equation for $a = 0.2$, $H = 5, 0.5, 0.05$. (b) The optimal control histories for $a = 0.2$, $H = 5, 0.5, 0.05$. (c) The optimal trajectories for $a = 0.2$, $H = 5, 0.5, 0.05$.

Another point of interest is the period of time in the interval [0, 15] during which the control signals are largest in magnitude. For the stable plant ($a = -0.2$) the largest controls are applied as $t \longrightarrow 15$. This is the case because the controller "waits" for the system to approach zero on its own before applying control effort. On the other hand, if the controller were to wait for the unstable plant to move toward zero, the instability would cause the value of x to grow larger; hence, the largest control magnitudes are applied in the initial stages of the interval of operation.

Example 5.2-2. Consider the second-order system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= 2x_1(t) - x_2(t) + u(t), \end{aligned} \tag{5.2-23}$$

which is to be controlled to minimize

$$J(u) = \int_0^T \left[x_1^2(t) + \tfrac{1}{2}x_2^2(t) + \tfrac{1}{4}u^2(t)\right] dt. \tag{5.2-24}$$

Find the optimal control law.

By expanding the Riccati equation with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad R = \tfrac{1}{2},$$

we obtain

$$\begin{aligned} \dot{k}_{11}(t) &= 2[k_{12}^2(t) - 2k_{12}(t) - 1] \\ \dot{k}_{12}(t) &= 2k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t) \\ \dot{k}_{22}(t) &= 2k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t) - 1. \end{aligned} \tag{5.2-25}$$

In arriving at (5.2-25) the symmetry of **K** has been used. The boundary conditions are $k_{11}(T) = k_{12}(T) = k_{22}(T) = 0$, and the optimal control law is

$$u^*(t) = -2[k_{12}(t) \quad k_{22}(t)]\mathbf{x}(t). \tag{5.2-26}$$

The solution of the Riccati equation and the optimal control and its trajectory are shown in Fig. 5-9 for $\mathbf{x}(0) = [-4 \quad 4]^T$.

The situation wherein the process is to be controlled for an interval of infinite duration merits special attention. Kalman [K-7] has shown that if (1) the plant is completely controllable, (2) $\mathbf{H} = \mathbf{0}$, and (3) **A**, **B**, **R**, and **Q** are constant matrices, $\mathbf{K}(t) \longrightarrow \mathbf{K}$ (a constant matrix) as $t_f \longrightarrow \infty$. The engineering implications of this result are very important. If the above hypotheses are satisfied, then the optimal control law for an infinite-duration process is stationary. This means that the implementation of the optimal controller is as shown in Fig. 5-6, *except* that $\mathbf{F}(t)$ is constant; thus, the controller consists of m fixed summing amplifiers, each having n inputs. From a practical viewpoint, it may be feasible to use the fixed control law even for processes of finite duration. For instance, in Example 5.2-2 k_{11}, k_{12}, and k_{22} are essentially constants for $0 \leq t \leq 12$. Looking at the state trajectory in Fig. 5-9(b), we see that the states have both essentially reached zero when $t = 5$. This means that perhaps the constant values $k_{11} = 6.03$, $k_{12} = 2.41$, $k_{22} = 1.28$ can be used without significant performance degradation—the designer should compare system performance using the steady-state gains with performance using the time-varying optimal gains to decide which should be implemented.

To determine the **K** matrix for an infinite-time process, we either integrate the Riccati equation backward in time until a steady-state solution is obtained [see Fig.5-9(a)] or solve the nonlinear algebraic equations

$$\mathbf{0} = -\mathbf{K}\mathbf{A} - \mathbf{A}^T\mathbf{K} - \mathbf{Q} + \mathbf{K}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}, \tag{5.2-27}$$

obtained by setting $\dot{\mathbf{K}}(t) = \mathbf{0}$ in Eq. (5.2-17).

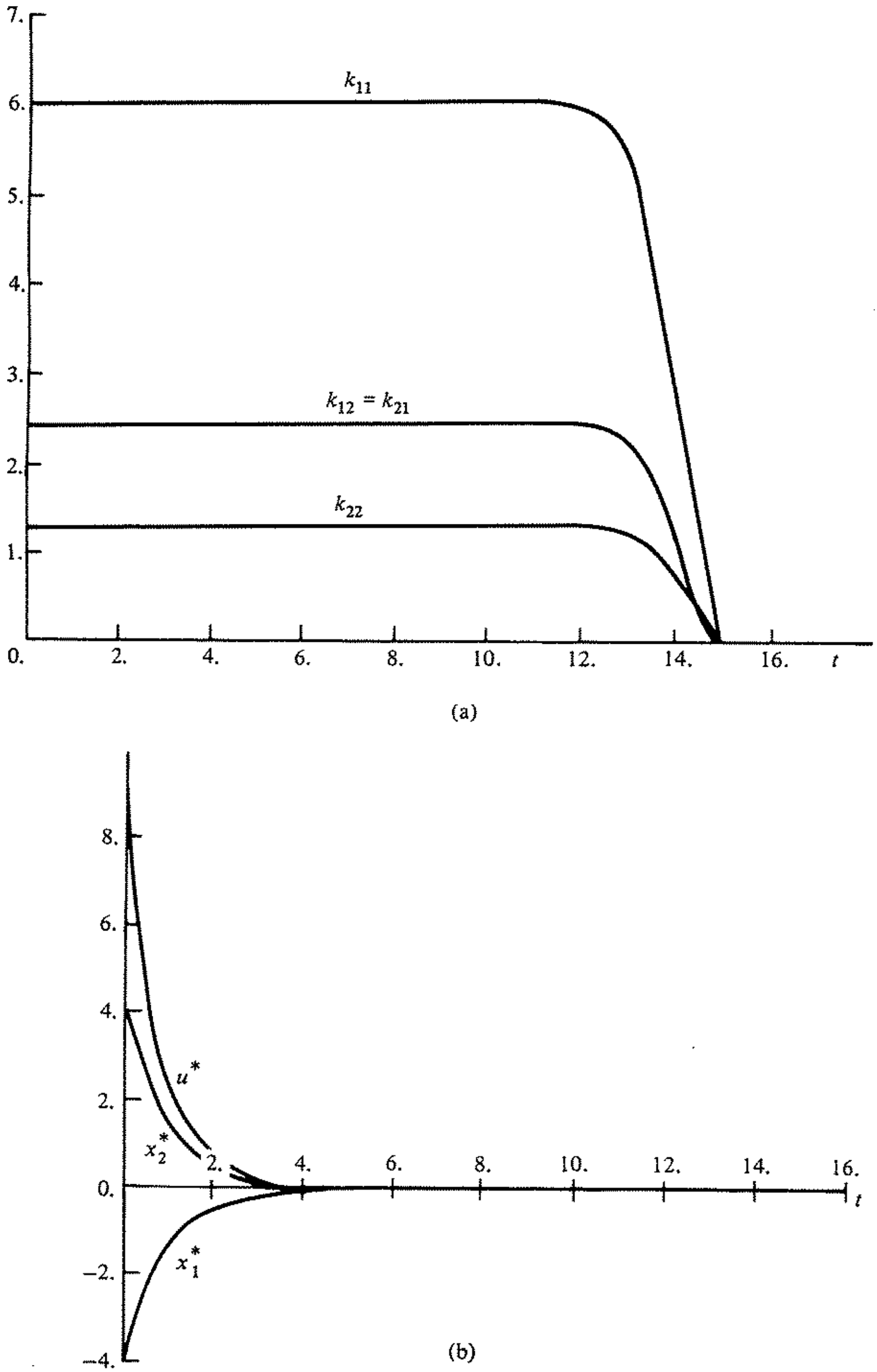


Figure 5-9 (a) The solution of the Riccati equation. (b) The optimal control and its trajectory

Linear Tracking Problems

Next, let us generalize the results obtained for the linear regulator problem to the tracking problem; that is, the desired value of the state vector is not the origin.

The state equations are

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{5.2-28}$$

and the performance measure to be minimized is

$$J = \tfrac{1}{2}[\mathbf{x}(t_f) - \mathbf{r}(t_f)]^T\mathbf{H}[\mathbf{x}(t_f) - \mathbf{r}(t_f)] + \tfrac{1}{2}\int_{t_0}^{t_f}\{[\mathbf{x}(t) - \mathbf{r}(t)]^T\mathbf{Q}(t)[\mathbf{x}(t) - \mathbf{r}(t)]$$
$$+ \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\}\,dt$$
$$\triangleq \tfrac{1}{2}\|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|_{\mathbf{H}}^2 + \tfrac{1}{2}\int_{t_0}^{t_f}\{\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2\}\,dt, \tag{5.2-29}$$

where $\mathbf{r}(t)$ is the desired or reference value of the state vector. The final time t_f is fixed, $\mathbf{x}(t_f)$ is free, and the states and controls are not bounded. **H** and **Q** are real symmetric positive semi-definite matrices, and **R** is real symmetric and positive definite.

The Hamiltonian is given by

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \tfrac{1}{2}\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \tfrac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2$$
$$+ \mathbf{p}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{B}(t)\mathbf{u}(t). \tag{5.2-30}$$

The costate equations are

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial\mathscr{H}}{\partial\mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t) + \mathbf{Q}(t)\mathbf{r}(t), \tag{5.2-31}$$

and the algebraic relations that must be satisfied are given by

$$\mathbf{0} = \frac{\partial\mathscr{H}}{\partial\mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t); \tag{5.2-32}$$

therefore,

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t). \tag{5.2-33}$$

Substituting (5.2-33) in the state equations yields the state and costate equations

$$\begin{bmatrix}\dot{\mathbf{x}}^*(t)\\ \hline \dot{\mathbf{p}}^*(t)\end{bmatrix} = \left[\begin{array}{c|c}\mathbf{A}(t) & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\\ \hline -\mathbf{Q}(t) & -\mathbf{A}^T(t)\end{array}\right]\begin{bmatrix}\mathbf{x}^*(t)\\ \hline \mathbf{p}^*(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \hline \mathbf{Q}(t)\mathbf{r}(t)\end{bmatrix} \tag{5.2-34}$$

Notice that the term $\mathbf{Q}(t)\mathbf{r}(t)$ is a forcing function; these differential equations are linear and time-varying, but not homogeneous. The solution of (5.2-34) is

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ \hline \mathbf{p}^*(t_f) \end{bmatrix} = \boldsymbol{\varphi}(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ \hline \mathbf{p}^*(t) \end{bmatrix} + \int_t^{t_f} \boldsymbol{\varphi}(t_f, \tau) \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{Q}(\tau)\mathbf{r}(\tau) \end{bmatrix} d\tau, \qquad (5.2\text{-}35)$$

where $\boldsymbol{\varphi}$ is the transition matrix of the system (5.2-34). If $\boldsymbol{\varphi}$ is partitioned, and the integral replaced by the $2n \times 1$ vector

$$\begin{bmatrix} \mathbf{f}_1(t) \\ \hline \mathbf{f}_2(t) \end{bmatrix},$$

these equations can be written

$$\mathbf{x}^*(t_f) = \boldsymbol{\varphi}_{11}(t_f, t)\mathbf{x}^*(t) + \boldsymbol{\varphi}_{12}(t_f, t)\mathbf{p}^*(t) + \mathbf{f}_1(t) \qquad (5.2\text{-}36a)$$

$$\mathbf{p}^*(t_f) = \boldsymbol{\varphi}_{21}(t_f, t)\mathbf{x}^*(t) + \boldsymbol{\varphi}_{22}(t_f, t)\mathbf{p}^*(t) + \mathbf{f}_2(t). \qquad (5.2\text{-}36b)$$

The boundary conditions are

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f) - \mathbf{H}\mathbf{r}(t_f). \qquad (5.2\text{-}37)$$

Replacing $\mathbf{p}^*(t_f)$ in (5.2-36b) by the right-hand side of (5.2-37) and then substituting $\mathbf{x}^*(t_f)$ from Eq. (5.2-36a) into (5.2-36b), we obtain

$$\begin{aligned}\mathbf{H}[\boldsymbol{\varphi}_{11}(t_f, t)\mathbf{x}^*(t) + \boldsymbol{\varphi}_{12}(t_f, t)\mathbf{p}^*(t) + \mathbf{f}_1(t)] - \mathbf{H}\mathbf{r}(t_f) &= \boldsymbol{\varphi}_{21}(t_f, t)\mathbf{x}^*(t) \\ &+ \boldsymbol{\varphi}_{22}(t_f, t)\mathbf{p}^*(t) + \mathbf{f}_2(t).\end{aligned} \qquad (5.2\text{-}38)$$

Solving for $\mathbf{p}^*(t)$ yields

$$\begin{aligned}\mathbf{p}^*(t) &= [\boldsymbol{\varphi}_{22}(t_f, t) - \mathbf{H}\boldsymbol{\varphi}_{12}(t_f, t)]^{-1}[\mathbf{H}\boldsymbol{\varphi}_{11}(t_f, t) - \boldsymbol{\varphi}_{21}(t_f, t)]\mathbf{x}^*(t) \\ &\quad + [\boldsymbol{\varphi}_{22}(t_f, t) - \mathbf{H}\boldsymbol{\varphi}_{12}(t_f, t)]^{-1}[\mathbf{H}\mathbf{f}_1(t) - \mathbf{H}\mathbf{r}(t_f) - \mathbf{f}_2(t)] \\ &\triangleq \mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{s}(t).\end{aligned} \qquad (5.2\text{-}39)$$

The definitions of $\mathbf{K}(t)$ and $\mathbf{s}(t)$ are apparent by inspection of Eq. (5.2-39); therefore, the optimal control law is

$$\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) - \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t) \\ &\triangleq \mathbf{F}(t)\mathbf{x}(t) + \mathbf{v}(t),\end{aligned} \qquad (5.2\text{-}40)$$

where $\mathbf{F}(t)$ is the feedback gain matrix and $\mathbf{v}(t)$ is the command signal.† Notice that $\mathbf{v}(t)$ depends on the system parameters and on the reference signal $\mathbf{r}(t)$. In fact, $\mathbf{v}(t)$ depends on future values of the reference signal, so we might say that the optimal control has an anticipatory quality. This is reinforced by physical reasoning, which tells us that we must determine our present strategy on the basis of where we are now *and* where we intend to go. (Actually, this same sort of situation was present, though in a more subtle way, in regulator problems, where we utilized our desire to be at the origin.) A diagram of the plant and controller is shown in Fig. 5-10. Notice that, as in the regulator problem, we must be able to measure all of the states in order to synthesize the optimal control law.

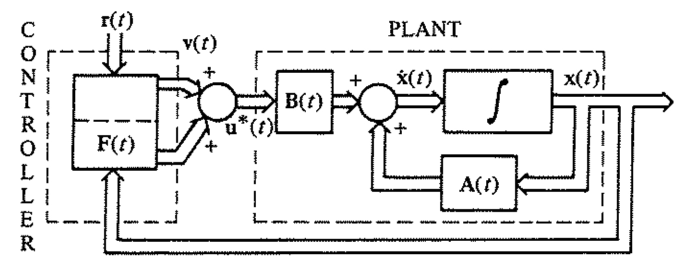


Figure 5-10 Plant and optimal feedback controller for linear tracking problems

Again we are confronted with the need to determine the transition matrix, but, as before, there is an easier computational route to travel. We begin with the equation

$$\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{s}(t). \qquad (5.2\text{-}41)$$

Differentiating both sides with respect to t, we obtain

$$\dot{\mathbf{p}}^*(t) = \dot{\mathbf{K}}(t)\mathbf{x}^*(t) + \mathbf{K}(t)\dot{\mathbf{x}}^*(t) + \dot{\mathbf{s}}(t). \qquad (5.2\text{-}42)$$

Substituting from (5.2-34) for $\dot{\mathbf{p}}^*(t)$ and $\dot{\mathbf{x}}^*(t)$, and using (5.2-41) to eliminate $\mathbf{p}^*(t)$, we obtain

$$\begin{aligned}&[\dot{\mathbf{K}}(t) + \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)]\mathbf{x}^*(t) \\ &\quad + [\dot{\mathbf{s}}(t) + \mathbf{A}^T(t)\mathbf{s}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t) - \mathbf{Q}(t)\mathbf{r}(t)] = \mathbf{0}.\end{aligned} \qquad (5.2\text{-}43)$$

† Strictly speaking, we have not shown that this extremal control does minimize J. It turns out, however, that this extremal control is indeed the optimal control.

Because this must be satisfied for all $\mathbf{x}^*(t)$ and $\mathbf{r}(t)$, we conclude that

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t) \tag{5.2-44}$$

and

$$\dot{\mathbf{s}}(t) = -[\mathbf{A}^T(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)]\,\mathbf{s}(t) + \mathbf{Q}(t)\mathbf{r}(t). \tag{5.2-45}$$

Since $\mathbf{K}$ is symmetric and $\mathbf{s}$ is an $n \times 1$ vector, (5.2-44) and (5.2-45) are a set of $[n[n+1]/2] + n$ first-order differential equations. Notice that (5.2-44) is the same Riccati equation that we obtained for linear regulator problems. To obtain the boundary conditions we have, from (5.2-37) and (5.2-39),

$$\begin{aligned}\mathbf{p}^*(t_f) &= \mathbf{H}\mathbf{x}^*(t_f) - \mathbf{H}\mathbf{r}(t_f)\\ &= \mathbf{K}(t_f)\mathbf{x}^*(t_f) + \mathbf{s}(t_f).\end{aligned} \tag{5.2-46}$$

Since these equations must be satisfied for all $\mathbf{x}^*(t_f)$ and $\mathbf{r}(t_f)$, the boundary conditions are

$$\mathbf{K}(t_f) = \mathbf{H} \tag{5.2-47}$$

and

$$\mathbf{s}(t_f) = -\mathbf{H}\mathbf{r}(t_f). \tag{5.2-48}$$

To determine $\mathbf{F}(t)$ and $\mathbf{v}(t)$, we then integrate (5.2-44) and (5.2-45) *from* t_f *to* t_0 using the boundary conditions (5.2-47) and (5.2-48), and store the values for $\mathbf{K}(t)$ and $\mathbf{s}(t)$. $\mathbf{F}(t)$ and $\mathbf{v}(t)$ can then be determined by using (5.2-40). The procedure is illustrated by the following examples.

Example 5.2-3. The system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= 2x_1(t) - x_2(t) + u(t)\end{aligned} \tag{5.2-49}$$

is to be controlled to minimize the performance measure

$$J(u) = [x_1(T) - 1]^2 + \int_0^T \{[x_1(t) - 1]^2 + 0.0025u^2(t)\}\,dt. \tag{5.2-50}$$

The final time T is specified, $\mathbf{x}(T)$ is free, and the admissible states and controls are not bounded. The optimal control law is to be found.

The performance measure indicates that the state x_1 is to be maintained close to 1.0 without excessive expenditure of control effort. In the nomenclature of linear tracking problems, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{H},$$

$$R = 0.005 \quad \text{and} \quad \mathbf{r}(t) = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix}.\dagger$$

The Riccati equation and the differential equations for $\mathbf{s}$ are found from Eqs. (5.2-44) and (5.2-45) with the result

$$\begin{aligned}\dot{k}_{11}(t) &= 2[100k_{12}^2(t) - 2k_{12}(t) - 1]\\ \dot{k}_{12}(t) &= 200k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t)\\ \dot{k}_{22}(t) &= 200k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t)\end{aligned} \tag{5.2-51}$$

$$\begin{aligned}\dot{s}_1(t) &= 2[100k_{12}(t) - 1]s_2(t) + 2\\ \dot{s}_2(t) &= -s_1(t) + [1 + 200k_{22}(t)]s_2(t),\end{aligned} \tag{5.2-52}$$

and, from Eqs. (5.2-47) and (5.2-48) the boundary conditions are

$$\mathbf{K}(T) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{s}(T) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

The optimal control law, obtained from (5.2-40), is

$$u^*(t) = -200[k_{12}(t)x_1(t) + k_{22}(t)x_2(t) + s_2(t)]. \tag{5.2-53}$$

Figure 5-11(a) shows the optimal control and its trajectory for $T = 15$, and $\mathbf{x}(0) = \mathbf{0}$. The "tail" on the x_1^* curve as $t \longrightarrow t_f$ results because the controller anticipates that the final time is near and reduces the control to values near zero at the expense of deviations in x_1^*. When the control is made small, $x_1^*(t)$ begins to increase; this occurs because the plant (5.2-49) is unstable. The solutions of the Riccati equation and of (5.2-52) are shown in Fig. 5-11(b), (c).

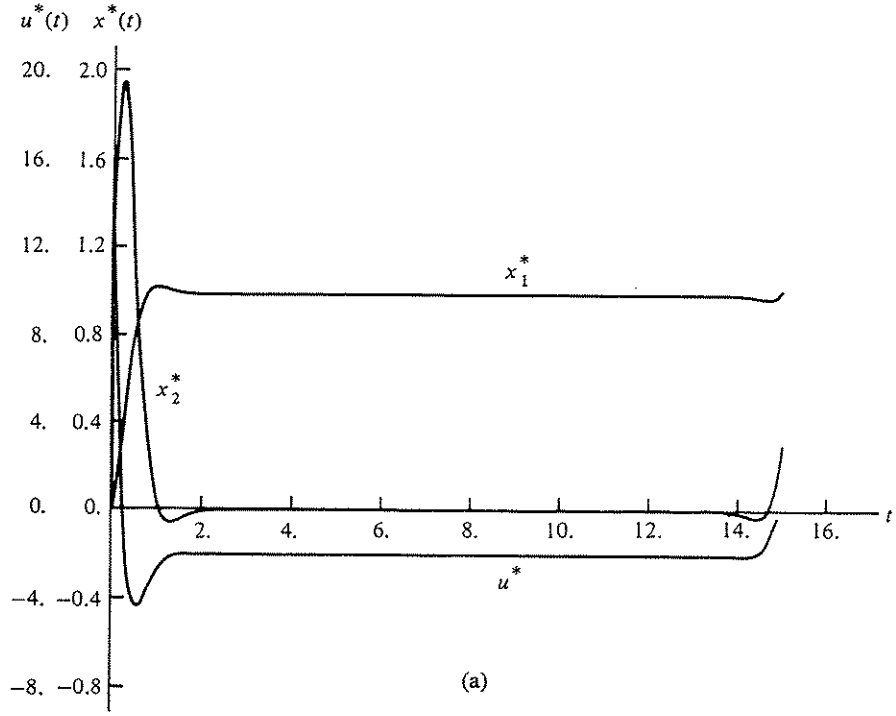
Example 5.2-4. The plant to be controlled is the same as in Example 5.2-3, but the performance measure is

$$J(u) = \int_0^T \{[x_1(t) - 0.2t]^2 + 0.025u^2(t)\}\,dt. \tag{5.2-54}$$

T is specified, $\mathbf{x}(T)$ is free, and the admissible controls are not bounded. The optimal control law is to be determined.

In this problem the objective is to maintain the state x_1 close to the ramp function $r_1(t) = 0.2t$, without excessive expenditure of control effort. By substituting

† For the matrices $\mathbf{H}$ and $\mathbf{Q}$ given in this example, $r_2(t)$ does not affect the solution and hence can be selected arbitrarily.



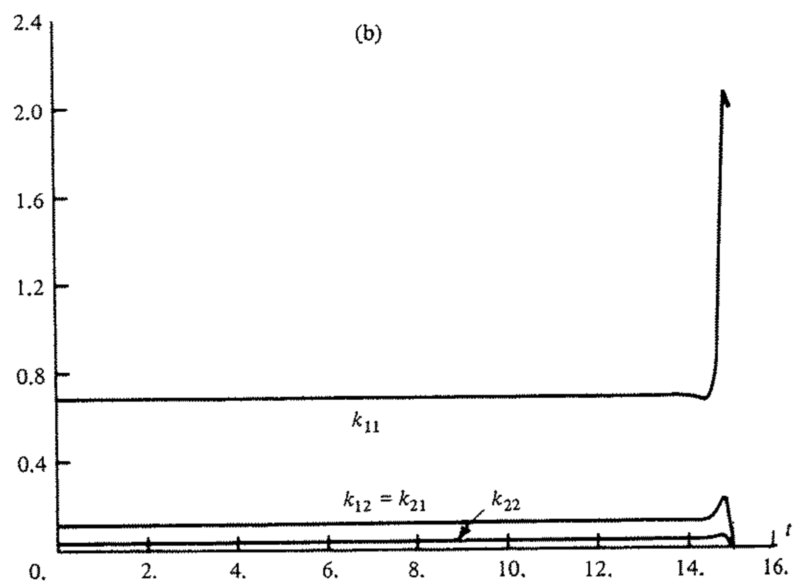


Figure 5-11 (a) The optimal control and trajectory for a linear tracking problem: $r_1(t) = 1.0$, $\mathbf{x}(0) = \mathbf{0}$. (b) Solution of the Riccati equation for Example 5.2-3. (c) s_1 and s_2 for Example 5.2-3

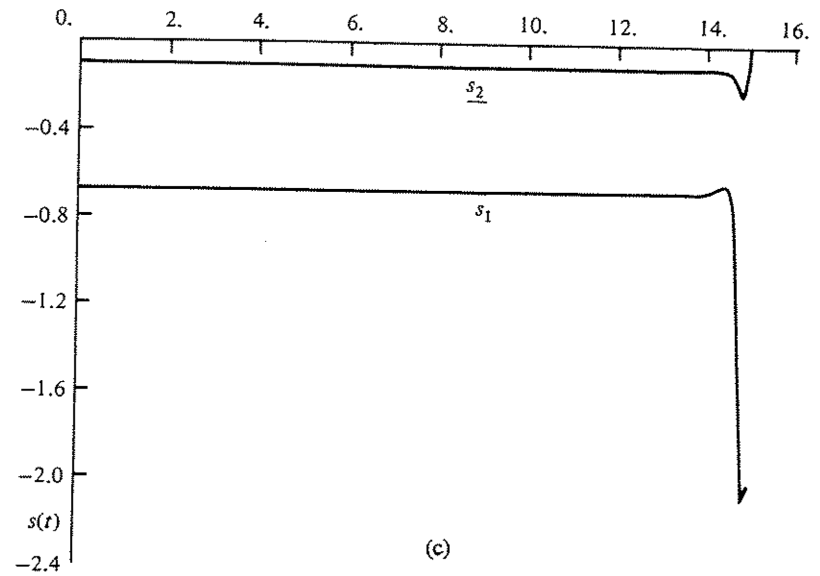


Figure 5-11 cont.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R = 0.05, \quad \text{and} \quad \mathbf{r}(t) = \begin{bmatrix} 0.2t \\ 0 \end{bmatrix}$$

into (5.2-44) and (5.2-45), we obtain the differential equations

$$\dot{k}_{11}(t) = 20k_{12}^2(t) - 4k_{12}(t) - 2$$

$$\dot{k}_{12}(t) = 20k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t) \qquad (5.2\text{-}55)$$

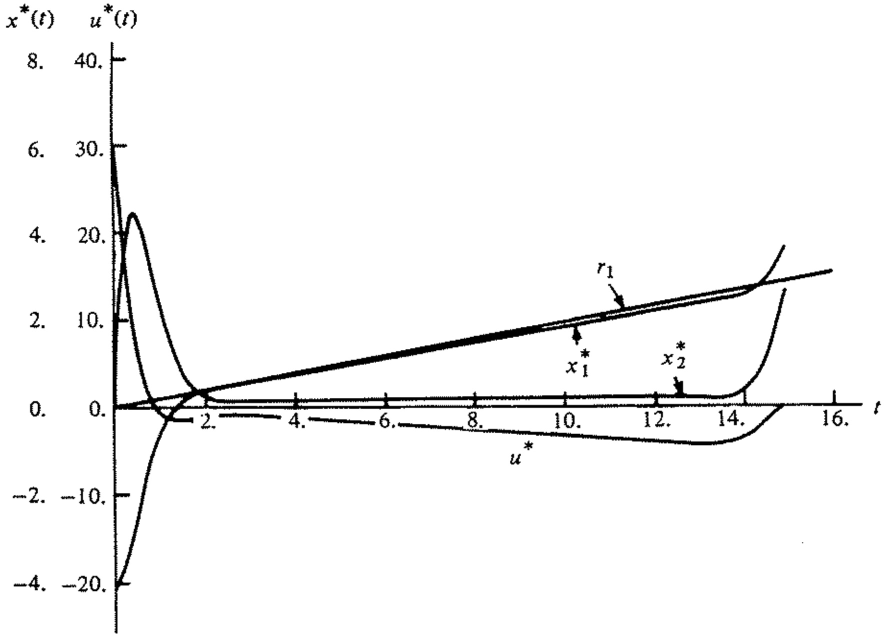
$$\dot{k}_{22}(t) = 20k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t)$$

$$\begin{aligned} \dot{s}_1(t) &= 2[10k_{12}(t) - 1]s_2(t) + 0.4t \\ \dot{s}_2(t) &= -s_1(t) + [20k_{22}(t) + 1]s_2(t). \end{aligned} \qquad (5.2\text{-}56)$$

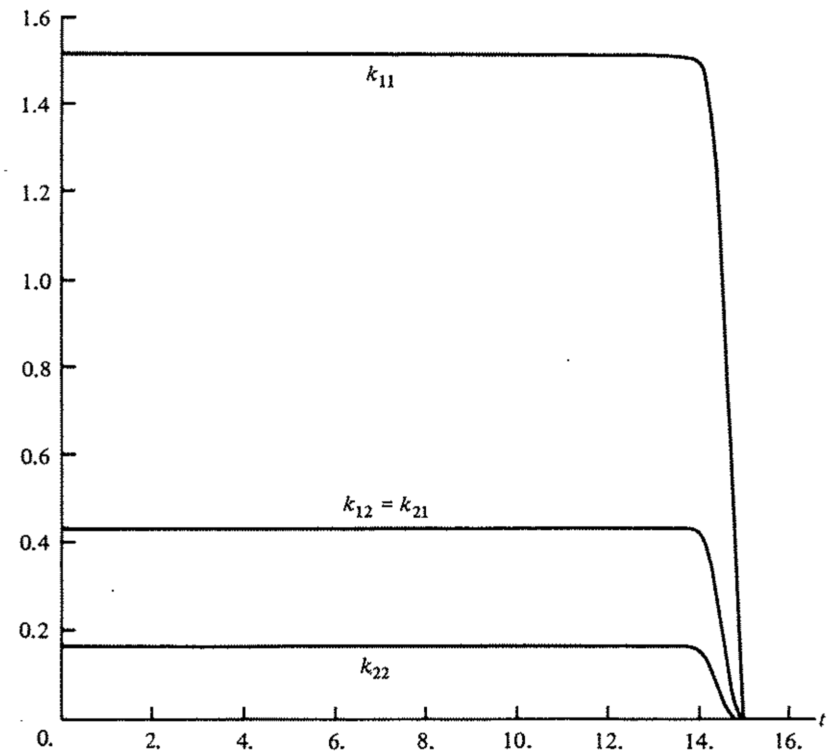
The boundary conditions for these five differential equations are $\mathbf{K}(T) = \mathbf{0}$, $\mathbf{s}(T) = \mathbf{0}$. Figures 5-12(b) and (c) show the solution of Eqs. (5.2-55) and (5.2-56) for $T = 15$. The optimal control law, obtained from Eq. (5.2-40), is

$$u^*(t) = -20[k_{12}(t)x_1(t) + k_{22}(t)x_2(t) + s_2(t)]. \qquad (5.2\text{-}57)$$

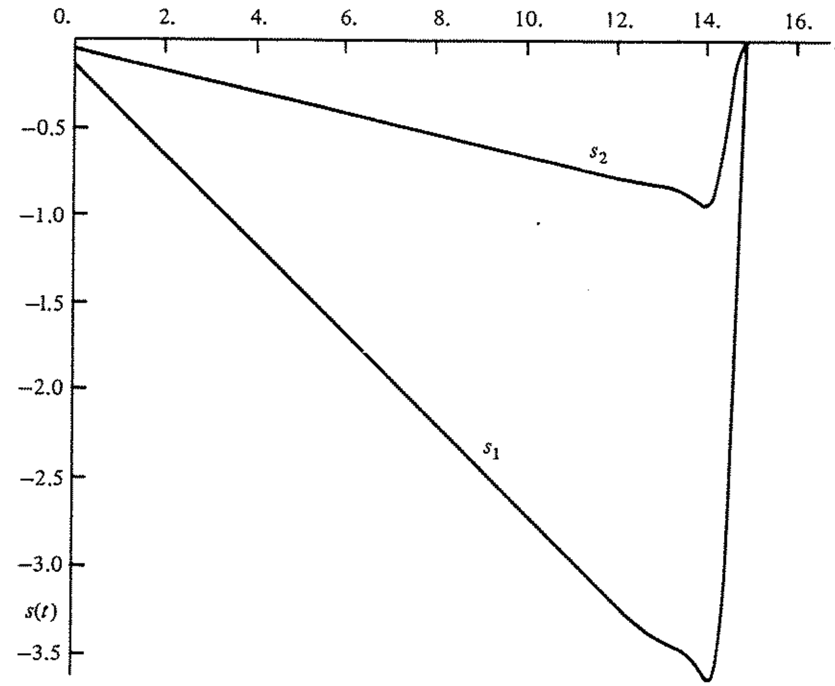
The optimal control and its trajectory for $\mathbf{x}(0) = [-4 \quad 0]^T$ are shown in Fig. 5-12(a). There is an initial transient period that is over at approximately $t = 2$. Thereafter, the difference between x_1^* and r_1 is small,



(a)



(b)



(c)

Figure 5-12 (a) The optimal control and trajectory for a linear tracking problem: $r_1(t) = 0.2t$, $\mathbf{x}(0) = [-4 \;\; 0]^T$. (b) Solution of the Riccati equation for Example 5.2-4. (c) s_1 and s_2 for Example 5.2-4

although the deviation does grow larger with increasing time. This is attributed to the penalty in the performance measure on control-effort expenditure; as time increases, the magnitude of the control signal required for tracking grows larger, so the contribution of control effort to the performance measure becomes more significant. The "tail" present as $t \longrightarrow 15$ occurs because the control law anticipates the end of the control interval and, as a result, conserves control effort, allowing x_1^* to deviate from its desired values.

5.3 PONTRYAGIN'S MINIMUM PRINCIPLE AND STATE INEQUALITY CONSTRAINTS

So far, we have assumed that the admissible controls and states are not constrained by any boundaries; however, in realistic systems such constraints do commonly occur. Physically realizable controls generally have magnitude limitations: the thrust of a rocket engine cannot exceed a certain value; motors, which provide torque, saturate; attitude control mass expulsion systems are capable of providing a limited torque. State constraints often arise because of safety, or structural restrictions: the current in an electric motor cannot exceed a certain value without damaging the windings; the turning radius of a maneuvering aircraft cannot be less than a specified minimum value; a spacecraft reentering the earth's atmosphere must satisfy certain attitude and velocity constraints to avoid burning up.

Let us first consider the effect of control constraints on the fundamental theorem derived in Section 4.1, and then show how the necessary conditions are modified.† This generalization of the fundamental theorem leads to Pontryagin's minimum principle.‡

Pontryagin's Minimum Principle

By definition, the control $\mathbf{u}^*$ causes the functional J to have a relative minimum if

$$J(\mathbf{u}) - J(\mathbf{u}^*) = \Delta J \geq 0 \tag{5.3-1}$$

for all admissible controls sufficiently close to $\mathbf{u}^*$. If we let $\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}$, the increment in J can be expressed as

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \delta J(\mathbf{u}^*, \delta\mathbf{u}) + \text{higher-order terms}; \tag{5.3-2}$$

δJ is linear in $\delta\mathbf{u}$ and the higher-order terms approach zero as the norm of $\delta\mathbf{u}$ approaches zero. If we were to re-prove the fundamental theorem *for unbounded controls* using control system notation, the reasoning would be exactly as given in Section 4.1. That is, if the control were unbounded, we could use the linearity of δJ with respect to $\delta\mathbf{u}$, and the fact that $\delta\mathbf{u}$ can vary arbitrarily to show that a necessary condition for $\mathbf{u}^*$ to be an extremal control is that the variation $\delta J(\mathbf{u}^*, \delta\mathbf{u})$ must be zero for all admissible $\delta\mathbf{u}$ having a sufficiently small norm. Since we are no longer assuming that the admissible controls are not bounded, $\delta\mathbf{u}$ is arbitrary only if the extremal control is strictly within the boundary for all time in the interval $[t_0, t_f]$. In this case, the boundary has no effect on the problem solution. If, however, an extremal control lies on a boundary during at least one subinterval $[t_1, t_2]$ of the interval $[t_0, t_f]$, as shown in Fig. 5-13(a), then admissible control variations $\delta\hat{\mathbf{u}}$ exist whose negatives $(-\delta\hat{\mathbf{u}})$ are not admissible. One such control variation is shown in Fig. 5-13(b). If only these variations are considered, a necessary condition for $\mathbf{u}^*$ to minimize J is that $\delta J(\mathbf{u}^*, \delta\hat{\mathbf{u}}) \geq 0$. On the other hand, for variations

† The derivation given here is heuristic; for rigorous proofs see [P-1], [R-1], and [A-2].
‡ In Pontryagin's original work, [P-1], this result is referred to as the maximum principle because of a sign difference in the definition of the Hamiltonian.

$\delta\tilde{\mathbf{u}}$, which are nonzero only for t not in the interval $[t_1, t_2]$, as, for example, in Fig. 5-13(c), it is necessary that $\delta J(\mathbf{u}^*, \delta\tilde{\mathbf{u}}) = 0$; the reasoning used in proving the fundamental theorem applies. Considering all admissible variations with $\|\delta\mathbf{u}\|$ small enough so that the sign of ΔJ is determined by δJ, we see that a necessary condition for $\mathbf{u}^*$ to minimize J is

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0. \tag{5.3-3}$$

It seems reasonable to ask if this result has an analog in calculus. To answer this question, refer to Fig. 4-4, where a function f, defined on a *closed*

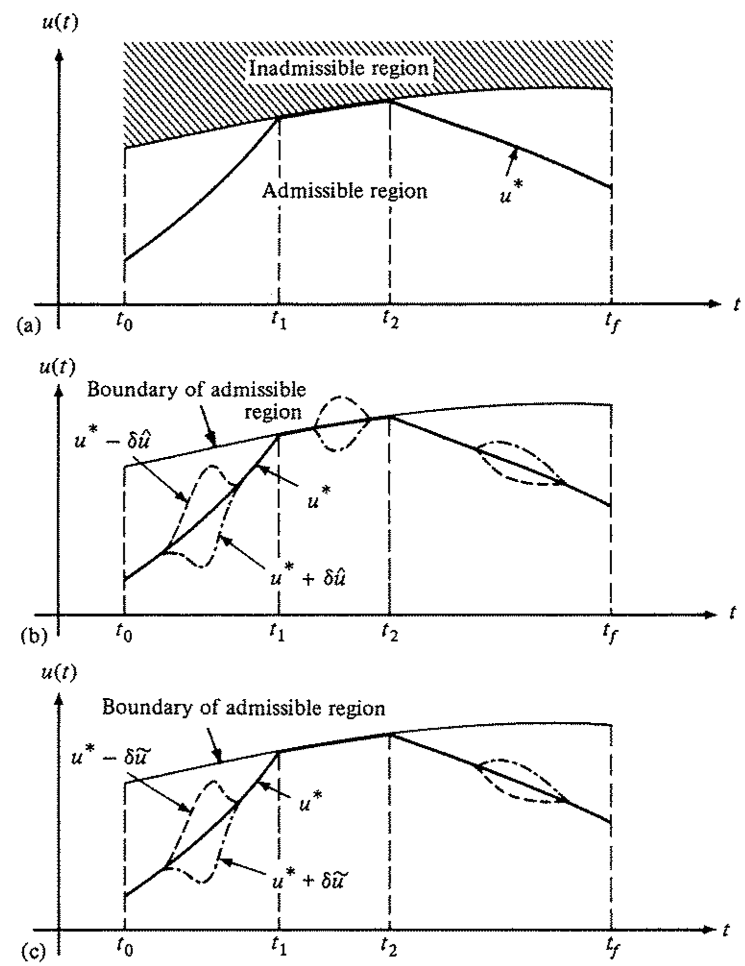


Figure 5-13 (a) An extremal control that is constrained by a boundary. (b) An admissible variation $\delta\hat{u}$ for which $-\delta\hat{u}$ is not admissible. (c) An admissible variation $\delta\tilde{u}$ for which $-\delta\tilde{u}$ is admissible

interval $[t_0, t_f]$, is shown. The differential df is the linear part of the increment Δf. Consider the end points t_0 and t_f of the interval, and admissible values of the time increment Δt, which are small enough so that the sign of Δf is determined by the sign of df. If t_0 is a point where f has a relative minimum, then $df(t_0, \Delta t)$ *must be* greater than or equal to zero. The same requirement applies for $f(t_f)$ to be a relative minimum. Thus, necessary conditions for the function f to have relative minima at the end points of the interval are

$$\begin{aligned} df(t_0, \Delta t) &\geq 0, \qquad \text{admissible } \Delta t \geq 0 \\ df(t_f, \Delta t) &\geq 0, \qquad \text{admissible } \Delta t \leq 0, \end{aligned} \tag{5.3-4}$$

and a necessary condition for f to have a relative minimum at an interior point t, $t_0 < t < t_f$, is

$$df(t, \Delta t) = 0. \tag{5.3-5}$$

For the control problem the analogous necessary conditions are

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0 \tag{5.3-6a}$$

if $\mathbf{u}^*$ lies on the boundary *during any portion of the time interval* $[t_0, t_f]$, and

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) = 0 \tag{5.3-6b}$$

if $\mathbf{u}^*$ lies within the boundary *during the entire time interval* $[t_0, t_f]$.

Next, let us see how this modification affects the necessary conditions, Eqs. (5.1-17) and (5.1-18), which were derived by using the assumption that the admissible control values were unconstrained. The increment of J is [if we use Eqs. (5.1-9), (5.1-13), and the definition of the Hamiltonian]

$$\begin{aligned} \Delta J(\mathbf{u}^*, \delta\mathbf{u}) = & \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f \\ & + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f \\ & + \int_{t_0}^{t_f} \left\{ \left[\dot{\mathbf{p}}^*(t) + \frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{x}(t) \right. \\ & + \left[\frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t) \\ & \left. + \left[\frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \dot{\mathbf{x}}^*(t)\right]^T \delta\mathbf{p}(t) \right\} dt \\ & + \text{higher-order terms.} \end{aligned} \tag{5.3-7}$$

If the state equations are satisfied, and $\mathbf{p}^*(t)$ is selected so that the coefficient of $\delta\mathbf{x}(t)$ in the integral is identically zero, and the boundary condition equation (5.1-18) is satisfied, we have

$$\begin{aligned} \Delta J(\mathbf{u}^*, \delta\mathbf{u}) = & \int_{t_0}^{t_f} \left[\frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t)\, dt \\ & + \text{higher-order terms.} \end{aligned} \tag{5.3-8}$$

The integrand is the first-order approximation to the change in $\mathscr{H}$ caused by a change in $\mathbf{u}$ alone; that is,

$$\begin{aligned} \left[\frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t) \doteq & \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) \\ & - \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t); \end{aligned} \tag{5.3-9}$$

therefore,

$$\begin{aligned} \Delta J(\mathbf{u}^*, \delta\mathbf{u}) = & \int_{t_0}^{t_f} [\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) \\ & - \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)]\, dt \\ & + \text{higher-order terms.} \end{aligned} \tag{5.3-10}$$

If $\mathbf{u}^* + \delta\mathbf{u}$ is in a sufficiently small neighborhood of $\mathbf{u}^*$ ($\|\delta\mathbf{u}\| < \beta$) then the higher-order terms are small, and the integral in Eq. (5.3-10) dominates the expression for ΔJ. Thus, for $\mathbf{u}^*$ to be a minimizing control it is necessary that

$$\int_{t_0}^{t_f} [\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) - \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)]\, dt \geq 0 \tag{5.3-11}$$

for all admissible $\delta\mathbf{u}$, such that $\|\delta\mathbf{u}\| < \beta$. We assert that in order for (5.3-11) to be satisfied for all admissible $\delta\mathbf{u}$ in the specified neighborhood, it is necessary that

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) \geq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \tag{5.3-12}$$

for all *admissible* $\delta\mathbf{u}(t)$ and for all $t \in [t_0, t_f]$. To show this, consider the control

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}^*(t); \qquad t \notin [t_1, t_2] \\ \mathbf{u}(t) &= \mathbf{u}^*(t) + \delta\mathbf{u}(t); \qquad t \in [t_1, t_2], \end{aligned} \tag{5.3-13}$$

where $[t_1, t_2]$ is an arbitrarily small, but nonzero, time interval, and $\delta\mathbf{u}(t)$ is an admissible control variation that satisfies $\|\delta\mathbf{u}\| < \beta$.† Suppose that

† Let

$$\|\delta\mathbf{u}\| = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |\delta u_i(t)|\right] dt.$$

Since $\mathbf{u}(t)$ is in a bounded region, each component of $\delta\mathbf{u}(t)$ is bounded and $\|\delta\mathbf{u}\|$ can be made less than β *for all admissible* $\delta\mathbf{u}(t)$ by making the interval $[t_1, t_2]$ small enough. Thus the control $\mathbf{u}(t)$ in Eq. (5.3-13) can be *any admissible control* in the interval $[t_1, t_2]$.

inequality (5.3-12) *is not satisfied* for the control described in Eq. (5.3-13); then in the interval $[t_1, t_2]$

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) < \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \qquad (5.3\text{-}14)$$

and, therefore,

$$\int_{t_0}^{t_f} [\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) - \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)]\, dt$$
$$= \int_{t_1}^{t_2} [\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) - \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)]\, dt < 0. \qquad (5.3\text{-}15)$$

Since the interval $[t_1, t_2]$ can be anywhere in the interval $[t_0, t_f]$, it is clear that if

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) < \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \qquad (5.3\text{-}16)$$

for any $t \in [t_0, t_f]$, then it is always possible to construct an admissible control, as in Eq. (5.3-13), which makes $\Delta J < 0$, thus contradicting the optimality of the control $\mathbf{u}^*$. Our conclusion is, therefore, that a necessary condition for $\mathbf{u}^*$ to minimize the functional J is

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \qquad (5.3\text{-}17)$$

for all $t \in [t_0, t_f]$ and for *all admissible controls*. Equation (5.3-17), which indicates that *an optimal control must minimize the Hamiltonian*, is called Pontryagin's minimum principle. Notice that we have established a necessary, but not (in general) sufficient, condition for optimality. An optimal control must satisfy Pontryagin's minimum principle; however, there may be controls that satisfy the minimum principle that are not optimal.

Let us now summarize the principal results of this section. A control $\mathbf{u}^* \in U$, which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (5.3\text{-}18)$$

to follow an admissible trajectory that minimizes the performance measure

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt, \qquad (5.3\text{-}19)$$

is sought. In terms of the Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)], \qquad (5.3\text{-}20)$$

necessary conditions for $\mathbf{u}^*$ to be an optimal control are

$$\left.\begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) && (5.3\text{-}21a) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) && (5.3\text{-}21b) \\ \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) &\leq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) && (5.3\text{-}21c) \\ &\text{for all admissible } \mathbf{u}(t) \end{aligned}\right\} \text{for all } t \in [t_0, t_f]$$

and

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f$$
$$+ \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0. \qquad (5.3\text{-}22)$$

It should be emphasized that

1. $\mathbf{u}^*(t)$ is a control that causes $\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t)$ to assume its *global*, or absolute, minimum.
2. Equations (5.3-21) and (5.3-22) constitute a set of *necessary* conditions for optimality; these conditions are not, in general, sufficient.

In addition, the minimum principle, although derived for controls with values in a closed and bounded region, can also be applied to problems in which the admissible controls are not bounded. This can be done by viewing the unbounded control region as having arbitrarily large bounds, thus ensuring that the optimal control will not be constrained by the boundaries. In this case, for $\mathbf{u}^*(t)$ to minimize the Hamiltonian it is necessary (but not sufficient) that

$$\frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) = \mathbf{0}. \qquad (5.3\text{-}23)$$

If Eq. (5.3-23) is satisfied, and the matrix

$$\frac{\partial^2 \mathscr{H}}{\partial \mathbf{u}^2}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

is positive definite, this is sufficient to guarantee that $\mathbf{u}^*(t)$ causes $\mathscr{H}$ to be a *local* minimum; if the Hamiltonian can be expressed in the form

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = f(\mathbf{x}(t), \mathbf{p}(t), t)$$
$$+ [\mathbf{c}(\mathbf{x}(t), \mathbf{p}(t), t)]^T \mathbf{u}(t) + \tfrac{1}{2}\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t), \qquad (5.3\text{-}24)$$

where $\mathbf{c}$ is an $m \times 1$ array that does not have any terms containing $\mathbf{u}(t)$, then satisfaction of (5.3-23) and $\partial^2\mathscr{H}/\partial\mathbf{u}^2 > 0$† are necessary and sufficient for $\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$ to be a *global* minimum.

For $\mathscr{H}$ of the form of (5.3-24),

$$\frac{\partial^2\mathscr{H}}{\partial\mathbf{u}^2}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) = \mathbf{R}(t); \tag{5.3-25}$$

thus, if $\mathbf{R}(t)$ is positive definite,

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{c}(\mathbf{x}^*(t), \mathbf{p}^*(t), t) \tag{5.3-26}$$

minimizes (globally) the Hamiltonian.

Example 5.3-1. Let us now illustrate the effect on the necessary conditions of constraining the admissible control values. Consider the system having the state equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_2(t) + u(t),\end{aligned} \tag{5.3-27}$$

with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$. The performance measure to be minimized is

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}[x_1^2(t) + u^2(t)]\,dt; \tag{5.3-28}$$

t_f is specified, and the final state $\mathbf{x}(t_f)$ is free.

a. Find necessary conditions for an *unconstrained* control to minimize J. The Hamiltonian is

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}x_1^2(t) + \tfrac{1}{2}u^2(t) + p_1(t)x_2(t)\\ - p_2(t)x_2(t) + p_2(t)u(t),\end{aligned} \tag{5.3-29}$$

from which the costate equations are

$$\begin{aligned}\dot{p}_1^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_1} = -x_1^*(t)\\ \dot{p}_2^*(t) &= -\frac{\partial\mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t).\end{aligned} \tag{5.3-30}$$

Since the control values are unconstrained, it is necessary that

$$\frac{\partial\mathscr{H}}{\partial u} = u^*(t) + p_2^*(t) = 0. \tag{5.3-31}$$

† The notation $\partial^2\mathscr{H}/\partial\mathbf{u}^2 > 0$ means that the $m \times m$ matrix $\partial^2\mathscr{H}/\partial\mathbf{u}^2$ is positive definite.

Notice that the Hamiltonian is of the form (5.3-24), and

$$\frac{\partial^2\mathscr{H}}{\partial u^2} = 1; \tag{5.3-32}$$

therefore,

$$u^*(t) = -p_2^*(t) \tag{5.3-33}$$

does minimize the Hamiltonian. The boundary conditions are (see Table 5-1, entry 2)

$$\mathbf{p}^*(t_f) = \mathbf{0}. \tag{5.3-34}$$

b. Find necessary conditions for optimal control if

$$-1 \le u(t) \le +1 \qquad \text{for all } t \in [t_0, t_f]. \tag{5.3-35}$$

The state and costate equations and the boundary condition for $\mathbf{p}^*(t_f)$ remains unchanged; however, now u must be selected to minimize

$$\begin{aligned}\mathscr{H}(\mathbf{x}^*(t), u(t), \mathbf{p}^*(t)) = \tfrac{1}{2}x_1^{*2}(t) + \tfrac{1}{2}u^2(t) + p_1^*(t)x_2^*(t)\\ - p_2^*(t)x_2^*(t) + p_2^*(t)u(t)\end{aligned} \tag{5.3-36}$$

subject to the constraining relation in Eq. (5.3-35).

To determine the control that minimizes $\mathscr{H}$, we first separate all of the terms containing $u(t)$,

$$\tfrac{1}{2}u^2(t) + p_2^*(t)u(t), \tag{5.3-37}$$

from the Hamiltonian. For times when the optimal control is unsaturated, we have

$$u^*(t) = -p_2^*(t) \tag{5.3-38}$$

as in part a; clearly, this will occur when $|p_2^*(t)| \le 1$. If, however, there are times when $|p_2^*(t)| > 1$, then from (5.3-37) the control that minimizes $\mathscr{H}$ is

$$u^*(t) = \begin{cases}-1, & \text{for } p_2^*(t) > 1\\ +1, & \text{for } p_2^*(t) < -1.\end{cases} \tag{5.3-39}$$

Thus, $u^*(t)$ is the saturation function of $p_2^*(t)$ pictured in Fig. 5-14.

In summary, then, we have for the *unconstrained* control—part a,

$$u^*(t) = -p_2^*(t), \tag{5.3-33}$$

and, for the *constrained* control—part b,

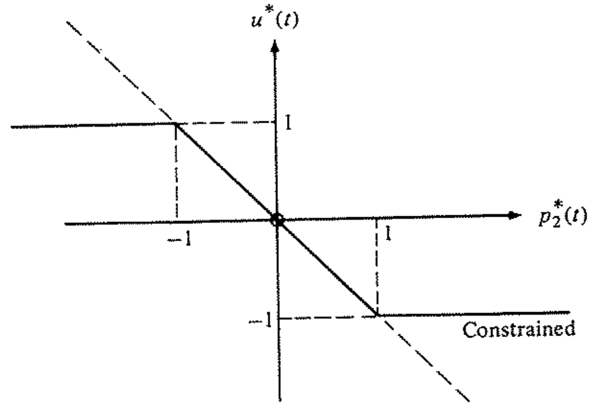


Figure 5-14 Constrained and unconstrained optimal controls for Example 5.3-1

$$u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \le p_2^*(t) \le 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases} \tag{5.3-39a}$$

To determine $u^*(t)$ explicitly, the state and costate equations must be solved. Because of the differences in Eqs. (5.3-33) and (5.3-39a), the state-costate trajectories in the two cases will be the same only if the initial state values are such that the bounded control does not saturate. If this situation occurs, the control constraints do not affect the solution. It must be emphasized that the optimal control history for part b *cannot* be determined, in general, by calculating the optimal control history for part a and allowing it to saturate whenever the stipulated boundaries are violated.

Additional Necessary Conditions

Pontryagin and his co-workers have also derived other necessary conditions for optimality that we will find useful. We now state, without proof, two of these necessary conditions:

1. If the final time is fixed and the Hamiltonian does not depend explicitly on time, then the Hamiltonian must be a constant when evaluated on an extremal trajectory; that is,

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) = c_1 \qquad \text{for } t \in [t_0, t_f]. \tag{5.3-40}$$

2. If the final time is free, and the Hamiltonian does not explicitly depend on time, then the Hamiltonian must be identically zero when evaluated on an extremal trajectory; that is,

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) = 0 \qquad \text{for } t \in [t_0, t_f]. \tag{5.3-41}$$

State Variable Inequality Constraints

Let us now consider problems in which there may be inequality constraints that involve the state variables as well as the controls. It will be assumed that the state constraints are of the form

$$\mathbf{f}(\mathbf{x}(t), t) \ge \mathbf{0},\dagger \tag{5.3-42}$$

where $\mathbf{f}$ is an l-vector function ($l \le m$) of the states and possibly time, which has continuous first and second partial derivatives with respect to $\mathbf{x}(t)$. It will also be assumed that the admissible control values lie in a closed and bounded region. Our approach will be to transform the l inequality constraints of (5.3-42) into a single equality constraint, and then to augment the performance measure with this equality constraint, as we have done previously with the state equations.

Let us define a new variable $\dot{x}_{n+1}(t)$ by

$$\begin{aligned}\dot{x}_{n+1}(t) \triangleq [f_1(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_1) + [f_2(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_2) \\ + \cdots + [f_l(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_l),\end{aligned} \tag{5.3-43}$$

where $\mathbb{1}(-f_i)$ is a unit Heaviside step function defined by

$$\mathbb{1}(-f_i) = \begin{cases} 0, & \text{for } f_i(\mathbf{x}(t), t) \ge 0 \\ 1, & \text{for } f_i(\mathbf{x}(t), t) < 0, \end{cases} \tag{5.3-44}$$

for $i = 1, 2, \ldots, l$. Notice that $\dot{x}_{n+1}(t) \ge 0$ for all t, and that $\dot{x}_{n+1}(t) = 0$ only for times when *all* of the constraints (5.3-42) are satisfied. Now let us require that the variable $x_{n+1}(t)$, given by

$$x_{n+1}(t) = \int_{t_0}^{t} \dot{x}_{n+1}(t)\,dt + x_{n+1}(t_0), \tag{5.3-45}$$

satisfy the two boundary conditions $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_f) = 0$. Since $\dot{x}_{n+1}(t) \ge 0$ for all t, satisfaction of these boundary conditions implies that $\dot{x}_{n+1}(t)$ must be zero throughout the interval $[t_0, t_f]$, but this occurs only if the constraints are satisfied for all $t \in [t_0, t_f]$.

Thus, to minimize the functional

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\,dt \tag{5.3-46}$$

subject to the state equation constraints

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{5.3-47}$$

† The notation $\mathbf{f}(\mathbf{x}(t), t) \ge \mathbf{0}$ means that each component of the vector $\mathbf{f}$ is ≥ 0.

admissibility constraints on the control variables, and state inequality constraints of the form

$$\mathbf{f}(\mathbf{x}(t), t) \geq \mathbf{0}, \tag{5.3-48}$$

first form the Hamiltonian

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) &= g(\mathbf{x}(t), \mathbf{u}(t), t) + p_1(t)a_1(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + \cdots + p_n(t)a_n(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + p_{n+1}(t)\{[f_1(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_1) + \cdots + [f_l(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_l)\} \\ &\triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t),\end{aligned} \tag{5.3-49}$$

where $x_{n+1}(t)$ is given by (5.3-45), and

$$a_{n+1}(\mathbf{x}(t), t) \triangleq [f_1(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_1) + \cdots + [f_l(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_l). \tag{5.3-50}$$

Using the notation of (5.3-49) means that $\mathbf{p}(t)$ and $\mathbf{x}(t)$ are $n + 1$ vectors. Notice that the Hamiltonian does not contain $x_{n+1}(t)$ explicitly. We can now apply Eqs. (5.3-21) to obtain necessary conditions for optimality:

$$\left.\begin{aligned}\dot{x}_1^*(t) &= a_1(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \\ &\ \ \vdots \\ \dot{x}_{n+1}^*(t) &= a_{n+1}(\mathbf{x}^*(t), t); \\ \dot{p}_1^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_1}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ &\ \ \vdots \\ \dot{p}_{n+1}^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_{n+1}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) = 0; \\ &\text{and} \\ \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) &\leq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \\ &\text{for all admissible } \mathbf{u}(t).\end{aligned}\right\} \begin{aligned}&\text{for all} \\ &t \in [t_0, t_f]\end{aligned} \tag{5.3-51}$$

$\dot{p}_{n+1}^*$ is zero because $x_{n+1}(t)$ does not appear explicitly in $\mathscr{H}$. The boundary conditions $\mathbf{x}^*(t_0)$ are specified $[x_{n+1}^*(t_0) = 0$ and $x_{n+1}^*(t_f) = 0]$; the remaining boundary conditions at $t = t_f$ can be determined by using the results obtained in Section 5.1.

Example 5.3-2. Let us now return to the problem discussed earlier in Example 5.3-1. The system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t)\end{aligned} \tag{5.3-52}$$

is to be controlled to minimize the performance measure

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}[x_1^2(t) + u^2(t)]\,dt. \tag{5.3-53}$$

$\mathbf{x}(t_0)$ is specified, the final state $\mathbf{x}(t_f)$ is free, and t_f is given. The admissible control values are constrained by

$$-1 \leq u(t) \leq 1 \qquad \text{for } t \in [t_0, t_f]. \tag{5.3-54}$$

In addition, it is required that

$$-2 \leq x_2(t) \leq 2 \qquad \text{for } t \in [t_0, t_f]. \tag{5.3-55}$$

We must first express (5.3-55) in the form of (5.3-48). To do this, observe that (5.3-55) implies

$$[x_2(t) + 2] \geq 0, \tag{5.3-56a}$$

and

$$[2 - x_2(t)] \geq 0. \tag{5.3-56b}$$

Writing (5.3-55) as these two inequalities gives

$$\begin{aligned}f_1(\mathbf{x}(t)) &= [x_2(t) + 2] \geq 0 \\ f_2(\mathbf{x}(t)) &= [2 - x_2(t)] \geq 0.\dagger\end{aligned} \tag{5.3-57}$$

The Hamiltonian is given by

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) &= \tfrac{1}{2}x_1^2(t) + \tfrac{1}{2}u^2(t) + p_1(t)x_2(t) \\ &\quad - p_2(t)x_2(t) + p_2(t)u(t) + p_3(t)\{[x_2(t) + 2]^2 \mathbb{1}(-x_2(t) - 2) \\ &\quad + [2 - x_2(t)]^2 \mathbb{1}(x_2(t) - 2)\}\end{aligned} \tag{5.3-58}$$

The necessary conditions for optimality, found from Eqs. (5.3-51), are

$$\begin{aligned}\dot{x}_1^*(t) &= x_2^*(t), \qquad x_1^*(t_0) = x_{1_0} \\ \dot{x}_2^*(t) &= -x_2^*(t) + u^*(t), \qquad x_2^*(t_0) = x_{2_0} \\ \dot{x}_3^*(t) &= [x_2^*(t) + 2]^2 \mathbb{1}(-x_2^*(t) - 2) \\ &\quad + [2 - x_2^*(t)]^2 \mathbb{1}(x_2^*(t) - 2), \qquad x_3^*(t_0) = 0\end{aligned} \tag{5.3-59}$$

$$\dot{p}_1^*(t) = -\frac{\partial \mathscr{H}}{\partial x_1} = -x_1^*(t)$$

† We could also combine the inequalities (5.3-56) by writing $[x_2(t) + 2][2 - x_2(t)] \geq 0$.

$$\dot{p}_2^*(t) = -\frac{\partial \mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t) - 2p_3^*(t)[x_2^*(t) + 2]\mathbb{1}(-x_2^*(t) - 2)$$
$$+ 2p_3^*(t)[2 - x_2^*(t)]\mathbb{1}(x_2^*(t) - 2)\dagger$$

$$\dot{p}_3^*(t) = -\frac{\partial \mathscr{H}}{\partial x_3} = 0 \Rightarrow p_3^*(t) = \text{a constant} \tag{5.3-60}$$

$$u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \leq p_2^*(t) \leq 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases} \tag{5.3-61}$$

The boundary conditions at the final time are $x_1^*(t_f) = 0$ (specified), and $p_1^*(t_f) = p_2^*(t_f) = 0$—from Table 5-1, or Eq. (5.1-18).

Comparing these necessary conditions with the results obtained in Example 5.3-1b, we see that the expressions for the optimal controls in terms of the extremal costates are the same; however, the equations for $\dot{p}_2^*(t)$ are different because of the presence of the state inequality constraints; hence, the optimal trajectories and control histories will generally not be the same.

In our discussion of state and control inequality constraints we have not considered constraints that include both the states and controls, that is, constraints of the form

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \geq \mathbf{0}. \tag{5.3-62}$$

For an explanation of how to handle constraints of this form, as well as an alternative derivation of the minimum principle, the interested reader can refer to Chapter 4 of [S-3].

In the remainder of this chapter we shall consider several examples of the application of Pontryagin's minimum principle. These examples will illustrate both the utility and the limitations of the variational approach to optimal control problems.

5.4 MINIMUM-TIME PROBLEMS

In this section we shall consider problems in which the objective is to transfer a system from an arbitrary initial state to a specified target set in minimum time. The target set (which may be moving) will be denoted by

† Performing the differentiation $\partial\mathscr{H}/\partial x_2$ formally also results in the presence of two unit impulse functions, which occur at $x_2^*(t) = \pm 2$; however, these terms are such that either the impulse functions or their coefficients are zero for all $t \in [t_0, t_f]$, so the impulses do not affect the solution.

$S(t)$, and the minimum time required to reach the target set by t^*. Mathematically, then, our problem is to transfer a system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{5.4-1}$$

from an arbitrary initial state $\mathbf{x}_0$ to the target set $S(t)$ and minimize

$$J(\mathbf{u}) = \int_{t_0}^{t_f} dt = t_f - t_0. \tag{5.4-2}$$

Typically, the control variables may be constrained by requirements such as

$$|u_i(t)| \leq 1, \qquad i = 1, 2, \ldots, m, \qquad t \in [t_0, t^*]. \tag{5.4-3}$$

Our approach will be to use the minimum principle to determine the optimal control law.†

To introduce several important aspects of minimum-time problems, let us consider the following simplified intercept problem.

Example 5.4-1. Figure 5-15 shows an aircraft that is initially at the point $x = 0$, $y = 0$ pursuing a ballistic missile that is initially at the point $x = a > 0$, $y = 0$. The missile flies the trajectory

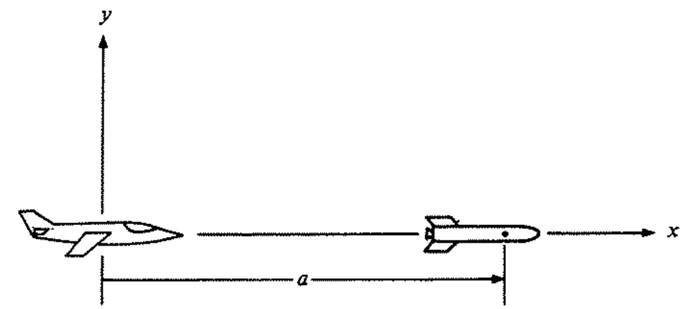


Figure 5-15 An intercept problem

$$x_M(t) = a + 0.1t^3$$
$$y_M(t) = 0 \tag{5.4-4}$$

for $t \geq 0$; thus, in this example the target set $S(t)$ is the position of the missile given by (5.4-4).

Neglecting gravitational and aerodynamic forces, let us model the aircraft as a point mass. Normalizing the mass to unity, we find that the motion of the aircraft in the x direction is described by

$$\ddot{x}(t) = u(t), \tag{5.4-5}$$

† For additional reading on time-optimal systems see [P-1] and [A-2].

or, in state form,

$$\dot{x}_1(t) = x_2(t) \qquad (5.4\text{-}6)$$
$$\dot{x}_2(t) = u(t),$$

where $x_1(t) \triangleq x(t)$ and $x_2(t) \triangleq \dot{x}(t)$. The thrust $u(t)$ is constrained by the relationship

$$|u(t)| \leq 1.0. \qquad (5.4\text{-}7)$$

By inspection of the geometry of the problem, it is clear that the optimal strategy for the pursuing aircraft is to accelerate with the maximum thrust possible in the positive x direction; therefore, $u^*(t)$ should be $+1.0$ for $t \in [0, t^*]$. To find t^*, we must determine the value(s) of t for which the x coordinate of the aircraft coincides with the target set $S(t)$; hence, assuming $\dot{x}(0) = 0$, we solve the equation

$$\tfrac{1}{2}[t^*]^2 = a + 0.1[t^*]^3 \qquad (5.4\text{-}8)$$

for t^*. Common sense indicates that there may not be a positive real value of $t^* \geq 0$ for which Eq. (5.4-8) is satisfied—if the missile is far enough away initially he can escape. It can be shown that interception is impossible if a is greater than 1.85. If $a = 1.85$, interception occurs at $t^* = 3.33$; for $a < 1.85$ the minimum interception times are less than 3.33.

Although greatly simplified, the preceding example illustrates two important characteristics that are typical of minimum-time problems:

1. For certain values of the initial condition a, a time-optimal control does not exist.
2. The optimal control, if it exists, is maximum effort during the entire time interval of operation.

In the subsequent development we shall generalize these concepts; let us first consider the question of existence of an optimal control.

The Set of Reachable States

If a system can be transferred from some initial state to a target set by applying admissible control histories, then an optimal control exists and may be found by determining the admissible control that causes the system to reach the target set most quickly. A description of the target set is assumed to be known; thus, to investigate the existence of an optimal control it is useful to introduce the concept of reachable states.

DEFINITION 5-1

If a system with initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is subjected to *all* admissible control histories for a time interval $[t_0, t]$, the collection of state values $\mathbf{x}(t)$ is called the set of states that are reachable (from $\mathbf{x}_0$) at time t, or simply *the set of reachable states*.

Although the set of reachable states depends on $\mathbf{x}_0$, t_0, and on t, we shall denote this set by $R(t)$. The following example illustrates the concept of reachable states.

Example 5.4-2. Find the set of reachable states for the system

$$\dot{x}(t) = u(t), \qquad (5.4\text{-}9)$$

where the admissible controls satisfy

$$-1 \leq u(t) \leq 1. \qquad (5.4\text{-}10)$$

The solution of Eq. (5.4-9) is

$$x(t) = x_0 + \int_{t_0}^{t} u(\tau)\, d\tau. \qquad (5.4\text{-}11)$$

If only admissible control values are used, Eq. (5.4-11) implies that

$$x_0 - [t - t_0] \leq x(t) \leq x_0 + [t - t_0]. \qquad (5.4\text{-}12)$$

Figure 5-16 shows the reachable sets for $t = t_1, t_2$, and t_3, where $t_1 < t_2 < t_3$.

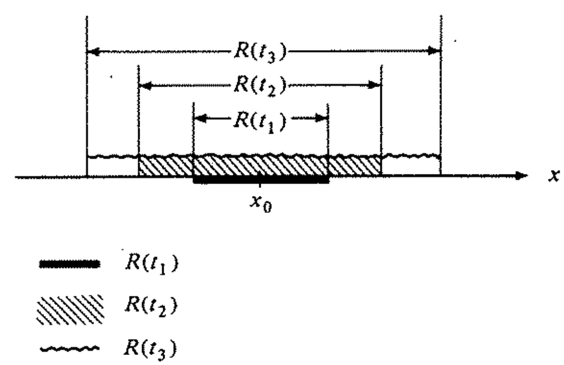


Figure 5-16 The reachable states for Example 5.4-2

The concept and properties of reachable sets are inextricably intertwined with the question of existence of time-optimal controls; if there is no value of t for which the target set $S(t)$ has at least one point in common with the set $R(t)$, then a time-optimal control does not exist. Conversely, it is helpful to visualize the minimum-time problem as a matter of finding the earliest time t^* when $S(t)$ and $R(t)$ meet, as shown in Fig. 5-17 for a second-order system. The target set is a moving point, and the boundary of the set of reachable states at time t_i is denoted by $\partial R(t_i)$. The target set and the set of reachable states first intersect at point p, where $t^* = t_2$.

Unfortunately, although it is conceptually satisfying to think of minimum-time problems in this fashion, it is generally not feasible to determine solutions by finding the intersections of reachable sets with the target set except in very simple problems (like Example 5.4-1). General theorems concerning the existence of time-optimal controls are unavailable at this time; however, later in this section we shall state an existence theorem that applies to an important class of minimum-time problems.

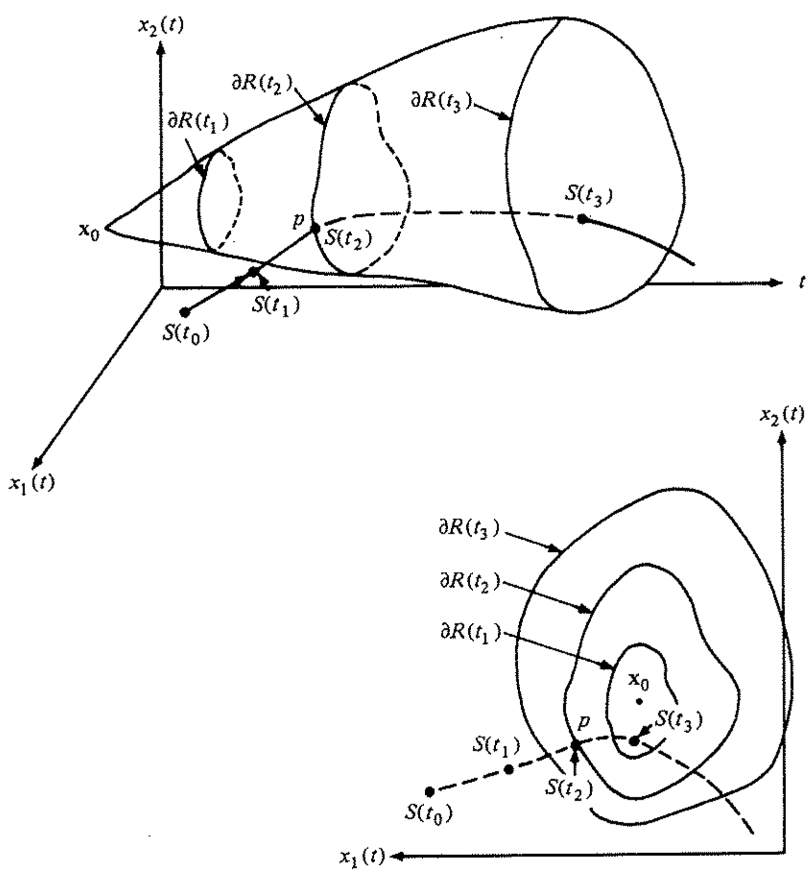


Figure 5-17 The minimum-time problem viewed as the intersection of a target set, $S(t)$, and the set of reachable states, $R(t)$

The Form of the Optimal Control for a Class of Minimum-Time Problems

Now let us determine the form of the optimal control for a particular class of systems by using the minimum principle. We shall assume that the state equations of the system are of the form

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t), \tag{5.4-13}$$

where $\mathbf{B}$ is an $n \times m$ array that may be explicitly dependent on the states and time. It is specified that the admissible controls must satisfy the inequality constraints

$$M_{i-} \le u_i(t) \le M_{i+}, \qquad i = 1, 2, \dots, m, \qquad t \in [t_0, t^*]; \tag{5.4-14}$$

M_{i+} and M_{i-} are known upper and lower bounds for the ith control component.

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = 1 + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t)]. \tag{5.4-15}$$

From the minimum principle, it is necessary that

$$\begin{aligned} &1 + \mathbf{p}^{*T}(t)[\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t)] \\ &\le 1 + \mathbf{p}^{*T}(t)[\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t)] \end{aligned} \tag{5.4-16}$$

for all admissible $\mathbf{u}(t)$, and for all $t \in [t_0, t^*]$. Equation (5.4-16) implies that

$$\mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t) \le \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t); \tag{5.4-17}$$

hence, $\mathbf{u}^*(t)$ is the control that causes $\mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t)$ to assume its minimum value. If the array $\mathbf{B}$ is expressed as

$$\mathbf{B}(\mathbf{x}^*(t), t) = \left[\mathbf{b}_1(\mathbf{x}^*(t), t) \,\vdots\, \mathbf{b}_2(\mathbf{x}^*(t), t) \,\vdots\, \cdots \,\vdots\, \mathbf{b}_m(\mathbf{x}^*(t), t)\right], \tag{5.4-18}$$

where $\mathbf{b}_i(\mathbf{x}^*(t), t)$, $i = 1, \dots, m$, is the ith column of the array, then the coefficient of the ith control component $u_i(t)$ in (5.4-17) is $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)$, and

$$\mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t) = \sum_{i=1}^{m} \mathbf{p}^{*T}(t)[\mathbf{b}_i(\mathbf{x}^*(t), t)]u_i(t). \tag{5.4-19}$$

Assuming that the control components are independent of one another, we then must minimize

$$\mathbf{p}^{*T}(t)[\mathbf{b}_i(\mathbf{x}^*(t), t)]u_i(t)$$

with respect to $u_i(t)$ for $i = 1, 2, \ldots, m$. If the coefficient of $u_i(t)$ is positive, $u_i^*(t)$ must be the smallest admissible control value M_{i-}. If the coefficient of $u_i(t)$ is negative, $u_i^*(t)$ must be the largest admissible control value M_{i+}; thus, the form of the optimal control is

$$u_i^*(t) = \begin{cases} M_{i+}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < 0 \\ M_{i-}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) > 0 \\ \text{Undetermined}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) = 0. \end{cases} \quad (5.4\text{-}20)$$

$$i = 1, 2, \ldots, m$$

If the extremal state and costate trajectories are such that the coefficient of $u_i(t)$ is as shown in Fig. 5-18(a), then the history of $u_i^*(t)$ will be as shown in Fig. 5-18(b).

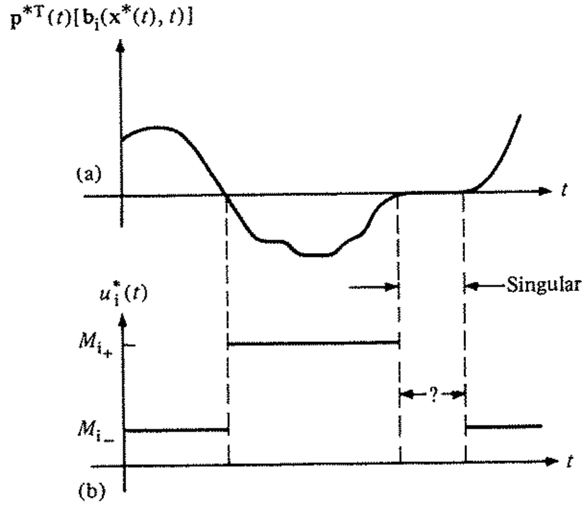


Figure 5-18 The relationship between a time-optimal control and its coefficient in the Hamiltonian

Notice that if $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)$ passes through zero, a switching of the control $u_i^*(t)$ is indicated. If $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)$ is zero for some finite time interval, then the coefficient of $u_i(t)$ in the Hamiltonian is zero, so the necessary condition that $u_i^*(t)$ minimize $\mathscr{H}$ provides no information about how to select $u_i^*(t)$; this signals the so-called *singular condition*, to be discussed in Section 5.6. Here we shall consider only problems in which the singular condition does not arise; such problems will be called *normal*.

Equation (5.4-20) is the mathematical statement of the well-known *bang-bang principle*, that is, if the state equations are of the form (5.4-13) and the admissible controls must satisfy constraints of the form (5.4-14),

then *the optimal control to obtain minimum-time response is maximum effort throughout the interval of operation.* The bang-bang concept is intuitively appealing as well. Certainly, the men who race automobiles come very close to bang-bang operation— they use the accelerator and brakes often; thus, their fuel consumption is large, tires and brakes do not last very long, and the cars are subjected to severe mechanical stresses, but barring accidents and mechanical failures, the drivers reach their destination quickly.

Before we move on to some problems that can be completely solved by using analytical methods, let us consider a nonlinear problem of the foregoing type.

Example 5.4-3.† Figure 5-19 shows a lunar rocket in the terminal phase of a minimum-time, soft landing on the surface of the moon. We shall make the following assumptions:

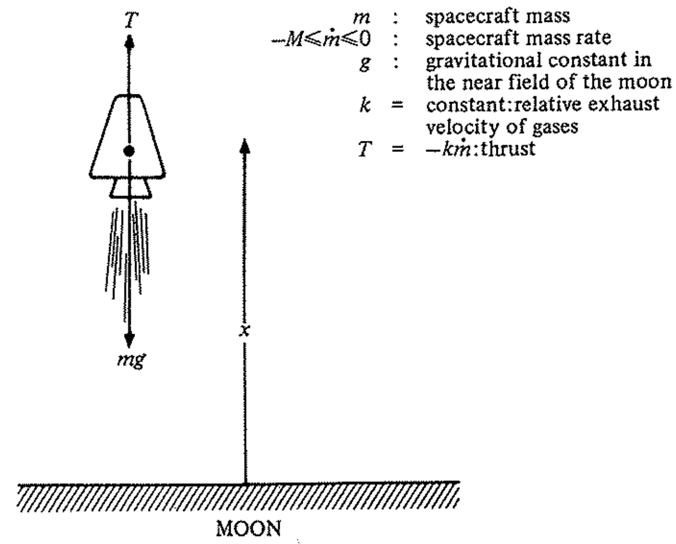


Figure 5-19 Lunar soft landing

a. Aerodynamic forces and gravitational forces of bodies other than the moon are negligible.
b. Lateral motion is ignored; thus, the descent trajectory is vertical and the thrust vector is tangent to the trajectory.
c. The acceleration of gravity is a constant, because of the nearness of the spacecraft to the moon.
d. The relative velocity of the exhaust gases with respect to the spacecraft is constant.

† See [M-2] and [M-3].

e. The mass rate is constrained by

$$-M \leq \dot{m} \leq 0. \tag{5.4-21}$$

The equation of motion is

$$\begin{aligned} m(t)\ddot{x}(t) &= -gm(t) + T(t) \\ &= -gm(t) - k\dot{m}(t). \end{aligned} \tag{5.4-22}$$

Defining the states of the system as $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$, $x_3 \triangleq m$ and the control as $u \triangleq \dot{m}$ leads to the state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -g - \frac{k}{x_3(t)}u(t) \\ \dot{x}_3(t) &= u(t). \end{aligned} \tag{5.4-23}$$

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)x_2(t) - gp_2(t) - \frac{kp_2(t)u(t)}{x_3(t)} + p_3(t)u(t), \tag{5.4-24}$$

and the optimal control must satisfy

$$\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) \leq \mathscr{H}(\mathbf{x}^*(t), u(t), \mathbf{p}^*(t))$$

for all admissible $u(t)$, and for all $t \in [t_0, t_f]$; therefore,

$$u^*(t) = \begin{cases} 0, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} < 0 \\ -M, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} > 0 \\ \text{Undetermined}, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} = 0. \end{cases} \tag{5.4-25}$$

To obtain an explicit solution for $u^*(t)$ we would have to solve a nonlinear two-point boundary-value problem (see Problem 5-31).

Minimum-Time Control of Time-Invariant Linear Systems

Armed with our knowledge about the form of time-optimal controls, for the remainder of this section we shall consider the following important class of problems: A linear, stationary system of order n having m controls is described by the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{5.4-26}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant $n \times n$ and $n \times m$ matrices, respectively. The components of the control vector are constrained by

$$|u_i(t)| \leq 1, \qquad i = 1, 2, \ldots, m. \tag{5.4-27}$$

Assuming that the system is completely controllable and normal (no singular intervals exist), find a control, if one exists, which transfers the system from an arbitrary initial state $\mathbf{x}_0$ at time $t = 0$ to the final state $\mathbf{x}(t_f) = \mathbf{0}$ in minimum time. We shall refer to this problem as the *stationary, linear regulator, minimum-time problem.*

From Eq. (5.4-20) we know that the optimal control, if it exists, is bang-bang. Let us now state without proof some important theorems due to Pontryagin et al. [P-1] which apply to stationary, linear regulator, minimum-time problems.

THEOREM 5.4-1 (EXISTENCE)

If *all* of the eigenvalues of $\mathbf{A}$ have nonpositive real parts, then an optimal control exists that transfers any initial state $\mathbf{x}_0$ to the origin.

THEOREM 5.4-2 (UNIQUENESS)

If an extremal control exists, then it is unique.†

Since an optimal control, if one exists, must be an extremal control, this theorem indicates that a control which satisfies the minimum principle and the required boundary conditions must be the optimal control. Thus, if an optimal control exists, satisfaction of the minimum principle is both necessary and sufficient for time-optimal control of stationary, linear regulator systems.

THEOREM 5.4-3 (NUMBER OF SWITCHINGS)

If the eigenvalues of $\mathbf{A}$ are all real, and a (unique) time-optimal control exists, then each control component can switch *at most* $(n - 1)$ times.

Thus, an nth-order system having all real, nonpositive eigenvalues has a unique time-optimal control with components that each switch at most $(n - 1)$ times.

Example 5.4-4. Find the optimal control satisfying

$$|u(t)| \leq 1 \tag{5.4-28}$$

which transfers the system

† Recall that a control which satisfies the necessary conditions in Eqs. (5.3-21) and the required boundary conditions is called an extremal control.

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned} \tag{5.4-29}$$

from any initial state $\mathbf{x}_0$ to the origin in minimum time. Here

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix}0\\ 1\end{bmatrix}. \tag{5.4-30}$$

Since the eigenvalues of $\mathbf{A}$ are both zero, we know from Theorems 5.4-1 through 5.4-3 that an optimal control exists, is unique, and has at most one switching.

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)x_2(t) + p_2(t)u(t); \tag{5.4-31}$$

thus, the minimum principle indicates that the optimal control $u^*(t)$ must satisfy

$$p_2^*(t)u^*(t) \leq p_2^*(t)u(t) \tag{5.4-32}$$

for all admissible $u(t)$ and for all $t \in [t_0, t_f]$. It can be shown that a singular interval cannot exist (see Section 5.6); therefore, the optimal control found from (5.4-32) is

$$u^*(t) = \begin{cases}-1, & \text{for } p_2^*(t) > 0\\ +1, & \text{for } p_2^*(t) < 0\end{cases} \triangleq -\operatorname{sgn}(p_2^*(t)). \tag{5.4-33}$$

From the Hamiltonian the costate equations are

$$\begin{aligned}\dot{p}_1^*(t) &= 0\\ \dot{p}_2^*(t) &= -p_1^*(t).\end{aligned} \tag{5.4-34}$$

The costate solution is of the form

$$\begin{aligned}p_1^*(t) &= c_1\\ p_2^*(t) &= -c_1 t + c_2,\end{aligned} \tag{5.4-35}$$

where c_1 and c_2 are constants of integration. Equation (5.4-35) indicates that p_2^*, and therefore u^*, can change sign at most once (this result also follows from Theorem 5.4-3).

Since there can be *at most* one switching, the optimal control for a specified initial state must be one of the forms:

$$u^*(t) = \begin{cases}+1, & \text{for all } t \in [t_0, t^*], \text{ or}\\ -1, & \text{for all } t \in [t_0, t^*], \text{ or}\\ +1, & \text{for } t \in [t_0, t_1),\dagger \ \text{ and } \ -1, \ \text{ for } t \in [t_1, t^*], \text{ or}\\ -1, & \text{for } t \in [t_0, t_1), \ \text{ and } \ +1, \ \text{ for } t \in [t_1, t^*].\end{cases} \tag{5.4-36}$$

† The notation $t \in [t_0, t_1)$ means $t_0 \leq t < t_1$.

Thus, segments of optimal trajectories can be found by integrating the state equations with $u = \pm 1$ to obtain

$$x_2(t) = \pm t + c_3 \tag{5.4-37}$$

$$x_1(t) = \pm\tfrac{1}{2}t^2 + c_3 t + c_4, \tag{5.4-38}$$

where c_3 and c_4 are constants of integration, and the upper sign corresponds to $u = +1$. Time can be eliminated from these equations by squaring the first equation, multiplying the result by $\frac{1}{2}$ and comparing with Eq. (5.4-38) to obtain

$$x_1(t) = \tfrac{1}{2}x_2^2(t) + c_5, \qquad \text{for } u = +1 \tag{5.4-39}$$

and

$$x_1(t) = -\tfrac{1}{2}x_2^2(t) + c_6, \quad \text{for } u = -1; \tag{5.4-40}$$

c_5 and c_6 are constants. Equations (5.4-39) and (5.4-40) each define a family of parabolas that are shown in Fig. 5-20(a) and (b)—the arrows indicate the direction of increasing time.

Now, let us consider each of the alternatives for the optimal control. From Fig. 5-20 we see that the controls given by Eq. (5.4-36) correspond to the following situations:

1. $u^*(t) = +1$ for $t \in [t_0, t^*]$. The initial state $\mathbf{x}_0$ must lie on segment A-0 in Fig. 5-20(a).
2. $u^*(t) = -1$ for $t \in [t_0, t^*]$. The initial state $\mathbf{x}_0$ must lie on segment B-0 in Fig. 5-20(b).
3. $u^*(t) = +1$ for $t \in [t_0, t_1)$, and $u^*(t) = -1$ for $t \in [t_1, t^*]$. Since the optimal control is -1 for $t \in [t_1, t^*]$, at time t_1 the system state must lie on segment B-0. This transfer has been accomplished by a control of $u^* = +1$; thus, the optimal trajectory consists of an initial segment like one of the trajectories in Fig. 5-20(a) followed by a switching of the control to -1 upon reaching B-0, and then on to the origin along B-0 with $u^* = -1$. Notice that B-0, in addition to being the terminal segment of the optimal trajectory, is the locus of state values where the control switches from $+1$ to -1; therefore, B-0 is referred to *as a switching curve*. Now, which initial states will have optimal trajectories as described above? Again referring to Fig. 5-20, we see that only the parabolic curves that have $c_5 < 0$ intersect B-0. In addition, only trajectories that begin below B-0 with $u^* = +1$ will ever intersect B-0. We conclude that for initial states lying below both A-0 and B-0 the optimal control will be $u^* = +1$ until B-0 is reached, followed by $u^* = -1$ thereafter.
4. $u^*(t) = -1$ for $t \in [t_0, t_1)$, and $u^*(t) = +1$ for $t \in [t_1, t^*]$. The same reasoning used in 3 leads to the conclusion that for states initially lying above A-0 and B-0 the optimal control will be $u^* = -1$ followed by $u^* = +1$; the switching occurs when the trajectory intersects A-0.

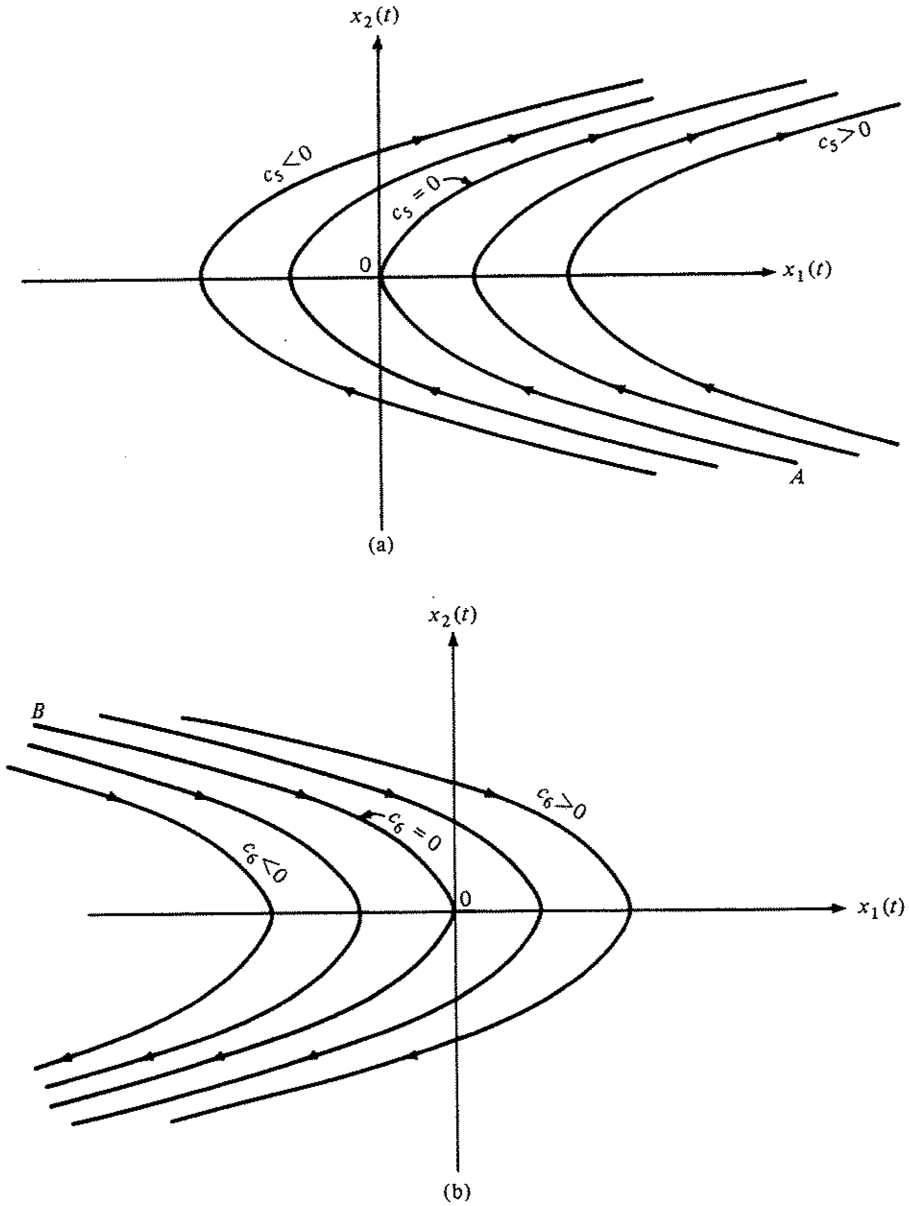


Figure 5-20 (a) Trajectories for $u = +1$. (b) Trajectories for $u = -1$

Thus, we see that A-0 and B-0, in addition to being terminal segments of optimal trajectories, together compose the switching curve A-0-B shown in Fig. 5-21(a). By putting $c_5 = c_6 = 0$ in Eqs. (5.4-39) and (5.4-40), we find the equation of this switching curve to be

$$x_1(t) = -\tfrac{1}{2}x_2(t)\,|x_2(t)|. \tag{5.4-41}$$

To summarize, for states above A-0-B the optimal control is $u^* = -1$ until the trajectory intersects A-0, where the optimal control switches to

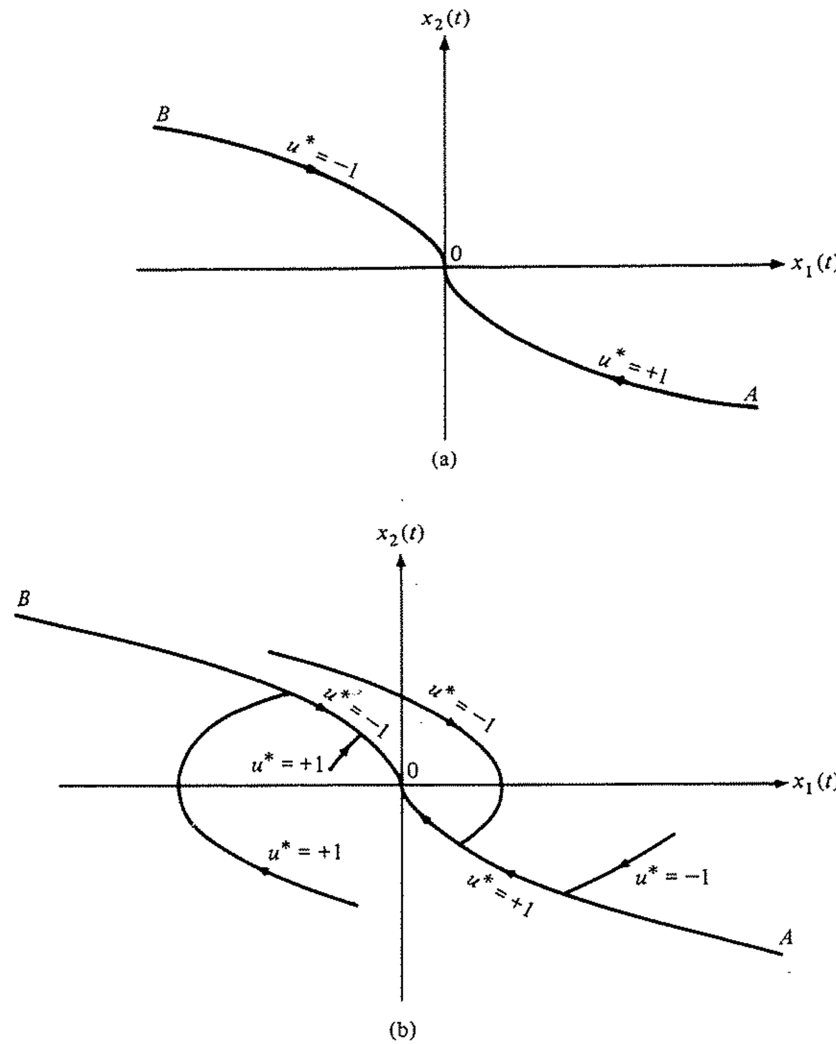


Figure 5-21 (a) The switching curve. (b) Optimal trajectories for several initial state values.

$u^* = +1$. The optimal control $u^* = +1$ is applied to transfer states below A-0-B to segment B-0, where the optimal control switches to $u^* = -1$. Once the system has reached the origin, it can be kept there by applying $u^*(t) = 0$ for $t > t^*$. Optimal trajectories for several initial state values are shown in Fig. 5-21(b).

It must be emphasized that we have succeeded in obtaining the optimal control *law*; that is, the optimal control at any time t is known as a function of the state value $\mathbf{x}(t)$. To express the optimal control law in a convenient form, let us define the *switching function* $s(\mathbf{x}(t))$, obtained from Eq. (5.4-41) as

$$s(\mathbf{x}(t)) \triangleq x_1(t) + \tfrac{1}{2}x_2(t)\,|x_2(t)|. \tag{5.4-42}$$

Notice that

$s(\mathbf{x}(t)) > 0$ implies $\mathbf{x}(t)$ lies above the switching curve A-0-B.

$s(\mathbf{x}(t)) < 0$ implies $\mathbf{x}(t)$ lies below the switching curve A-0-B.

$s(\mathbf{x}(t)) = 0$ implies $\mathbf{x}(t)$ lies on the switching curve A-0-B.

Thus, in terms of this switching function the optimal control law is

$$u^*(t) = \begin{cases} -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) < 0 \\ -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) < 0 \\ 0, & \text{for } \mathbf{x}(t) = \mathbf{0}. \end{cases} \tag{5.4-43}$$

An implementation of this optimal control law is shown in Fig. 5-22; the required hardware consists of a summing device, a sign changer, a nonlinear function generator, and an ideal relay.

The procedure used in solving the preceding example can be generalized to include nth-order, stationary, linear regulator systems controlled by one input. Let us assume that all of the eigenvalues of $\mathbf{A}$ are real and nonpositive; thus, for all initial states a unique time-optimal control exists and has at most $(n - 1)$ switchings. To obtain the optimal control law:

1. (a) We first determine the set of points from which the origin can be reached with $u = +1$ (call this set O_+), and the set of points from which the origin can be reached with $u = -1$ (call this set O_-). Let O_1 denote the set of points from which the origin can be reached with no control switchings; then

$$O_1 = O_+ \cup O_- \tag{5.4-44}$$

where $\cup$ denotes "the union of."†

† O_1 is the union of O_+ and O_-; this means that every element of O_1 is an element of either O_+, O_-, or both O_+ and O_-.

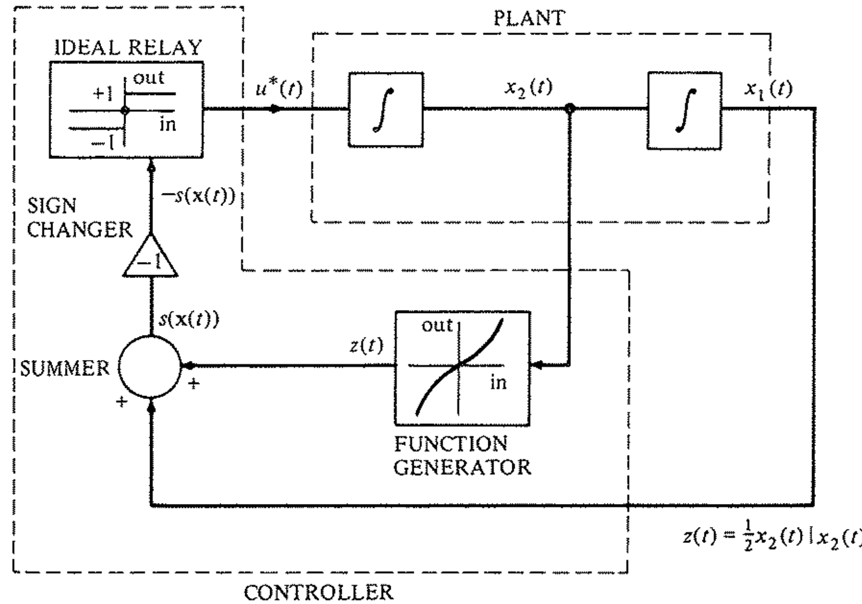


Figure 5-22 Implementation of the time-optimal control law for Example 5.4-4

(b) Next, we determine the set of points O_{-+} from which O_+ can be reached by applying $u = -1$; the origin can be reached from O_{-+} by applying $u = -1$ until reaching O_+, followed by $u = +1$. Similarly, we find the set of points O_{+-} from which O_- can be reached by applying $u = +1$. To reach the origin from O_{+-}, we apply $u = +1$ until reaching O_-, followed by $u = -1$. The set of points from which the origin can be reached with at most one switching (two control values) is given by

$$\begin{aligned} O_2 &= O_+ \cup O_- \cup O_{+-} \cup O_{-+} \\ &= O_1 \cup O_{+-} \cup O_{-+}\dagger \end{aligned} \tag{5.4-45}$$

(c) We continue until the set of points O_{n-1} from which the origin can be reached with at most $(n - 2)$ switchings is determined. All points not in the set O_{n-1} require $(n - 1)$ switchings to reach the origin. By eliminating time from the trajectory equations, express O_{n-1} in the form

$$s(\mathbf{x}(t)) = 0. \tag{5.4-46}$$

† $O_1 \cup O_{+-} \cup O_{-+}$ means the set of points which are in at least one of the sets O_1, O_{+-}, O_{-+}.

2. Next, we determine the optimal control to be applied at any point in the state space. The switching function $s(\mathbf{x}(t))$ defines a switching hypersurface that divides the state space into two half-spaces. From one half-space the control $u^* = +1$ is applied to drive the system to O_{n-1}, where the control switches to -1, until the system reaches O_{n-2}, where the control again switches to $+1$, etc., until the origin is reached. From the other half-space the control sequence is reversed; $u^* = -1$ is applied to transfer the system to O_{n-1}, where the control switches to $+1$, and so on, until reaching the origin.
3. Finally, we determine a combination of physical devices to implement the time-optimal control law.

Before concluding our consideration of time-optimal problems, let us solve another second-order example that illustrates the procedure we have just summarized.

Example 5.4-5. Find the control law for transferring the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -ax_2(t) + u(t)\end{aligned} \tag{5.4-47}$$

from an arbitrary initial state $\mathbf{x}_0$ to the origin in minimum time. The admissible controls are constrained by

$$|u(t)| \leq 1, \tag{5.4-48}$$

and a is a positive real number.

The eigenvalues of this system are 0 and $-a$; thus, since both eigenvalues are real and nonpositive, the hypotheses of Theorems 5.4-1 through 5.4-3 are satisfied and we know that an optimal control exists that is unique and has at most $(n-1)$ switchings.

We are again dealing with a second-order system; thus, we know that the optimal control law is determined by a switching *curve*—in higher-dimensional problems switchings occur on hypersurfaces in the state space.

From the minimum principle and Theorem 5.4-3, we find that the possible forms for the optimal control are the same as given for Example 5.4-4 in Eq. (5.4-36).

Next, let us proceed to find the sets O_+ and O_- (from which the origin can be reached by applying only $u = +1$, or $u = -1$) by solving the differential equations (5.4-47) with $u = \pm 1$. The solutions are

$$x_2(t) = c_1\epsilon^{-at} \pm \frac{1}{a}[1 - \epsilon^{-at}] \tag{5.4-49}$$

$$x_1(t) = -\frac{c_1}{a}\epsilon^{-at} \pm \frac{1}{a}t \pm \frac{1}{a^2}\epsilon^{-at} + c_2 \tag{5.4-50}$$

These equations define two families of curves; to determine the curves which pass through the origin, set $x_1(t) = x_2(t) = 0$ and $t = 0$ (since the system is time invariant, $t = 0$ is an arbitrary reference time), solve for c_1 and c_2, and substitute in (5.4-49) and (5.4-50) to obtain

$$x_2(t) = \pm\frac{1}{a}[1 - \epsilon^{-at}] \tag{5.4-51}$$

$$x_1(t) = \pm\frac{1}{a}t \pm \frac{1}{a^2}\epsilon^{-at} \mp \frac{1}{a^2}. \tag{5.4-52}$$

To determine O_+, use the upper sign (which corresponds to $u = +1$), solve (5.4-51) for t, and substitute in (5.4-52) to obtain the relationship

$$x_1(t) = -\frac{1}{a^2}\ln\left(-a\left[x_2(t) - \frac{1}{a}\right]\right) - \frac{1}{a}x_2(t).\dagger \tag{5.4-53}$$

The set of points in the x_1- x_2 plane for which this equation is satisfied is O_+. Similar reasoning yields as the expression for O_-

$$O_- = \left\{x_1(t), x_2(t)\colon x_1(t) = \frac{1}{a^2}\ln\left(a\left[x_2(t) + \frac{1}{a}\right]\right) - \frac{1}{a}x_2(t)\right\}.\ddagger \tag{5.4-54}$$

Since Eq. (5.4-53) applies for $x_2(t) < 0$ and (5.4-54) applies for $x_2(t) > 0$, the expression for O_1 (the set of all points that are in either O_+ or O_-) is given by

$$O_1 = \left\{x_1(t), x_2(t)\colon x_1(t) = \frac{x_2(t)}{|x_2(t)|}\frac{1}{a^2}\ln\left(a\left[|x_2(t)| + \frac{1}{a}\right]\right) - \frac{1}{a}x_2(t)\right\}. \tag{5.4-55}$$

The switching function is then

$$s(\mathbf{x}(t)) = x_1(t) - \frac{x_2(t)}{|x_2(t)|}\frac{1}{a^2}\ln\left(a\left[|x_2(t)| + \frac{1}{a}\right]\right) + \frac{1}{a}x_2(t). \tag{5.4-56}$$

The switching curves for $a = 0.5$, 1.0, and 2.0 are shown in Fig. 5-23, and some typical trajectories for $a = 0.5$ are shown in Fig. 5-24. It is left as an exercise for the reader to verify that for points *above* the switching curve the optimal control is $u^* = -1$ until reaching the switching curve, where u^* switches to $+1$, and remains at $+1$ until the origin is reached, at which time $u^* = 0$ is applied to keep the system at the origin. Similar reasoning gives the optimal control law for points below the switching curve. In summary, the optimal control law is

† ln denotes the natural logarithm, or $\log_\epsilon$.

‡ This notation means that O_- is the set of points that satisfy the equation

$$x_1(t) = \frac{1}{a^2}\ln\left(a\left[x_2(t) + \frac{1}{a}\right]\right) - \frac{1}{a}x_2(t).$$

$$u^*(t) = \begin{cases} -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) < 0 \\ -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) < 0 \\ 0, & \text{for } \mathbf{x}(t) = \mathbf{0}. \end{cases} \qquad (5.4\text{-}57)$$

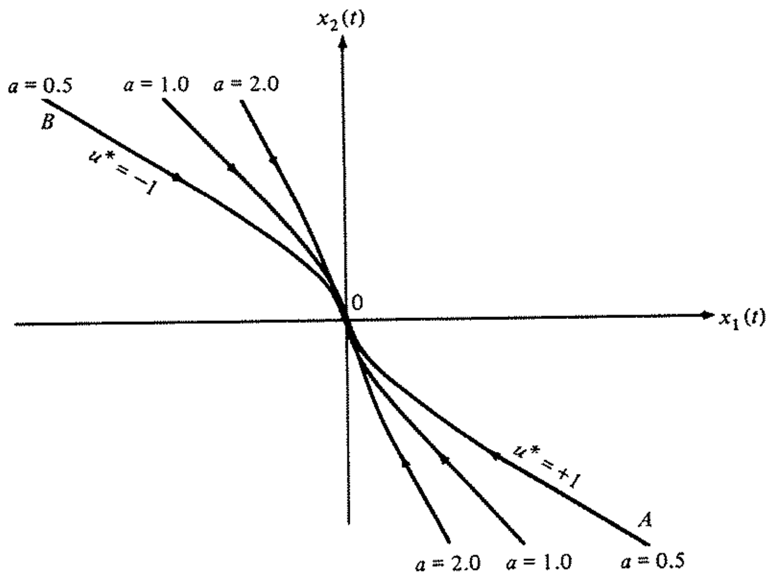


Figure 5-23 Time-optimal switching curves for Example 5.4-5 with $a = 0.5, 1.0, 2.0$

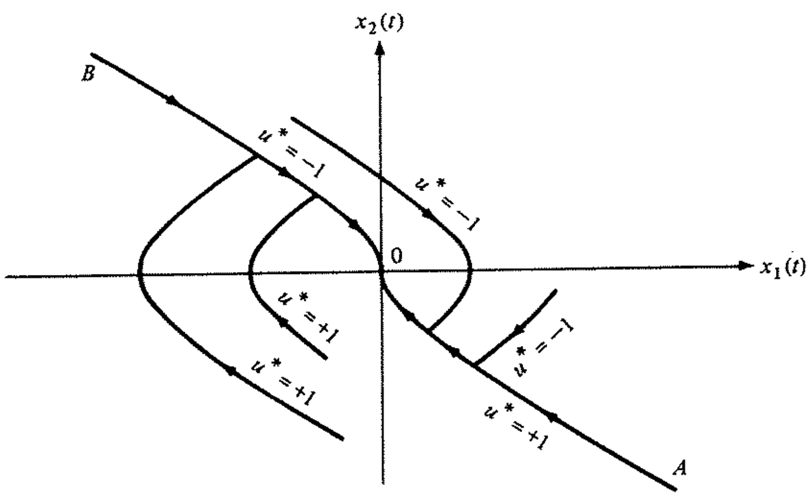


Figure 5-24 Several optimal trajectories for Example 5.4-5 with $a = 0.5$

Summary

In this section we have found that time-optimal controls for a rather general class of systems are "bang-bang"; that is, the optimal control switches between its maximum and minimum admissible values.

A procedure for finding time-optimal control laws for time-invariant, linear regulator systems was discussed and demonstrated for two second-order systems. Although this procedure is conceptually straightforward, it does have serious limitations:

1. For higher-order systems ($n \geq 3$) it is generally difficult, if not impossible, to obtain an analytical expression for the switching hypersurface.
2. Even in cases where an expression for the switching hypersurface can be found, physical implementation of the optimal control law may be quite complicated, indicating that a suboptimal, but easier-to-implement, control law may be preferable.
3. The procedure is generally not applicable to nonlinear systems, because of the difficulty of analytically integrating the differential equations.

5.5 MINIMUM CONTROL-EFFORT PROBLEMS

In the preceding section we considered problems in which the objective was to transfer a system from an arbitrary initial state to a specific target set as quickly as possible. Let us now consider problems in which control effort required, rather than elapsed time, is the criterion of optimality. Such problems arise frequently in aerospace applications, where often there are limited control resources available for achieving desired objectives.

The class of problems we will discuss is the following: Find a control $\mathbf{u}^*(t)$ satisfying constraints of the form

$$M_{i-} \leq u_i(t) \leq M_{i+}, \qquad i = 1, 2, \dots, m, \qquad (5.5\text{-}1)$$

which transfers a system described by

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (5.5\text{-}2)$$

from an arbitrary initial state $\mathbf{x}_0$ to a specified target set $S(t)$ with a minimum expenditure of control effort.

As measures of control effort we shall consider the two performance indices

$$J_1(\mathbf{u}) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} \beta_i \left| u_i(t) \right| \right] dt \qquad (5.5\text{-}3)$$

and

$$J_2(\mathbf{u}) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} r_i u_i^2(t)\right] dt, \tag{5.5-4}$$

where β_i and r_i, $i = 1, \ldots, m$, are nonnegative weighting factors. As discussed in Chapter 2, the fuel consumed by a mass-expulsion thrusting system is often expressed by an integral of the form (5.5-3); thus, if a performance measure to be minimized has the form given by J_1, we shall refer to the problem as a *minimum-fuel problem*. The total electrical energy supplied to a network of resistors by several voltage and current sources is given by an integral of the form (5.5-4); hence, if a performance measure of this form is to be minimized, we shall say that we wish to solve a *minimum-energy problem*. The reader must be cautioned that in a particular problem (5.5-3) may not represent fuel expenditure, or control energy required may not be given by (5.5-4); therefore, the results obtained in this section will apply to the performance measure J_1 or J_2, not necessarily to the problems of minimizing fuel or energy consumption.

Our discussion will be primarily devoted to solving several example problems that are rather elementary, but nonetheless indicative of the characteristics of fuel and energy-optimal systems.†

Minimum-Fuel Problems

In our discussion of minimum-time problems in Section 5.4 the concept of reachable states was introduced. Recall that $R(t)$ was used to denote the set of states that can be reached at time t by starting from an initial state $\mathbf{x}_0$ at time t_0. Minimum-fuel problems may also be visualized in terms of reachable states; that is, the minimum-fuel solution is given by the intersection of the target set $S(t)$ with the set of reachable states $R(t)$, which requires the *smallest amount of consumed fuel*. To represent this idea geometrically we could use a state–time–consumed-fuel coordinate system and determine the intersections (if any) of $S(t)$ and $R(t)$. Unfortunately, although such a geometric representation is helpful as a conceptual device, it is of limited value in actually obtaining solutions. Instead of pursuing this avenue further, we shall approach minimum control-effort problems by starting with the necessary conditions provided by Pontryagin's minimum principle.

The Form of the Optimal Control for a Class of Minimum-Fuel Problems. Let us assume that the state equations of a system are of the form

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t), \tag{5.5-5}$$

† For additional reading on fuel- and energy-optimal systems see [A-2], [L-3], and [L-4].

where $\mathbf{B}$ is an $n \times m$ array that may be explicitly dependent on the states and time. The performance measure to be minimized is

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |u_i(t)|\right] dt, \tag{5.5-6}$$

and the admissible controls are to satisfy the constraints

$$-1 \le u_i(t) \le +1, \qquad i = 1, 2, \ldots, m, \qquad t \in [t_0, t_f].\dagger \tag{5.5-7}$$

The Hamiltonian is

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \sum_{i=1}^{m} |u_i(t)| + \mathbf{p}^T(t)\mathbf{a}(\mathbf{x}(t), t)\\ + \mathbf{p}^T(t)\mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t),\end{aligned} \tag{5.5-8}$$

and the minimum principle requires that

$$\begin{aligned}&\sum_{i=1}^{m} |u_i^*(t)| + \mathbf{p}^{*T}(t)\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t)\\ &\le \sum_{i=1}^{m} |u_i(t)| + \mathbf{p}^{*T}(t)\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t),\end{aligned} \tag{5.5-9}$$

or

$$\sum_{i=1}^{m} |u_i^*(t)| + \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t) \le \sum_{i=1}^{m} |u_i(t)| + \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t) \tag{5.5-10}$$

for all admissible $\mathbf{u}(t)$, and for all $t \in [t_0, t_f]$. As in Section 5.4 let us express $\mathbf{B}$ in the form

$$\mathbf{B}(\mathbf{x}^*(t), t) = [\mathbf{b}_1(\mathbf{x}^*(t), t) \,\vdots\, \mathbf{b}_2(\mathbf{x}^*(t), t) \,\vdots\, \cdots \,\vdots\, \mathbf{b}_m(\mathbf{x}^*(t), t)],$$

where $\mathbf{b}_i(\mathbf{x}^*(t), t)$ is the ith column of the $n \times m$-dimensional $\mathbf{B}$ array. Assuming that the components of $\mathbf{u}$ are independent of one another, we have from (5.5-10) that

$$\begin{aligned}&|u_i^*(t)| + \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)u_i^*(t)\\ &\le |u_i(t)| + \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)u_i(t), \qquad i = 1, 2, \ldots, m.\end{aligned} \tag{5.5-11}$$

The definition of $|u_i(t)|$ is

† For simplicity we have assumed that $M_{i-} = -1$, $M_{i+} = +1$, and $\beta_i = 1$ for $i = 1, 2, \ldots, m$. The derivation is easily modified if these assumptions are not made.

$$|u_i(t)| \triangleq \begin{cases} u_i(t), & \text{for } u_i(t) \geq 0 \\ -u_i(t), & \text{for } u_i(t) \leq 0; \end{cases} \tag{5.5-12}$$

therefore,

$$|u_i(t)| + \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)u_i(t)$$
$$= \begin{cases} [1 + \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)]u_i(t), & \text{for } u_i(t) \geq 0 \quad (5.5\text{-}13a) \\ [-1 + \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)]u_i(t), & \text{for } u_i(t) \leq 0. \quad (5.5\text{-}13b) \end{cases}$$

If $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) > 1.0$, the minimum value of expression (5.5-13a) is 0, because $u_i(t) \geq 0$; the minimum value of (5.5-13b) is attained for $u_i(t) = -1$ and is equal to $[+1 - \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t)] < 0$.

If $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) = 1.0$, (5.5-13a) can be made equal to 0 by selecting $u_i(t) = 0$; on the other hand, (5.5-13b) will be 0 for all $u_i(t) \leq 0$; therefore, any nonpositive $u_i(t)$ will minimize (5.5-13).†

If $0 \leq \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < 1.0$, the minimum values of both (5.5-13a) and (5.5-13b) are zero and are attained for $u_i(t) = 0$.

The same reasoning is used for $\mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < 0$. In summary, the form of the optimal control is

$$u_i^*(t) = \begin{cases} 1.0, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < -1.0 \\ 0, & \text{for } -1.0 < \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < 1.0 \\ -1.0, & \text{for } 1.0 < \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) \\ \text{an undetermined nonnegative value if } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) = -1.0 \\ \text{an undetermined nonpositive value if } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) = +1.0. \end{cases} \tag{5.5-14}$$

Figure 5-25 illustrates the dependence of the optimal control on its coefficient in the Hamiltonian. Notice that whereas in minimum-time problems the optimal control is "bang-bang" (see Fig. 5-18) the minimum-fuel control may be described as "bang-off-bang" (if we assume no singular intervals).

In the remainder of this section we shall consider problems in which the plant dynamics are linear.

Free Final Time. Let us now consider some examples of linear minimum-fuel problems in which the final time t_f is not specified.

Example 5.5-1. The system

$$\dot{x}(t) = u(t) \tag{5.5-15}$$

is to be transferred from an arbitrary initial state x_0 to the origin. The performance measure to be minimized is

† If $\mathbf{p}^{*T}\mathbf{b}_i = \pm 1$ for a nonzero time interval, a singular solution exists; otherwise, a control switching is indicated.

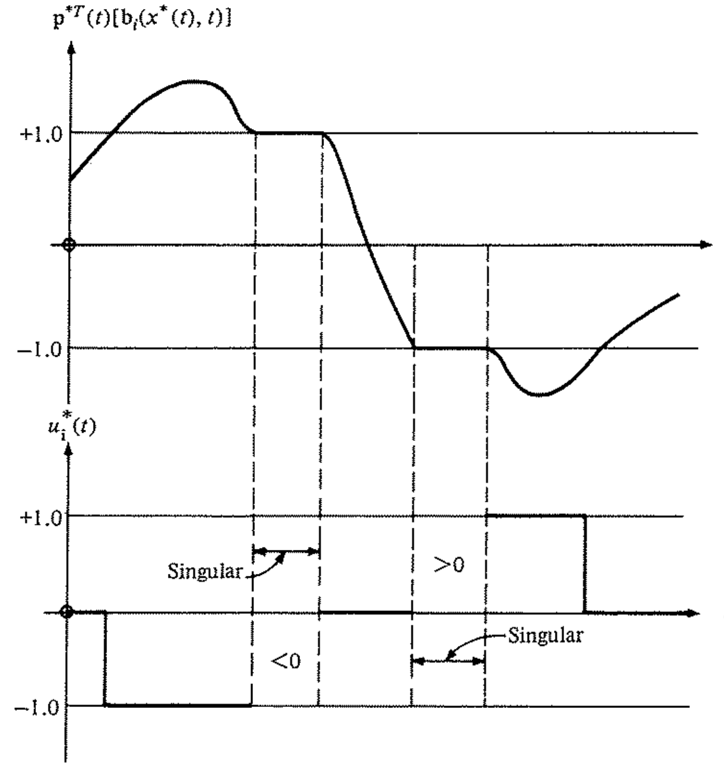


Figure 5-25 The relationship between a fuel-optimal control and its coefficient in the Hamiltonian

$$J(u) = \int_0^{t_f} |u(t)|\, dt, \tag{5.5-16}$$

where t_f is free, and the admissible controls satisfy

$$|u(t)| \leq 1.0. \tag{5.5-17}$$

It is desired to determine the optimal control law.

From (5.5-15) and (5.5-16) the Hamiltonian is

$$\mathscr{H}(x(t), u(t), p(t)) = |u(t)| + p(t)u(t). \tag{5.5-18}$$

The costate equation

$$\dot{p}^*(t) = -\frac{\partial \mathscr{H}}{\partial x} = 0 \tag{5.5-19}$$

has a solution of the form

$$p^*(t) = c_1, \tag{5.5-20}$$

where c_1 is a constant.

From Eq. (5.5-14) with $\mathbf{b}_i = \mathbf{B} = 1$, we have

$$u^*(t) = \begin{cases} 1.0, & \text{for } p^*(t) = c_1 < -1.0 \\ 0, & \text{for } -1.0 < c_1 < 1.0 \\ -1.0, & \text{for } \quad 1.0 < c_1 \\ \text{an undetermined nonnegative value if } c_1 = -1.0 \\ \text{an undetermined nonpositive value if } c_1 = +1.0. \end{cases} \tag{5.5-21}$$

The solution of the state equation is

$$x(t) = x_0 + \int_0^t u(t)\,dt; \tag{5.5-22}$$

thus, for $x(t_f) = 0$

$$0 = x_0 + \int_0^{t_f} u(t)\,dt, \tag{5.5-23}$$

or

$$x_0 = -\int_0^{t_f} u(t)\,dt. \tag{5.5-24}$$

Clearly, from (5.5-24) the control $u(t) = 0$, $t \in [0, t_f]$ can be optimal only if $x_0 = 0$—a trivial case. Suppose that $x_0 = 5.0$; then each of the controls

$$\begin{aligned} u(t) &= -1, & t &\in [0, 5] \\ u(t) &= -0.5 & t &\in [0, 10] \\ u(t) &= -0.2, & t &\in [0, 25] \\ u(t) &= -0.1, & t &\in [0, 50] \\ u(t) &= \begin{cases} -1, \\ -0.5 \end{cases} & t &\in [0, 2] \\ & & t &\in (2, 8] \end{aligned} \tag{5.5-25}$$

satisfies (5.5-24) and each makes $J = 5.0$. Now suppose we calculate a lower limit on the fuel required to force this system from x_0 to the origin. From (5.5-24)

$$|x_0| = \left| \int_0^{t_f} u(t)\,dt \right| \leq \int_0^{t_f} |u(t)|\,dt = J. \tag{5.5-26}$$

But each of the controls of (5.5-25) satisfies $J = |x_0|$; therefore, each of these controls is optimal. In this example, the optimal controls are nonunique. Notice, however, that the optimal controls of Eq. (5.5-25) each require a different amount of time to transfer the system to the origin.

In the preceding example there were many optimal controls (an infinite number); let us now consider an example in which an optimal control does not exist.

Example 5.5-2. It is desired to transfer the system

$$\dot{x}(t) = -ax(t) + u(t) \tag{5.5-27}$$

from an arbitrary initial state x_0 to the origin with admissible controls satisfying

$$|u(t)| \leq 1, \tag{5.5-28}$$

and $a > 0$.

The performance measure to be minimized is

$$J(u) = \int_0^{t_f} |u(t)|\,dt, \tag{5.5-29}$$

where t_f is free.

Using the state equation and the performance measure, we find that the Hamiltonian is

$$\mathscr{H}(x(t), u(t), p(t)) = |u(t)| - p(t)ax(t) + p(t)u(t); \tag{5.5-30}$$

thus, the costate equation is

$$\dot{p}^*(t) = -\frac{\partial \mathscr{H}}{\partial x} = ap^*(t), \tag{5.5-31}$$

which implies that

$$p^*(t) = c_1 \epsilon^{at}, \tag{5.5-32}$$

where c_1 is a constant of integration.

From Eq. (5.5-14) with $\mathbf{b}_i = \mathbf{B} = 1$, the form of the optimal control is

$$u^*(t) = \begin{cases} +1.0, & \text{for } \quad p^*(t) < -1.0 \\ \;\;0, & \text{for } \quad -1.0 < p^*(t) < 1.0 \\ -1.0, & \text{for } \quad 1.0 < p^*(t). \end{cases} \tag{5.5-33}$$

Notice that when $|p^*(t)|$ passes through the value 1.0, a switching of the control is indicated. Another possibility is that $|p^*(t)|$ might remain equal to 1.0 for a finite time interval; however, since

$$p^*(t) = c_1 \epsilon^{at}$$

and $a > 0$, it is clear that this situation cannot occur. It should be emphasized that the foregoing development tacitly assumes that an optimal control exists; we shall test the validity of this assumption shortly.

$p^*(t)$ will be one of the five forms shown in Fig. 5-26, depending on the value of c_1. The optimal controls, given by Eq. (5.5-33), which correspond to Fig. 5-26 are

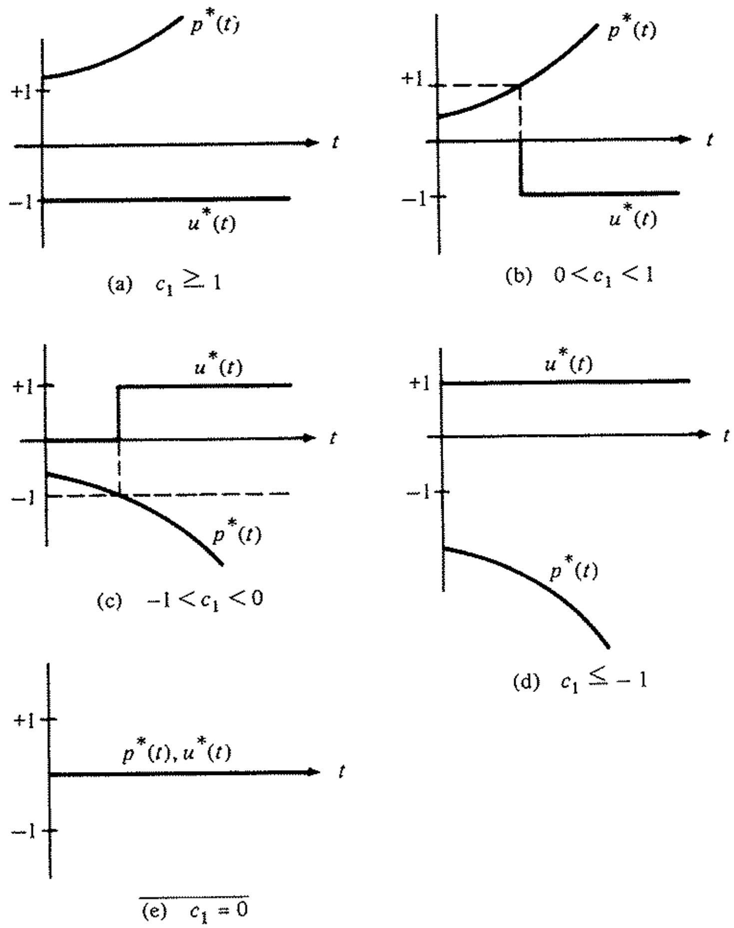


Figure 5-26 Possible forms for the costate and the corresponding fuel-optimal controls

$$
\begin{aligned}
u^*(t) &= -1, & t \in [0, t_f], &\quad \text{for } 1 \leq c_1 \\
u^*(t) &= \begin{cases} 0, & t \in [0, t_1) \\ -1, & t \in [t_1, t_f], \end{cases} & &\quad \text{for } 0 < c_1 < 1 \\
u^*(t) &= \begin{cases} 0, & t \in [0, t_1') \\ +1, & t \in [t_1', t_f], \end{cases} & &\quad \text{for } -1 < c_1 < 0 \qquad (5.5\text{-}34) \\
u^*(t) &= +1, & t \in [0, t_f], &\quad \text{for } c_1 \leq -1 \\
u^*(t) &= 0, & t \in [0, t_f], &\quad \text{for } c_1 = 0.
\end{aligned}
$$

We shall denote these five forms by $u^* = \{-1\}, \{0, -1\}, \{0, +1\}, \{+1\}$, and $\{0\}$, respectively.

The solution of the state equation is

$$x(t) = \epsilon^{-at}x_0 + \epsilon^{-at}\int_0^t \epsilon^{a\tau}u(\tau)\,d\tau. \qquad (5.5\text{-}35)$$

Notice that if the control is identically zero, then at $t = t_f$

$$x(t_f) = \epsilon^{-at_f}x_0. \qquad (5.5\text{-}36)$$

Since the system is stable, it naturally moves toward zero when no control is applied. If we are willing to wait long enough, the system will come arbitrarily close to (but never precisely reach) zero—and without the expenditure of any control effort at all. However, the problem statement stipulated that $x(t_f) = 0$, not $|x(t_f)| < \eta$, where η is some arbitrarily small positive number. If $x_0 > 0$, then clearly $u^* = \{-1\}, \{0, -1\}$ are the only possible choices for the optimal control (why?). If $u(t) = -1$ for $t \in [0, t_f]$, it can be shown from (5.5-35) that $x(t_f) = 0$ implies

$$t_f = \frac{1}{a}\ln(ax_0 + 1); \qquad (5.5\text{-}37)$$

thus, the fuel consumption using this control would be $[\ln(ax_0 + 1)]/a$.

Now, suppose $u(t) = 0$ is applied for $0 \leq t < t_1$ and $u(t) = -1$ for $t_1 \leq t \leq t_f$. From (5.5-35)

$$x(t_f) = \epsilon^{-at_f}x_0 + \epsilon^{-at_f}\int_{t_1}^{t_f} \epsilon^{a\tau}[-1]\,d\tau; \qquad (5.5\text{-}38)$$

setting $x(t_f) = 0$ and performing the indicated integration, we obtain

$$0 = \epsilon^{-at_f}x_0 - \frac{1}{a}[1 - \epsilon^{-a[t_f - t_1]}]. \qquad (5.5\text{-}39)$$

Solving for $t_f - t_1$ gives

$$t_f - t_1 = -\frac{1}{a}\ln(1 - ax_0\epsilon^{-at_f}), \qquad (5.5\text{-}40)$$

but since t_f is free, $ax_0\epsilon^{-at_f}$ can be made arbitrarily small by letting $t_f \longrightarrow \infty$, so

$$[t_f - t_1] \longrightarrow 0 \quad \text{as} \quad t_f \longrightarrow \infty. \qquad (5.5\text{-}41)$$

But $t_f - t_1$ is the interval during which $u = -1$ is applied, and by making t_f very large the consumed fuel can be made arbitrarily small (but not zero). Our conclusion is that if t_f is free an *optimal control does not exist*—given any candidate for an optimal control, it is always possible to find a control that transfers the system to the zero state with less fuel.

It is left as an exercise for the reader to verify that the same conclusions hold when $x_0 < 0$.

In the preceding example we have simply verified mathematically what common sense tells us; if elapsed time is not penalized, and the system moves toward the desired final state without consuming any fuel, the optimal strategy is to let the system drift as long as possible before any control is applied. At this point the reader might wonder: what did the minimum principle do for us? Could we not have deduced the same conclusions without using it at all? The answer to these questions is that quite likely the same conclusions could have been reached by intuitive reasoning alone, but the minimum principle, by specifying the possible forms of the optimal control, greatly reduced the number of control histories that had to be examined. In addition, we must remember that our interest is in solving problems that generally require more than physical reasoning and common sense.

Let us next discuss minimum-fuel problems with fixed final times.

Fixed Final Time. First, let us reconsider the preceding examples with the final time specified; that is, $t_f = T$. The value of T must be at least as large as t^*, the minimum time required to reach the specified target set from the initial state x_0.

In Example 5.5-1 we found that the optimal control was nonunique—there were an infinite number of controls that would transfer the system to $x(t_f) = 0$ with the minimum possible amount of fuel. The situation with $t_f = T$ is much the same unless $T = t^*$. In this case, the minimum-fuel and minimum-time controls are the same and unique. If, however, $T > t^*$, there are again an infinite number of controls that are optimal; it is left as an exercise for the reader to verify that this is the case. Fixing the final time does not alter the nonuniqueness of the optimal controls for the system of Example 5.5-1.

Let us now see if fixing the final time has any effect on the existence of fuel-optimal controls for the system of Example 5.5-2.

Example 5.5-3. The possible forms for optimal controls and the solution of the state equation are given in (5.5-34) and (5.5-35). If the fixed final time T is equal to the minimum time t^* required to reach the origin from the initial state x_0, then $u^*(t)$ is either $+1$ or -1 throughout the entire interval $[0, T]$, and

$$x(T) = 0 = \epsilon^{-aT}x_0 + \epsilon^{-aT}\int_0^T \epsilon^{a\tau}[\pm 1]\,d\tau, \tag{5.5-42}$$

or

$$x_0 = \mp\frac{1}{a}[\epsilon^{aT} - 1]. \tag{5.5-42a}$$

This expression defines the largest and smallest values of x_0 from which the origin can be reached in a (specified) time T. Initial states that satisfy

$$\frac{1}{a}[\epsilon^{aT} - 1] < |x_0| \tag{5.5-43}$$

cannot be transferred to the origin in time T; therefore, we shall assume in what follows that

$$|x_0| \leq \frac{1}{a}[\epsilon^{aT} - 1]. \tag{5.5-44}$$

If (5.5-44) is an equality, this means that $T = t^*$; otherwise, $T > t^*$, and the form of the optimal control must be as shown in Fig. 5-26(b) or (c). The optimal control must be nonzero during some part of the time interval, because we have previously shown that the system will not reach the origin in the absence of control.

If $x_0 > 0$, the optimal control must have the form $u^* = \{0, -1\}$ shown in Fig. 5-26(b). Substituting $u(t) = 0$, $t \in [0, t_1)$, $u(t) = -1$, $t \in [t_1, T]$, in (5.5-35) and performing the integration, we obtain

$$x(T) = 0 = \epsilon^{-aT}x_0 - \frac{1}{a}\epsilon^{-aT}[\epsilon^{aT} - \epsilon^{at_1}]. \tag{5.5-45}$$

Solving this equation for t_1, the time when the control switches from 0 to -1, gives

$$t_1 = \frac{1}{a}\ln(\epsilon^{aT} - ax_0). \tag{5.5-46}$$

Similarly, if $x_0 < 0$, the optimal control has the form $u^* = \{0, +1\}$ shown in Fig. 5-26(c), and

$$x(T) = 0 = \epsilon^{-aT}x_0 + \frac{1}{a}\epsilon^{-aT}[\epsilon^{aT} - \epsilon^{at_1'}]. \tag{5.5-47}$$

Solving for the switching time t_1' yields

$$t_1' = \frac{1}{a}\ln(\epsilon^{aT} + ax_0). \tag{5.5-48}$$

From (5.5-46) and (5.5-48) the optimal control is

$$u^*(t) = \begin{cases} 0, & \text{for } x_0 > 0 \text{ and } t < \dfrac{1}{a}\ln(\epsilon^{aT} - ax_0) \\ -1, & \text{for } x_0 > 0 \text{ and } \dfrac{1}{a}\ln(\epsilon^{aT} - ax_0) \leq t \leq T \\ 0, & \text{for } x_0 < 0 \text{ and } t < \dfrac{1}{a}\ln(\epsilon^{aT} + ax_0) \\ +1, & \text{for } x_0 < 0 \text{ and } \dfrac{1}{a}\ln(\epsilon^{aT} + ax_0) \leq t \leq T. \end{cases} \tag{5.5-49}$$

Notice that the optimal control expressed by (5.5-49) is in open-loop form, because $u^*(t)$ has been expressed in terms of x_0 and t; that is,

$$u^*(t) = e(x_0, t). \tag{5.5-50}$$

From an engineering point of view we would prefer to have the optimal control in feedback form; that is,

$$u^*(t) = f(x(t), t). \tag{5.5-51}$$

To obtain the optimal control law, we observe that

$$x(T) = \epsilon^{-a[T-t]}x(t) + \epsilon^{-aT}\int_t^T \epsilon^{a\tau}u(\tau)\,d\tau \tag{5.5-52}$$

for all t. We know that during the last part of the time interval the control is either $+1$ or -1, depending on whether $x(t)$ is less than zero or greater than zero; thus, assuming $x(t) > 0$, we have

$$x(T) = 0 = \epsilon^{-a[T-t]}x(t) - \epsilon^{-aT}\int_t^T \epsilon^{a\tau}\,d\tau, \qquad t \geq t_1. \tag{5.5-53}$$

Performing the indicated integration and solving for $x(t)$ gives

$$x(t) = \frac{1}{a}[\epsilon^{a[T-t]} - 1], \qquad t \geq t_1. \tag{5.5-54}$$

During the initial part of the time interval, the optimal control is zero; consequently,

$$x(t) = \epsilon^{-at}x_0, \qquad t < t_1. \tag{5.5-55}$$

The switching of the control from 0 to -1 occurs when the solution (5.5-55) for the *coasting interval* ($u = 0$) intersects the solution (5.5-54) for the *on*-negative *interval* ($u = -1$). Figure 5-27 shows these solutions. Defining

$$z(T-t) \triangleq \frac{1}{a}[\epsilon^{a[T-t]} - 1], \tag{5.5-56}$$

we observe that the control should switch from 0 to -1 when the state $x(t)$ is equal to $z(T-t)$. It is left as an exercise for the reader to verify that if $-[\epsilon^{aT} - 1]/a < x_0 < 0$ the optimal control switches from 0 to $+1$ when

$$x(t) = -z(T-t). \tag{5.5-57}$$

To summarize, the optimal control law is

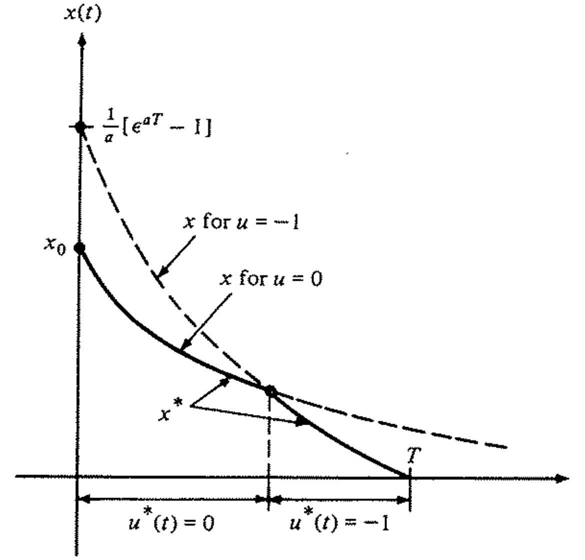


Figure 5-27 The two segments of a fuel-optimal trajectory $0 < x_0 < [\epsilon^{aT} - 1]/a$

$$u^*(t) = \begin{cases} -1, & \text{for } x(t) \geq z(T-t) \\ 0, & \text{for } |x(t)| < z(T-t) \\ +1, & \text{for } x(t) \leq -z(T-t) \end{cases} \tag{5.5-58}$$

or, more compactly,

$$u^*(t) = \begin{cases} 0, & \text{for } |x(t)| < z(T-t) \\ -\operatorname{sgn}(x(t)), & \text{for } |x(t)| \geq z(T-t).\dagger \end{cases} \tag{5.5-58a}$$

An implementation of this optimal control law is shown in Fig. 5-28. The logic element shown controls the switch. Notice that the controller requires a clock to tell it the current value of the time—the control law is time-varying. Naturally, this complicates the implementation; a time-invariant control law would be preferable.

Selecting the Final Time. In the preceding example fixing the final time led to a time-varying control law. We next ask: "How is the final time specified?" To answer this question, let us see how the minimum fuel required depends on the value of the final time T. Equations (5.5-46) and (5.5-48) indicate that the control switches from 0 at

† Here we define $\operatorname{sgn}(0) \triangleq 0$.

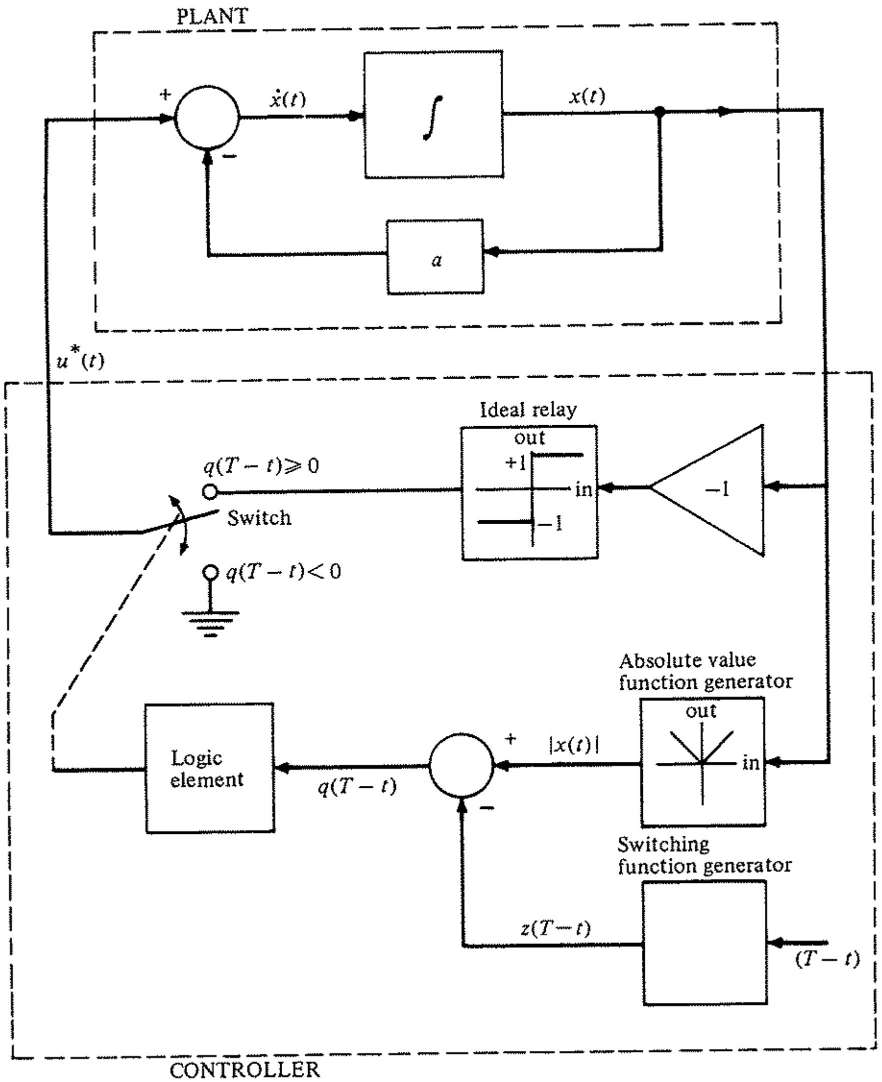


Figure 5-28 Implementation of a time-varying fuel-optimal control law.

$$t_1 = \frac{1}{a}\ln(\epsilon^{aT} - a\,|x_0|) \tag{5.5-59}$$

and remains at ± 1 until the final time is reached; thus, the fuel consumed is

$$T - t_1 = T - \frac{1}{a}\ln(\epsilon^{aT} - a\,|x_0|). \tag{5.5-60}$$

Using Eq. (5.5-60), the designer can obtain a plot of consumed fuel versus final time for several values of x_0 selected from the range of expected initial conditions; one such curve is shown in Fig. 5-29 for $|x_0| = 10.0$ and $a = 1.0$. The selection of T is then made by *subjectively* evaluating the information contained in these curves. Figure 5-29 indicates that in this particular example the value chosen for T will reflect the relative importance of consumed fuel and elapsed time.

The reader may have noticed that in Examples 5.5-1 through 5.5-3 a "trade-off" existed between fuel expenditure and elapsed time. The reason for this is that in each case the target set was the origin, and with no control applied the state of these systems either moved closer to the origin (Examples 5.5-2 and 5.5-3) or remained constant (Example 5.5-1). If the plants were of such a form that the states moved away from the target set with no control applied, the solutions obtained could have been quite different—see Problem 5-28.

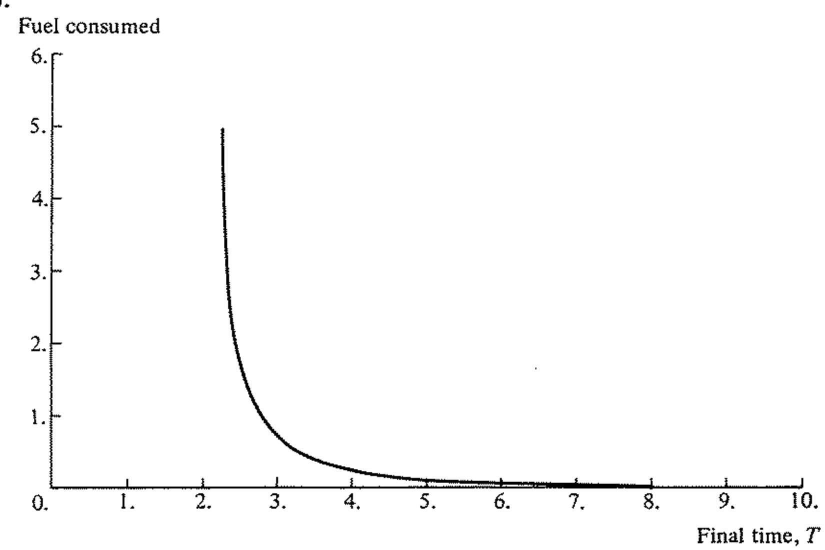


Figure 5-29 Dependence of consumed fuel on specified final time T, $|x_0| = 10$

A Weighted Combination of Elapsed Time and Consumed Fuel as the Performance Measure

The preceding examples in this section illustrated a trade-off between elapsed response time and consumed fuel; that is, the fuel expended to accomplish a specified state transfer was inversely proportional to the time required for the transfer. One technique for handling problems in which this trade-off is present is to include both elapsed time and consumed fuel in the performance measure. For a system with one control, such a performance measure would have the form

$$J(u) = \int_{t_0}^{t_f} [\lambda + |u(t)|]\, dt. \tag{5.5-61}$$

The final time t_f is free, and $\lambda > 0$ is chosen to weight the relative importance of elapsed time and fuel expended. For $\lambda \longrightarrow 0$ the optimal system will resemble a free-final-time, fuel-optimal system, whereas for $\lambda \longrightarrow \infty$ the optimal solution will resemble a time-optimal solution. Let us now reconsider Example 5.5-2 with (5.5-61) as the performance measure.

Example 5.5-4. The state equation and control constraint are given in Eqs. (5.5-27) and (5.5-28). The Hamiltonian is

$$\mathscr{H}(x(t), u(t), p(t)) = \lambda + |u(t)| - p(t)ax(t) + p(t)u(t), \tag{5.5-62}$$

and the costate equation is (again)

$$\dot{p}^*(t) = ap^*(t); \tag{5.5-63}$$

thus,

$$p^*(t) = c_1\epsilon^{at}, \tag{5.5-64}$$

where c_1 is a constant of integration. The requirement that $u^*(t)$ minimize the Hamiltonian on an extremal trajectory is unaffected by the presence of λ in the performance measure; therefore,

$$u^*(t) = \begin{cases} 1.0, & \text{for } p^*(t) < -1.0 \\ 0, & \text{for } -1.0 < p^*(t) < 1.0 \\ -1.0, & \text{for } p^*(t) > 1.0 \\ \text{undetermined, but nonnegative for } p^*(t) = -1.0\dagger \\ \text{undetermined, but nonpositive for } p^*(t) = +1.0.\dagger \end{cases} \tag{5.5-65}$$

If we recall that $a > 0$, Eq. (5.5-64) ensures that $p^*(t)$ cannot equal ± 1.0 for a nonzero time interval; hence, there are no intervals of singular control.

Equations (5.5-64) and (5.5-65) indicate that the optimal control must again be one of the forms shown in Fig. 5-26. Let us now examine the various alternatives.

Suppose that $t_0 = 0$ and $u^*(t) = 0$, $t \in [0, t_f]$; this implies that

$$\mathscr{H}(x^*(t), 0, p^*(t)) = \lambda - p^*(t)ax^*(t) \qquad \text{for all } t \in [0, t_f]. \tag{5.5-66}$$

† If $p^*(t) = \pm 1.0$ for a nonzero time interval, this signals the singular condition.

In this problem the final time is free and t does not appear explicitly in the Hamiltonian; therefore, from Eq. (5.3-41),

$$\mathscr{H}(x^*(t), u^*(t), p^*(t)) = 0 \qquad \text{for all } t \in [0, t_f]. \tag{5.5-67}$$

If Eq. (5.5-67) is to be satisfied, then Eq. (5.5-66) implies that

$$x^*(t) = \frac{\lambda}{ap^*(t)}, \tag{5.5-68}$$

or

$$x^*(t) = \frac{\lambda}{ac_1\epsilon^{at}} \qquad \text{for all } t \in [0, t_f]. \tag{5.5-68a}$$

Since $x^*(t_f) = 0$, Eq. (5.5-68a) can be satisfied for $\lambda > 0$ at t_f only if $t_f \longrightarrow \infty$, but this implies that the minimum cost approaches ∞. From our earlier discussion of this example, however, we know that controls can be found for which $J < \infty$; therefore, we conclude that $u(t) = 0$, $t \in [0, t_f]$, cannot be an optimal control.

If $u^* = \{0, -1\}$ is the form of the optimal control, $p^*(t)$ must pass through the value $+1.0$ at the time t_1, when the control switches [see Eq. (5.5-65)]. In addition, we know from Eq. (5.5-65) that $u^*(t_1)$ is some nonpositive value, so $|u^*(t_1)| = -u^*(t_1)$. The Hamiltonian must be zero for all t; thus, at time t_1

$$\mathscr{H}(x^*(t_1), u^*(t_1), p^*(t_1)) = \lambda - u^*(t_1) - ax^*(t_1) + u^*(t_1) = 0, \tag{5.5-69}$$

which implies that

$$x^*(t_1) = \frac{\lambda}{a}. \tag{5.5-70}$$

This equation is an important result, for it indicates that if there is a switching of control from 0 to -1 it occurs when $x^*(t)$ passes through the value λ/a. From Eq. (5.5-35)—the solution of the state equations—and Eq. (5.5-70) we obtain the family of optimal trajectories

$$x(t) = x_0\epsilon^{-at} \qquad \text{for } x(t) > \frac{\lambda}{a} \tag{5.5-71a}$$

$$x(t) = \frac{\lambda}{a}\epsilon^{-a[t-t_1]} - \frac{1}{a}[1 - \epsilon^{-a[t-t_1]}] \qquad \text{for } 0 < x(t) \leq \frac{\lambda}{a}. \tag{5.5-71b}$$

A control of the form $\{0, -1\}$ cannot transfer the system $\dot{x}(t) = -ax(t) + u(t)$ from a negative initial state value to the origin; hence, Eq. (5.5-71) applies for $x_0 > 0$.

Optimal trajectories for several different values of x_0 are shown in Fig. 5-30. Notice that if $0 < x_0 \leq \lambda/a$, the optimal strategy is to apply

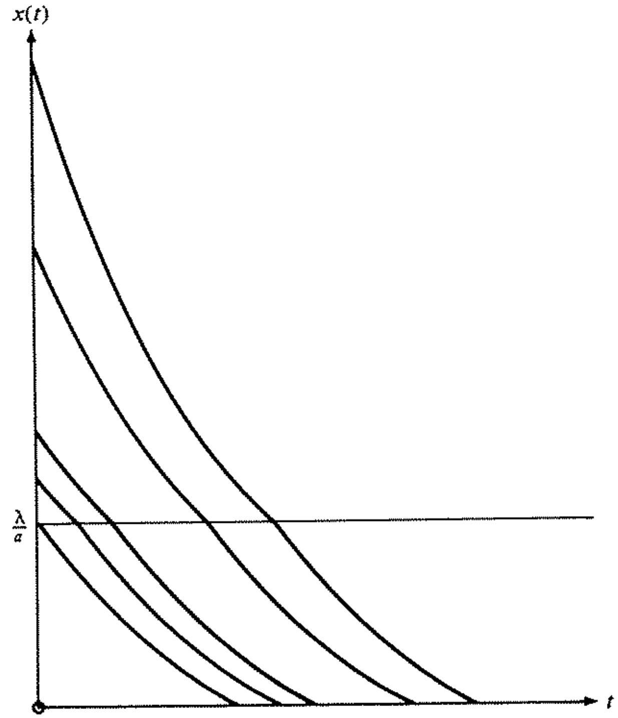


Figure 5-30 Several optimal trajectories for a weighted-time-fuel performance measure.

$u^*(t) = -1$ until the system reaches $x(t) = 0$. This is an intuitively reasonable result, because if $\lambda \longrightarrow \infty$, all trajectories begin with $x_0 \leq \lambda/a$ and will thus be minimum-time solutions. On the other hand, if $\lambda \longrightarrow 0$ the line λ/a moves very close to zero, and the optimal strategy approaches that indicated by Example 5.5-2 with free final time; let the system coast to as near the origin as possible before applying control.

The reader can show that for $x_0 < -\lambda/a$, the optimal strategy is to allow the system to coast [with $u^*(t) = 0$] until it reaches $x(t) = -\lambda/a$, where the optimal control switches to $u^*(t) = +1$.

The optimal control law—which is *time-invariant*—is summarized by

$$u^*(t) = \begin{cases} 0, & \text{for } \dfrac{\lambda}{a} < x(t) \\ -1.0, & \text{for } 0 < x(t) \leq \dfrac{\lambda}{a} \\ +1.0, & \text{for } -\dfrac{\lambda}{a} \leq x(t) < 0 \\ 0, & \text{for } x(t) < -\dfrac{\lambda}{a} \\ 0, & \text{for } x(t) = 0. \end{cases} \tag{5.5-72}$$

Figure 5-31 illustrates this optimal control law and its implementation. In solving this example the reader should note that we were able to determine the optimal control law using only the *form* of the costate solution—there was no need to solve for the constant of integration c_1. We also exploited the necessary condition that

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) = 0, \qquad t \in [0, t_f], \tag{5.5-73}$$

for t_f free and $\mathscr{H}$ not explicitly dependent on t, to determine the optimal control law and to show that the singular condition could not arise.

Let us now consider a somewhat less elementary example, which further illustrates the use of a weighted-time-fuel performance measure.

Example 5.5-5. Find the optimal control law for transferring the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \end{aligned} \tag{5.5-74}$$

from an arbitrary initial state $\mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}$ to the final state $\mathbf{x}(t_f) = \mathbf{0}$ with a minimum value of the performance measure

$$J(u) = \int_0^{t_f} [\lambda + |u(t)|]\, dt. \tag{5.5-75}$$

The admissible controls are constrained by

$$|u(t)| \leq 1.0; \tag{5.5-76}$$

the final time t_f is free, and $\lambda > 0$.

The reader can easily verify that the presence of λ in the Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \lambda + |u(t)| + p_1(t)x_2(t) + p_2(t)u(t) \tag{5.5-77}$$

does not alter the form of the optimal control given by Eq. (5.5-14); therefore, we have

$$u^*(t) = \begin{cases} 1.0, & \text{for } p_2^*(t) < -1.0 \\ 0, & \text{for } -1.0 < p_2^*(t) < 1.0 \\ -1.0, & \text{for } 1.0 < p_2^*(t) \\ \text{undetermined, but } \geq 0 \text{ for } p_2^*(t) = -1.0 \\ \text{undetermined, but } \leq 0 \text{ for } p_2^*(t) = +1.0. \end{cases} \tag{5.5-78}$$

The costate equations

$$\begin{aligned} \dot{p}_1^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_1} = 0 \\ \dot{p}_2^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_2} = -p_1^*(t) \end{aligned} \tag{5.5-79}$$

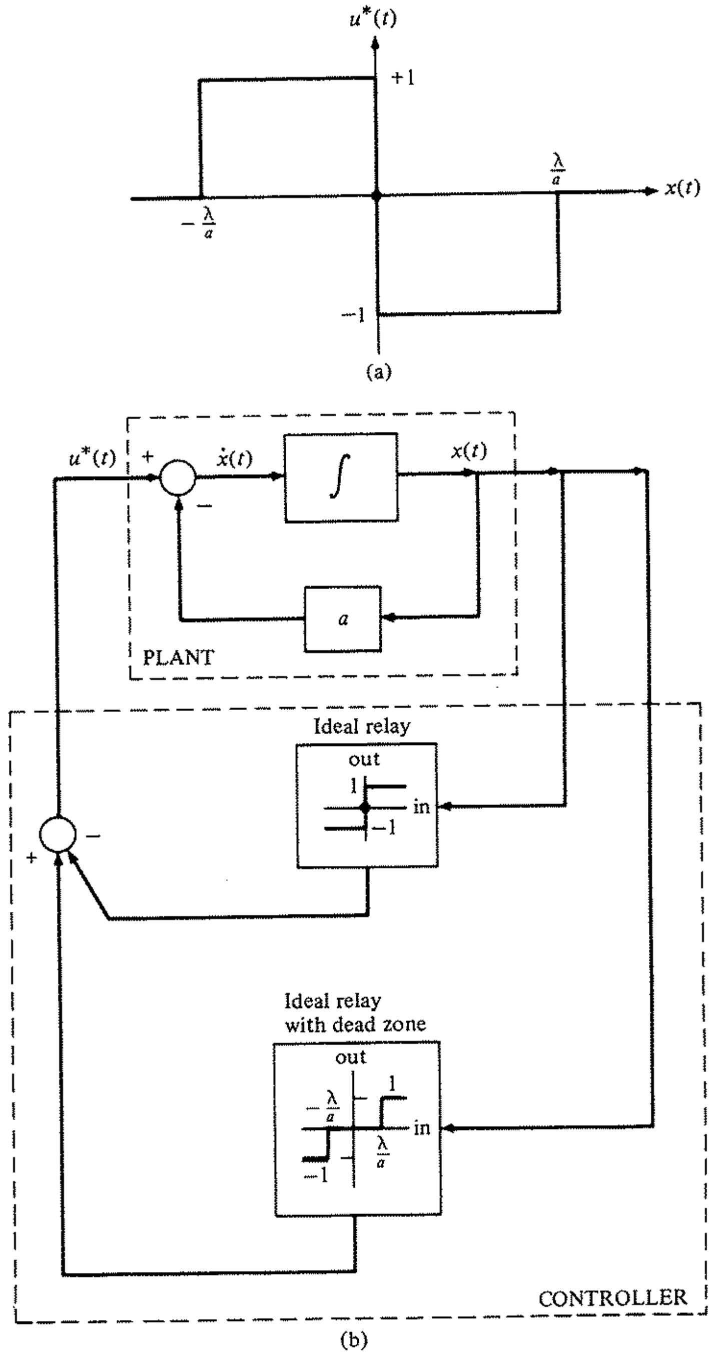


Figure 5-31 (a) The optimal control law for Example 5.5-4. (b) Implementation of the weighted-time-fuel optimal control law of Example 5.5-4

have solutions of the form

$$\begin{aligned} p_1^*(t) &= c_1 \\ p_2^*(t) &= -c_1 t + c_2. \end{aligned} \tag{5.5-80}$$

Clearly, p_2^* can change sign at most once, so the optimal control must have one of the forms (excluding singular solutions)

$$\begin{aligned} u^* = \{0\}, \{+1\}, \{-1\}, \{0, +1\}, \{0, -1\}, \{+1, 0\}, \\ \{-1, 0\}, \{+1, 0, -1\}, \{-1, 0, +1\}. \end{aligned} \tag{5.5-81}$$

First let us see whether or not there can be any singular solutions. For $p_2^*(t)$ to be equal to ± 1.0 during a finite time interval, it is necessary that $c_1 = 0$ and $c_2 = \pm 1.0$. Substituting $p_2^*(t) = \pm 1$ in (5.5-77), and using (5.5-78) and the definition of the absolute value function, we obtain

$$\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) = \lambda > 0 \tag{5.5-82}$$

if the singular condition is to occur, but we know (since $\mathscr{H}$ is explicitly independent of time and t_f is free) that the Hamiltonian must be zero on an optimal trajectory. We conclude, then, that the singular condition cannot arise in this problem.

Let us now investigate the control alternatives given by Eq. (5.5-81). First, observe that none of the alternatives that ends with an interval of $u = 0$ can be optimal because the system (5.5-74) does not move to the origin with no control applied. Next, consider the optimal control candidates

$$u^* = \{-1\}, \{0, -1\}, \{+1, 0, -1\}. \tag{5.5-83}$$

To be optimal, the trajectories resulting from these three control forms must terminate at the origin with an interval of $u^* = -1$ control. The system differential equations are the same in this problem as in the minimum-time problem discussed in Example 5.4-4, so the terminal segments of these trajectories all lie on the curve B-0 in Fig. 5-20(b). Now, for any interval during which $u(t) = 0$ the state equations are

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= 0, \end{aligned} \tag{5.5-84}$$

which implies that

$$\begin{aligned} x_2(t) &= c_3 = \text{a constant} \\ x_1(t) &= c_3 t + c_4. \end{aligned} \tag{5.5-85}$$

Thus, as time increases, $x_1(t)$ increases or decreases, depending on whether $x_2(t)$ is greater or less than zero when the control switches to $u = 0$.

Several trajectories for $u = 0$ are shown in Fig. 5-32; the direction of increasing time is indicated by the arrows. Notice that if $x_2(t) = 0$ when the control switches to zero, the value of x_1 does not change until the control becomes nonzero.

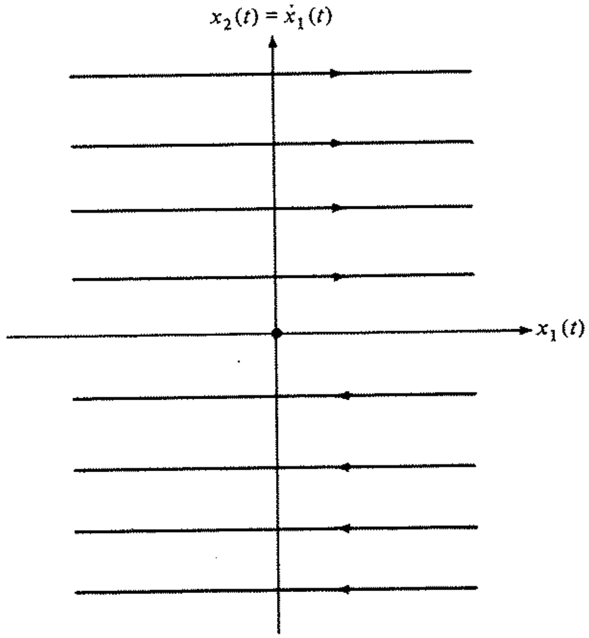


Figure 5-32 Trajectories for $u = 0$

Trajectory segments generated by $u = +1$ are the same as trajectories shown in Fig. 5-20(a).

To draw the candidates for an optimal trajectory we simply piece together segments of the trajectories shown in Figs. 5-20 and 5-32. The trajectories *C-D-E-0*, *C-F-G-0*, and *C-H-I-0* shown in Fig. 5-33 are three *candidates* for an optimal trajectory which has the initial state $\mathbf{x}_0$. Our task now is to determine the point on segment *C-K*, where the optimal control switches from $+1$ to 0. Once this point is known, we can easily determine the entire optimal trajectory.

Let t_1 be the time when the optimal control switches from $+1$ to 0, and let t_2 be the time when the optimal control switches from 0 to -1. Clearly, t_1 occurs somewhere on segment *C-K* and t_2 on segment *K-0*. We know from Eq. (5.4-40) that on *K-0*

$$x_1^*(t) = -\tfrac{1}{2}x_2^{*2}(t), \tag{5.5-86}$$

so

$$x_1^*(t_2) = -\tfrac{1}{2}x_2^{*2}(t_2). \tag{5.5-87}$$

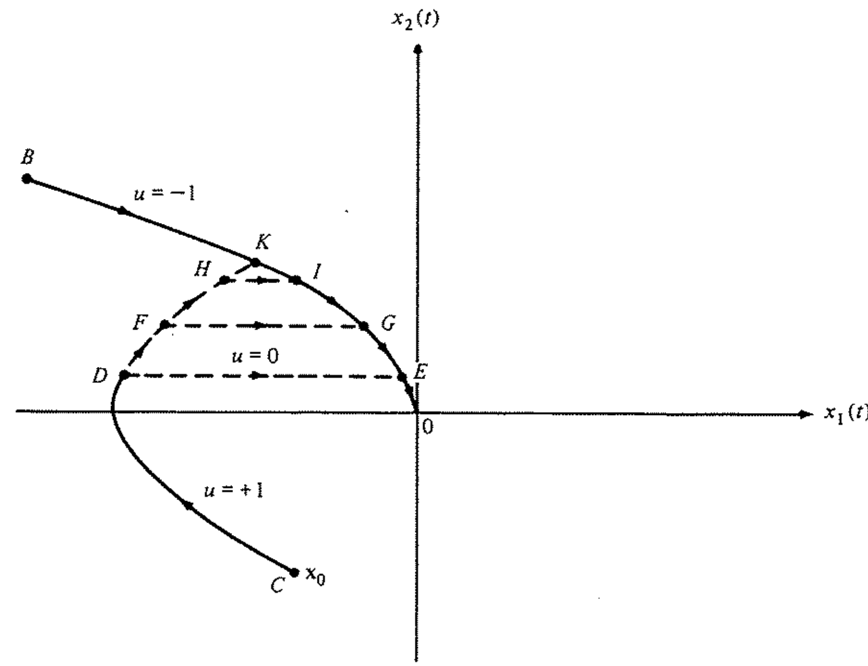


Figure 5-33 Three candidates for the optimal trajectory with initial state $\mathbf{x}_0$

In addition, integrating Eq. (5.5-84) gives

$$x_1^*(t_2) = x_1^*(t_1) + x_2^*(t_1)[t_2 - t_1], \tag{5.5-88}$$

and from Eqs. (5.5-80) and (5.5-78) we obtain

$$p_2^*(t_1) = -c_1 t_1 + c_2 = -1.0 \tag{5.5-89}$$

$$p_2^*(t_2) = -c_1 t_2 + c_2 = +1.0. \tag{5.5-90}$$

Because $p_2^*(t_1) = -1$ and $p_2^*(t_2) = +1$, the necessary condition that $\mathscr{H}$ be identically zero requires that

$$\lambda + c_1 x_2^*(t_1) = 0 \tag{5.5-91}$$

and

$$\lambda + c_1 x_2^*(t_2) = 0. \tag{5.5-92}$$

Let us now solve Eqs. (5.5-87) through (5.5-92) for $x_1^*(t_1)$.

First we observe that Eqs. (5.5-91) and (5.5-92) imply that $x_2^*(t_1) = x_2^*(t_2)$ and that

$$c_1 = \frac{-\lambda}{x_2^*(t_1)}. \tag{5.5-93}$$

Subtracting (5.5-90) from (5.5-89) gives

$$[t_2 - t_1] = -\frac{2}{c_1}, \tag{5.5-94}$$

which, if we use (5.5-93), becomes

$$[t_2 - t_1] = \frac{2x_2^*(t_1)}{\lambda}. \tag{5.5-95}$$

Putting this in (5.5-88) yields

$$x_1^*(t_2) = x_1^*(t_1) + \frac{2x_2^{*2}(t_1)}{\lambda}. \tag{5.5-96}$$

Substituting the right side of Eq. (5.5-87) for $x_1^*(t_2)$ and using the fact that $x_2^*(t_2) = x_2^*(t_1)$, yields

$$-\frac{1}{2}x_2^{*2}(t_1) = x_1^*(t_1) + \frac{2x_2^{*2}(t_1)}{\lambda}. \tag{5.5-97}$$

Collecting terms, we obtain

$$\boxed{x_1^*(t_1) = -\frac{\lambda + 4}{2\lambda}x_2^{*2}(t_1).} \tag{5.5-98a}$$

This is the sought-after result. The values of x_1 and x_2 that satisfy Eq. (5.5-98a) are the locus of points where the control switches from $+1$ to 0. It is left to the reader to show that for $u^* = \{-1, 0, +1\}$ the locus of points which defines the switching from $u^* = -1$ to $u^* = 0$ is given by

$$\boxed{x_1^*(t_1') = +\frac{\lambda + 4}{2\lambda}x_2^{*2}(t_1').} \tag{5.5-98b}$$

Notice particularly that Eq. (5.5-98) together with Eq. (5.5-87) and its counterpart for $u^*(t) = +1$ define the optimal control *law*. Furthermore, this optimal control law is time-invariant. The switching curves for $\lambda = 0.1$, 1.0, and 10.0 and several optimal trajectories for $\lambda = 1.0$ are shown in Fig. 5-34. Observe that if $\lambda \longrightarrow \infty$ the switching curves merge together—the interval of $u^* = 0$ approaches zero, and trajectories

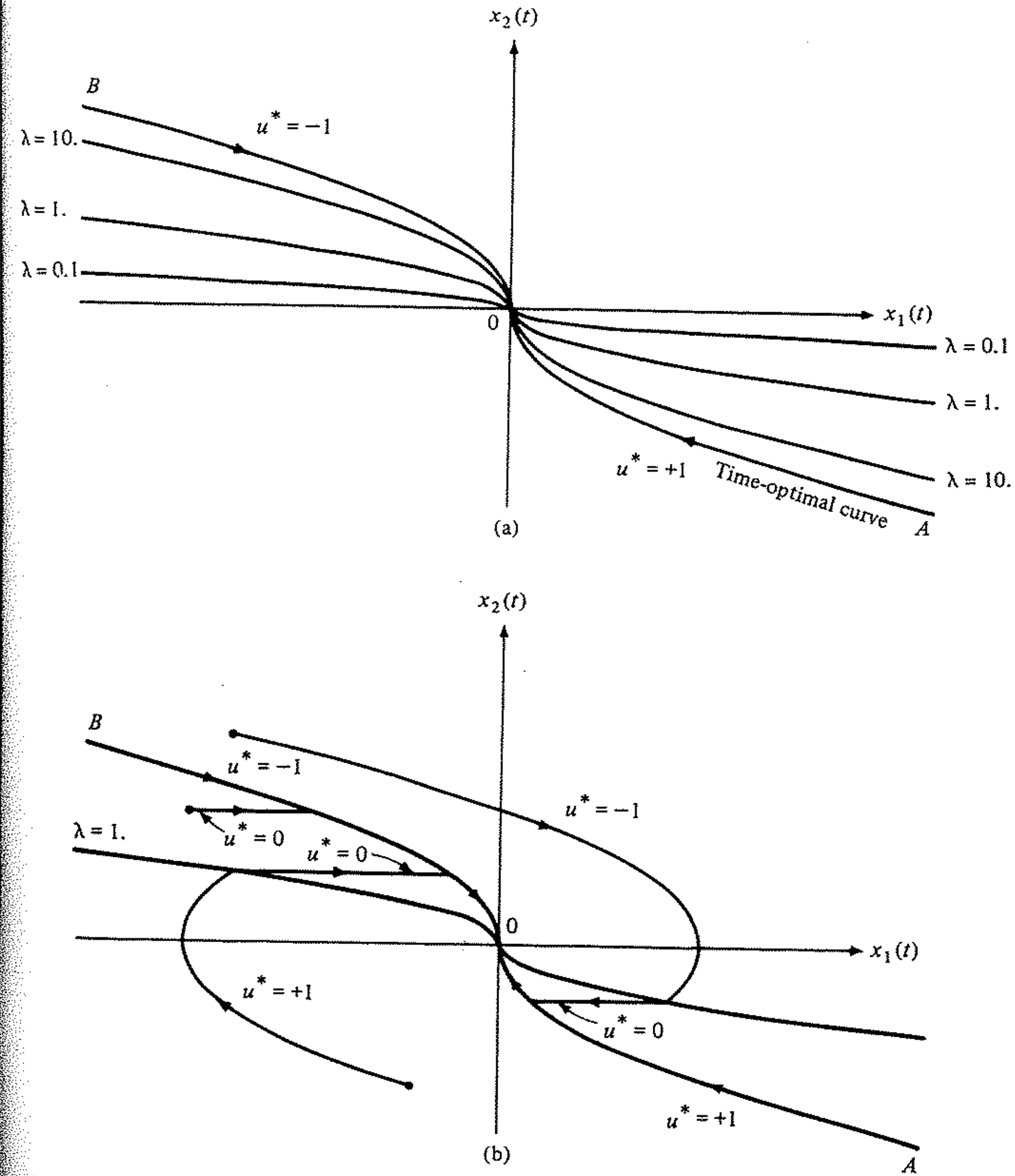


Figure 5-34 (a) Switching curves for weighted-time-fuel optimal performance. (b) Weighted-time-fuel optimal trajectories for three initial conditions ($\lambda = 1.0$)

approach the time-optimal trajectories of Example 5.4-4. On the other hand, if $\lambda \to 0$, the interval of $u^* = 0$ approaches infinity, and the trajectories approach fuel-optimal trajectories.

The numerical value of λ must be decided upon subjectively by the designer. To help in making this decision, curves showing the dependence of elapsed time and consumed fuel on λ—such as Fig. 5-35—could be plotted for several initial conditions.

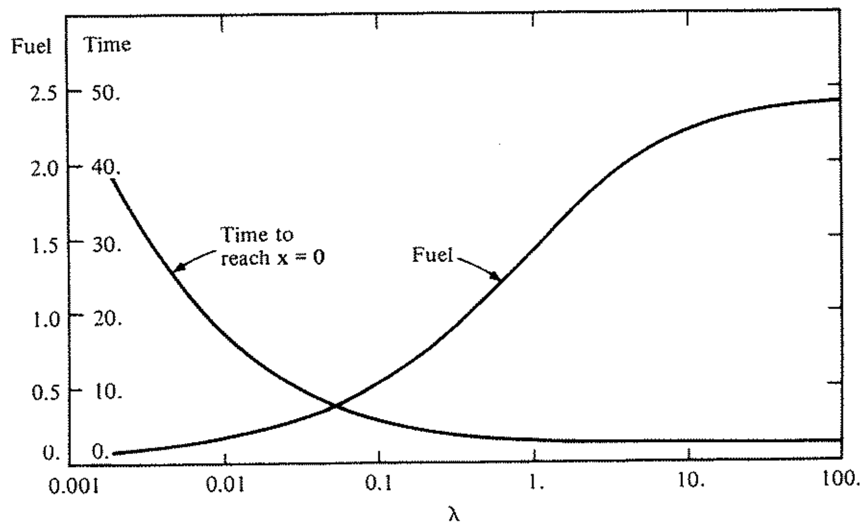


Figure 5-35 The dependence of elapsed time and consumed fuel on the weighting parameter λ, $\mathbf{x}(0) = \begin{bmatrix} -1.5 \\ 0.0 \end{bmatrix}$

Minimum-Energy Problems

The characteristics of fuel-optimal problems and energy-optimal problems are similar; therefore, the following discussion will be limited to one example, which illustrates some of the differences between these two types of systems.

Example 5.5-6. The plant of Examples 5.5-2 and 5.5-4

$$\dot{x}(t) = -ax(t) + u(t) \tag{5.5-99}$$

is to be transferred from an arbitrary initial state, $x(0) = x_0$, to the origin by a control that minimizes the performance measure

$$J(u) = \int_0^{t_f} [\lambda + u^2(t)]\, dt; \tag{5.5-100}$$

the admissible controls are constrained by

$$|u(t)| \le 1. \tag{5.5-101}$$

The plant parameter a and the weighting factor λ are greater than zero, and the final time t_f is free. The objective is to find the optimal control law.

The first step, as usual, is to form the Hamiltonian,

$$\mathscr{H}(x(t), u(t), p(t)) = \lambda + u^2(t) - p(t)ax(t) + p(t)u(t). \tag{5.5-102}$$

The costate equation and its solution are

$$\dot{p}^*(t) = ap^*(t) \tag{5.5-103}$$

and

$$p^*(t) = c_1\epsilon^{at}. \tag{5.5-104}$$

For $|u(t)| < 1$, the control that minimizes $\mathscr{H}$ is the solution of the equation

$$\frac{\partial \mathscr{H}}{\partial u} = 2u^*(t) + p^*(t) = 0. \tag{5.5-105}$$

Notice that $\mathscr{H}$ is quadratic in $u(t)$ and

$$\frac{\partial^2 \mathscr{H}}{\partial u^2} = 2 > 0, \tag{5.5-106}$$

so

$$u^*(t) = -\tfrac{1}{2}p^*(t) \tag{5.5-107}$$

does globally minimize the Hamiltonian for $|u^*(t)| < 1$, or, equivalently, for

$$|p^*(t)| < 2. \tag{5.5-108}$$

If $|p^*(t)| \ge 2$, then the control that minimizes $\mathscr{H}$ is

$$u^*(t) = \begin{cases} +1.0, & \text{for } p^*(t) \le -2.0 \\ -1.0, & \text{for } 2.0 \le p^*(t). \end{cases} \tag{5.5-109}$$

Putting Eqs. (5.5-107) and (5.5-109) together, we obtain

$$u^*(t) = \begin{cases} 1.0, & \text{for } p^*(t) \le -2.0 \\ -\tfrac{1}{2}p^*(t), & \text{for } -2.0 < p^*(t) < 2.0 \\ -1.0, & \text{for } \quad 2.0 \le p^*(t). \end{cases} \tag{5.5-110}$$

This relationship between an extremal control and an extremal costate is illustrated in Fig. 5-36. There is no possibility of singular solutions in this example, since there are no values of $p^*(t)$ for which the Hamiltonian is unaffected by $u(t)$.

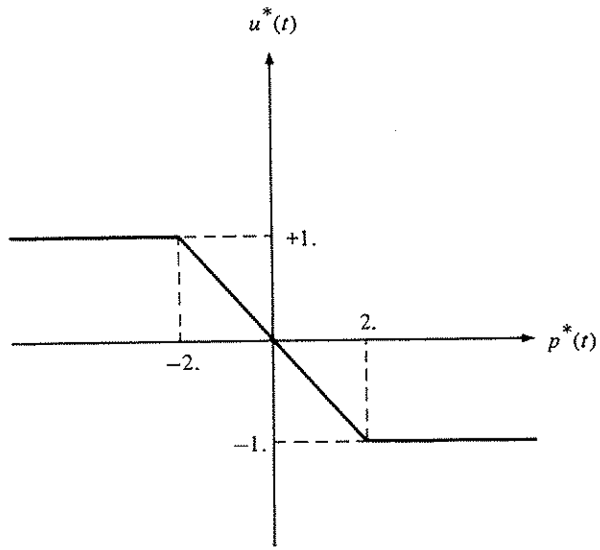


Figure 5-36 The relationship between an extremal control and costate

We rule out the possibility that $p^*(t) = 0$ for $t \in [0, t_f]$ {since this implies $u^*(t) = 0$ for $t \in [0, t_f]$ and the system would never reach the origin}; the possible forms for $p^*(t)$ are shown in Fig. 5-37. Corresponding to the costate curves labeled 1, 2, 3, and 4 are the optimal control possibilities:

1. $u^* = \{-\tfrac{1}{2}p^*\}$, or $\{-\tfrac{1}{2}p^*, -1.0\}$, (5.5-111a)

   depending on whether or not the system reaches the origin before p^* attains the value 2.0.

2. $u^* = \{-1.0\}$. (5.5-111b)

3. $u^* = \{-\tfrac{1}{2}p^*\}$, or $\{-\tfrac{1}{2}p^*, +1.0\}$, (5.5-111c)

   depending on whether or not the system reaches the origin before p^* attains the value −2.0.

4. $u^* = \{+1.0\}$. (5.5-111d)

The controls given by (5.5-111a) and (5.5-111b) are nonpositive for all $t \in [0, t_f]$ and correspond to positive state values. This can be seen from the solution of the state equation

$$x(t_f) = 0 = \epsilon^{-a[t_f - t]}x(t) + \epsilon^{-at_f}\int_t^{t_f} \epsilon^{a\tau}u(\tau)\,d\tau, \qquad (5.5\text{-}112)$$

which implies that

$$-\epsilon^{at}x(t) = \int_t^{t_f} \epsilon^{a\tau}u(\tau)\,d\tau. \qquad (5.5\text{-}113)$$

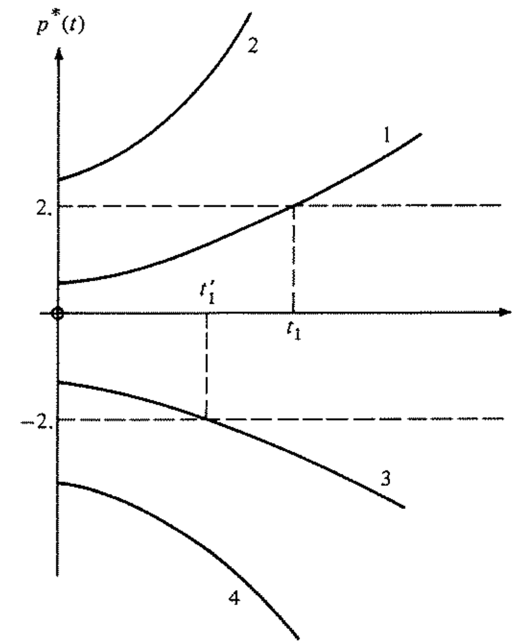


Figure 5-37 Possible forms for an extremal costate trajectory

For $u(\tau)$ nonpositive when $\tau \in [t, t_f]$, the integral is negative; therefore, $x(t)$ must be positive. Similarly, the nonnegative controls specified by Eqs. (5.5-111c) and (5.5-111d) correspond to negative values of $x(t)$.

Since t_f is free, and the Hamiltonian does not contain t explicitly, it is also necessary that

$$\mathscr{H}(x^*(t), u^*(t), p^*(t)) = 0, \qquad t \in [t_0, t_f]. \qquad (5.5\text{-}114)$$

If the control saturates at the value −1 when $t = t_1$, then from (5.5-110), $p^*(t_1) = 2.0$; substituting $u^*(t_1) = -1$ and $p^*(t_1) = 2$ in $\mathscr{H}$, we obtain

$$\mathscr{H}(x^*(t_1), u^*(t_1), p^*(t_1)) = \lambda + 1 - 2ax^*(t_1) - 2 = 0, \qquad (5.5\text{-}115)$$

which implies

$$x^*(t_1) = \frac{\lambda - 1}{2a}. \qquad (5.5\text{-}116)$$

If the control saturates at −1 when $t = t_1$, then from (5.5-111) $u^*(t) = -1$ for $t \in [t_1, t_f]$, and $x^*(t) < x^*(t_1)$ for $t > t_1$; thus,

$$u^*(t) = -1 \qquad \text{for } 0 < x^*(t) < \frac{\lambda - 1}{2a}. \qquad (5.5\text{-}117a)$$

Using similar reasoning, we can show that if the control saturates at the value +1 when $t = t_1'$, then

$$x^*(t_1') = \frac{\lambda - 1}{-2a} \tag{5.5-118}$$

and

$$u^*(t) = +1 \qquad \text{for} \quad \frac{\lambda - 1}{-2a} < x^*(t) < 0. \tag{5.5-117b}$$

Notice that if $\lambda \leq 1$, $x^*(t_1) \leq 0$ in (5.5-116), and $x^*(t_1') \geq 0$ in (5.5-118), but (5.5-116) applies for positive state values and (5.5-118) applies for negative state values; hence the optimal control does not saturate for $\lambda \leq 1$.

Let us now examine the unsaturated region where $u^*(t) = -\frac{1}{2}p^*(t)$. Again using the necessary condition of Eq. (5.5-114), by substituting $u^*(t) = -\frac{1}{2}p^*(t)$, we obtain

$$\mathscr{H}(x^*(t), u^*(t), p^*(t)) = \lambda + \tfrac{1}{4}p^{*2}(t) - p^*(t)ax^*(t) - \tfrac{1}{2}p^{*2}(t) = 0. \tag{5.5-119}$$

Solving for $p^*(t)$ yields

$$p^*(t) = 2\left[-ax^*(t) \pm \sqrt{[ax^*(t)]^2 + \lambda}\right], \tag{5.5-120}$$

which implies

$$u^*(t) = \left[ax^*(t) \pm \sqrt{[ax^*(t)]^2 + \lambda}\right]. \tag{5.5-121}$$

If $x^*(t) > 0$, $u^*(t)$ must be negative, so the minus sign applies; for $x^*(t) < 0$ the positive sign applies. The optimal control law, if we put together Eqs. (5.5-121) and (5.5-117), is

$$u^*(t) = \begin{cases} \left[ax(t) - \sqrt{[ax(t)]^2 + \lambda}\right], & \text{for } 0 < \dfrac{\lambda - 1}{2a} < x(t) \\ -1.0, & \text{for } 0 < x(t) \leq \dfrac{\lambda - 1}{2a} \\ +1.0, & \text{for } -\dfrac{\lambda - 1}{2a} \leq x(t) < 0 \\ \left[ax(t) + \sqrt{[ax(t)]^2 + \lambda}\right], & \text{for } x(t) < -\dfrac{\lambda - 1}{2a} < 0 \\ 0, & \text{for } x(t) = 0. \end{cases} \tag{5.5-122†}$$

Figure 5-38 illustrates this optimal control law and its implementation. Comparing Figs. 5-31(a) and 5-38(a), the reader will note that the weighted-time-fuel-optimal controls are either "on" (±1) or "off" (0), whereas the weighted-time-energy-optimal controls can assume all values from −1 to +1.

† The optimal control law is valid for all state values, so we write $x(t)$ instead of $x^*(t)$.

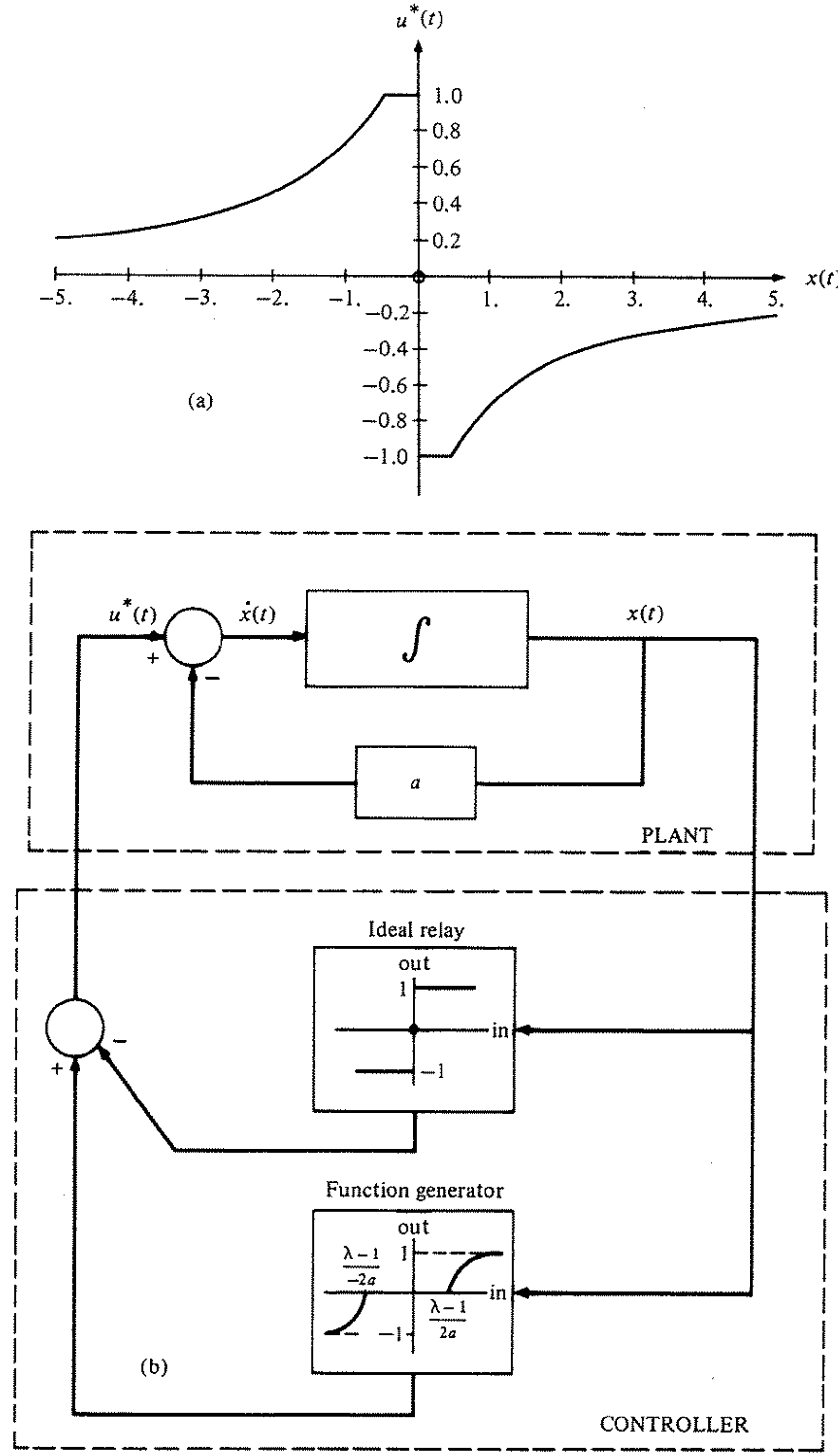


Figure 5-38 (a) The weighted-time-energy optimal control law for Example 5.5-6: $\lambda = 2.0$, $a = 1.0$. (b) Implementation of the weighted-time-energy optimal control law for Example 5.5-6

To provide an additional basis for comparing this energy-optimal system with the fuel-optimal system of Example 5.5-4, several optimal trajectories for each system are shown in Fig. 5-39. For the weighted-time-energy-optimal system λ was 2; the value of λ for the weighted-time-fuel-optimal system was adjusted for each initial condition to make the two systems require the same amount of time to reach the origin. The fuel and energy requirements for the two systems are summarized in Table 5-2.

Table 5-2 FUEL AND ENERGY REQUIREMENTS FOR THE SYSTEMS OF EXAMPLES 5.5-4 AND 5.5-6

Initial condition $x(0)$	*Time required to reach* $x(t_f) = 0$	*Fuel for time-fuel-optimal system*	*Fuel for time-energy-optimal system*	*Energy for time-energy-optimal system*	*Energy for time-fuel-optimal system*
1.5	0.982	0.8252	0.8434	0.7473	0.8252
2.0	1.205	0.9138	0.9559	0.8043	0.9138
2.5	1.393	0.9688	1.0326	0.8356	0.9688
3.0	1.555	1.0038	1.0883	0.8548	1.0038
5.0	2.034	1.0612	1.2090	0.8858	1.0612
7.0	2.361	1.0788	1.2636	0.8950	1.0788

Summary

In this section we have considered the optimization of systems whose control effort is to be conserved. Although our discussion was primarily concerned with the solution of several example problems, it was shown that the form of fuel-optimal controls for a class of nonlinear systems is "bang-off-bang"; it was left as an exercise for the reader (Problem 5-30) to show that the form of energy-optimal controls for the same class of nonlinear systems is a continuous, saturating function.

In all of the examples considered a trade-off existed between conservation of control effort and rapid action. It was found that such problems may be characterized by nonunique or nonexistent optimal controls when the final time is free, and that fixing the final time may still result in nonunique optimal controls or in a time-varying optimal control law. To circumvent these difficulties, a performance measure consisting of a weighted combination of elapsed-time and control-effort expended was introduced. In the problems solved, this form of performance measure resulted in time-invariant optimal control laws, and, in addition, reflected the trade-off between conservation of control effort and rapid action. It should be emphasized that there are alternative formulations of minimum-control-effort problems (see Problem 5-33)

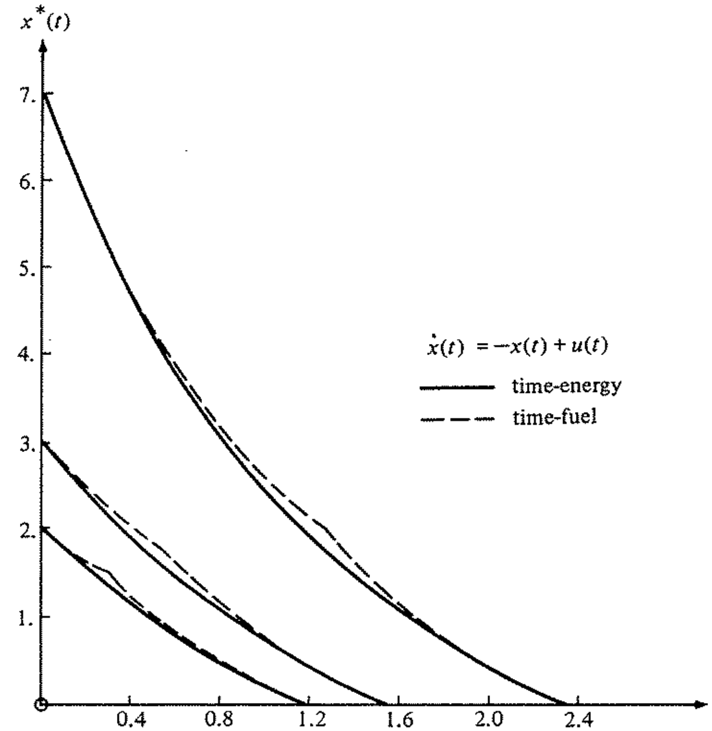


Figure 5-39 Weighted-time-fuel and weighted-time-energy optimal trajectories

and that conserving control effort and obtaining rapid action may not always be conflicting objectives (see Problems 5-28 and 5-31).

No attempt was made to generalize the results of the examples to a "design procedure." The reason for this omission is that unless the system is of low order, time-invariant, and linear, we have little hope of analytically determining the optimal control law. The difficulties mentioned at the end of Section 5.4 for time-optimal systems also apply to the energy- and fuel-optimal systems considered here—only more so. The primary virtue of the discussion in this section is that it provides insight into the form of the optimal control and furnishes a starting point for numerical determination of the optimal control law.

5.6 SINGULAR INTERVALS IN OPTIMAL CONTROL PROBLEMS

In discussing minimum-time and minimum-control-effort problems we have used Pontryagin's necessary condition,

$$\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \tag{5.6-1}$$

for all $t \in [t_0, t_f]$ and for all admissible $\mathbf{u}(t)$, to determine $\mathbf{u}^*(t)$ in terms of the extremal states and costates. If, however, there is a time interval $[t_1, t_2]$ of finite duration during which the necessary condition (5.6-1) provides no information about the relationship between $\mathbf{u}^*(t)$, $\mathbf{x}^*(t)$, and $\mathbf{p}^*(t)$, then we say that the problem is singular. The interval $[t_1, t_2]$ is called an interval of singularity, or simply a *singular interval.*

We shall now investigate the conditions that allow singular intervals to occur, and the effects of singular intervals on optimal controls and trajectories. To begin our investigation, let us return to a minimum-time problem discussed in Section 5.4.

Example 5.6-1. In Example 5.4-4 we considered the problem of transferring the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned} \tag{5.6-2}$$

from an arbitrary initial state to the origin in minimum time. The admissible controls were required to satisfy the inequality

$$|u(t)| \leq 1.0. \tag{5.6-3}$$

In solving this problem we assumed that a singular interval did not exist; let us now verify that this assumption was correct.

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)\,x_2(t) + p_2(t)u(t), \tag{5.6-4}$$

and application of the minimum principle gives

$$1 + p_1^*(t)x_2^*(t) + p_2^*(t)u^*(t) \leq 1 + p_1^*(t)x_2^*(t) + p_2^*(t)u(t). \tag{5.6-5}$$

If there exists a time interval $[t_1, t_2]$ during which

$$p_2^*(t) = 0, \tag{5.6-6}$$

then (5.6-5) provides no information about the relationship between $u^*(t)$, $\mathbf{x}^*(t)$, and $\mathbf{p}^*(t)$. Therefore, if

$$p_2^*(t) = 0 \qquad \text{for } t \in [t_1, t_2], \tag{5.6-7}$$

then $[t_1, t_2]$ is a singular interval.† Let us investigate further to see if this condition can occur. The costate equations

† Isolated times when $p_2^*(t)$ *passes through* zero indicate a switching of the control, not a singular interval.

$$\begin{aligned}\dot{p}_1^*(t) &= 0\\ \dot{p}_2^*(t) &= -p_1^*(t)\end{aligned} \tag{5.6-8}$$

have solutions of the form

$$\begin{aligned}p_1^*(t) &= c_1\\ p_2^*(t) &= -c_1 t + c_2.\end{aligned} \tag{5.6-9}$$

But for $p_2^*(t) = 0$ for $t \in [t_1, t_2]$ it is necessary that

$$c_1 = 0 \tag{5.6-10a}$$

and

$$c_2 = 0. \tag{5.6-10b}$$

Substituting these values in the Hamiltonian gives

$$\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) = 1 \qquad \text{for all } t \in [0, t_f], \tag{5.6-11}$$

but since the final time is free and $\mathscr{H}$ is explicitly independent of time, Eq. (5.6-11) violates the necessary condition that

$$\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) = 0 \qquad \text{for all } t \in [0, t_f]. \tag{5.6-12}$$

We conclude that $p_2^*(t)$ cannot be zero during a finite time interval, and, thus, that a singular interval cannot exist.

Let us now discuss in more generality the possibility of singular intervals occurring in linear minimum-time problems.

Singular Intervals in Linear Time-Optimal Problems

Consider the minimum-time transfer of the linear, stationary system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{5.6-13}$$

from an arbitrary initial state $\mathbf{x}_0$ at $t = 0$ to some target set $S(t)$. For simplicity we shall assume that the control is a scalar. The admissible controls satisfy the inequality

$$|u(t)| \leq 1.0. \tag{5.6-14}$$

Let us attempt to find conditions that are necessary for the existence of a singular interval.

The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + \mathbf{p}^T(t)\mathbf{A}\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{b}u(t), \tag{5.6-15}$$

and from the minimum principle we know that if an optimal control u^* exists it must satisfy

$$1 + \mathbf{p}^{*T}(t)\mathbf{A}\mathbf{x}^*(t) + \mathbf{p}^{*T}(t)\mathbf{b}u^*(t) \leq 1 + \mathbf{p}^{*T}(t)\mathbf{A}\mathbf{x}^*(t) + \mathbf{p}^{*T}(t)\mathbf{b}u(t) \quad (5.6\text{-}16)$$

for all $t \in [0, t_f]$ and for all admissible $u(t)$. Since the final time is free and $\mathscr{H}$ does not contain t explicitly, we also know that

$$\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) = 1 + \mathbf{p}^{*T}(t)\,\mathbf{A}\mathbf{x}^*(t) + \mathbf{p}^{*T}(t)\,\mathbf{b}u^*(t) = 0 \quad (5.6\text{-}17)$$

for all $t \in [0, t_f]$. From (5.6-16) we observe that $[t_1, t_2]$ is a singular interval if

$$\mathbf{p}^{*T}(t)\,\mathbf{b} = 0 \qquad \text{for all } t \in [t_1, t_2]. \quad (5.6\text{-}18)$$

Clearly, this condition occurs *if* $\mathbf{p}^*(t) = \mathbf{0}$ for $t \in [t_1, t_2]$. But this cannot happen, because substituting $\mathbf{p}^*(t) = \mathbf{0}$ in Eq. (5.6-17) leads to the contradiction $1 = 0$; therefore,

$$\mathbf{p}^*(t) \neq \mathbf{0} \qquad \textit{for any } t \in [0, t_f]. \quad (5.6\text{-}19)$$

Equation (5.6-18) is also satisfied (for all t) if

$$\mathbf{b} = \mathbf{0}, \quad (5.6\text{-}20)$$

but this indicates that the control does not affect the system at all; we might say that the system is "completely uncontrollable." This is our first hint that perhaps controllability has something to do with the existence of singular intervals.

Having ruled out $\mathbf{p}^*(t) = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$ as possibilities, let us consider the remaining alternative, namely that the product $\mathbf{p}^{*T}(t)\,\mathbf{b} = 0$ for $t \in [t_1, t_2]$. If $\mathbf{p}^{*T}(t)\,\mathbf{b}$ is to be zero for a finite time interval, this implies that derivatives of all orders of $\mathbf{p}^{*T}(t)\,\mathbf{b}$ are zero during this interval; that is,

$$\begin{aligned} &\mathbf{p}^{*T}(t)\mathbf{b} = 0 \\ &\frac{d^k}{dt^k}[\mathbf{p}^{*T}(t)\mathbf{b}] = 0, \qquad k = 1, 2, \ldots \end{aligned} \quad (5.6\text{-}21)$$

Since $\mathbf{b}$ is an $n \times 1$ matrix of constants,

$$\begin{aligned} \frac{d^k}{dt^k}[\mathbf{p}^{*T}(t)\mathbf{b}] &= \frac{d^k}{dt^k}[\mathbf{p}^{*T}(t)]\mathbf{b} \\ &\triangleq \overset{(k)}{\mathbf{p}}{}^{*T}(t)\mathbf{b}. \end{aligned} \quad (5.6\text{-}22)$$

From the Hamiltonian the costate equation is

$$\dot{\mathbf{p}}^*(t) = -\mathbf{A}^T\mathbf{p}^*(t); \quad (5.6\text{-}23)$$

hence the costate solution is

$$\mathbf{p}^*(t) = \epsilon^{-\mathbf{A}^T t}\mathbf{c}, \quad (5.6\text{-}24)$$

where $\mathbf{c}$ is the vector of initial costate values.

Let us write out a few of the derivatives in Eq. (5.6-21); we have

$$\begin{aligned} &\mathbf{p}^{*T}(t)\mathbf{b} = 0 \\ &\dot{\mathbf{p}}^{*T}(t)\mathbf{b} = 0 \\ &\ddot{\mathbf{p}}^{*T}(t)\mathbf{b} = 0 \\ &\quad\vdots \\ &\overset{(k)}{\mathbf{p}}{}^{*T}(t)\mathbf{b} = 0 \qquad \text{for } t \in [t_1, t_2]. \end{aligned} \quad (5.6\text{-}25)$$

Now, using Eqs. (5.6-23) and (5.6-24), we have

$$\dot{\mathbf{p}}^{*T}(t)\mathbf{b} = -[\mathbf{A}^T\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{b} = 0. \quad (5.6\text{-}26)$$

By applying the matrix identity

$$[\mathbf{M}_1\mathbf{M}_2]^T = \mathbf{M}_2^T\mathbf{M}_1^T\dagger \quad (5.6\text{-}27)$$

Eq. (5.6-26) becomes

$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}\mathbf{b} = 0. \quad (5.6\text{-}28)$$

Similarly, differentiating Eq. (5.6-23) gives

$$\ddot{\mathbf{p}}^*(t) = -\mathbf{A}^T\dot{\mathbf{p}}^*(t), \quad (5.6\text{-}29)$$

so

$$\ddot{\mathbf{p}}^{*T}(t)\mathbf{b} = [[-\mathbf{A}^T][-\mathbf{A}^T]\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{b} = 0. \quad (5.6\text{-}30)$$

Using (5.6-27) twice on the term in brackets gives

$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}^2\mathbf{b} = 0. \quad (5.6\text{-}31)$$

The pattern is now clear; continuing to write out the terms of Eq. (5.6-21), using Eqs. (5.6-23), (5.6-24), and (5.6-27), we obtain for the kth derivative

$$\overset{(k)}{\mathbf{p}}{}^{*T}(t)\mathbf{b} = [-1]^k[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}^k\mathbf{b} = 0, \qquad k = 0, 1, 2, \ldots. \quad (5.6\text{-}32)$$

Cancelling the minus signs, we find that the first n equations are‡

† See Appendix 1.
‡ Recall that n is the order of the system.

$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{b} = 0$$
$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}\mathbf{b} = 0$$
$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}^2\mathbf{b} = 0$$
$$\vdots$$
$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\mathbf{A}^{n-1}\mathbf{b} = 0, \qquad (5.6\text{-}33)$$

or, written together,

$$[\epsilon^{-\mathbf{A}^T t}\mathbf{c}]^T\left[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \mathbf{A}^2\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}\right] = \mathbf{0}^T. \qquad (5.6\text{-}33a)$$

Taking the transpose of both sides and again using Eq. (5.6-27), we find that this becomes

$$\left[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \mathbf{A}^2\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}\right]^T \epsilon^{-\mathbf{A}^T t}\mathbf{c} = \mathbf{0}. \qquad (5.6\text{-}34)$$

But

$$\epsilon^{-\mathbf{A}^T t}\mathbf{c} = \mathbf{p}^*(t), \qquad (5.6\text{-}24)$$

and we have already shown [see Eq. (5.6-19)] that $\mathbf{p}^*(t) \neq \mathbf{0}$ for any $t \in [0, t_f]$; therefore, if Eq. (5.6-34) is to be satisfied, the matrix

$$\mathbf{E} \triangleq \left[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \mathbf{A}^2\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}\right]$$

must be singular. From Section 1.2 we know that the matrix **E** is nonsingular if and only if the system (5.6-13) is completely controllable.

To summarize, we have found that in linear, stationary, minimum-time problems:

1. For a singular interval to exist, it is necessary that the system be uncontrollable.
2. Conversely, if the system is completely controllable, a singular interval cannot exist.

It can also be shown that if **E** is singular, a singular interval must exist.

In conclusion, the problem of transferring the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \qquad (5.6\text{-}13)$$

from an arbitrary initial state $\mathbf{x}_0$ to a specified target set in minimum time has a singular interval if and only if the system (5.6-13) is *not completely controllable*. This necessary and sufficient condition for the existence of an interval of singularity can also be extended to the situation where the system has several inputs (see Problem 5-39).

Singular Intervals in Linear Fuel-Optimal Problems

Let us now investigate minimum-fuel systems to see whether or not singular intervals can exist. We begin by considering the fuel-optimal control of the system in Example 5.6-1.

Example 5.6-2. Determine whether the problem of transferring the system

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t) \qquad (5.6\text{-}35)$$

from an arbitrary initial state $\mathbf{x}_0$ to a specified target set $S(t)$ with minimum fuel expenditure has any singular intervals. The final time is free.

The Hamiltonian is given by

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = |u(t)| + p_1(t)x_2(t) + p_2(t)u(t) \qquad (5.6\text{-}36)$$

and from the minimum principle,

$$|u^*(t)| + p_1^*(t)x_2^*(t) + p_2^*(t)u^*(t) \leq |u(t)| + p_1^*(t)x_2^*(t) + p_2^*(t)u(t). \qquad (5.6\text{-}37)$$

It is also necessary that on an extremal $\mathscr{H} \equiv 0$, so

$$|u^*(t)| + p_1^*(t)x_2^*(t) + p_2^*(t)u^*(t) = 0. \qquad (5.6\text{-}38)$$

If $[t_1, t_2]$ is a singular interval, Eq. (5.6-37) indicates that either

$$p_2^*(t) = +1.0 \qquad \text{for all } t \in [t_1, t_2] \qquad (5.6\text{-}39a)$$

or

$$p_2^*(t) = -1.0 \qquad \text{for all } t \in [t_1, t_2]. \qquad (5.6\text{-}39b)$$

In either case, if (5.6-37) is satisfied, Eq. (5.6-38) reduces to

$$p_1^*(t)x_2^*(t) = 0 \qquad \text{for all } t \in [t_1, t_2]. \qquad (5.6\text{-}40)$$

The costate solution, found earlier, is

$$p_1^*(t) = c_1$$
$$p_2^*(t) = -c_1 t + c_2. \qquad (5.6\text{-}9)$$

In order that $p_2^*(t) = \pm 1$ for a finite time interval, c_1 must equal zero, and c_2 must equal ± 1.0. If $c_1 = 0$, $p_1^*(t) = 0$ for all t and Eq. (5.6-40) will be satisfied. From this analysis, we have determined that a singular interval can exist, even though this system is completely controllable. Notice that if a singular interval occurs it will persist for all $t \in [0, t_f]$; thus, if the optimal control is singular at all, it is singular throughout the interval of operation of the system.

It is left as an exercise for the reader (Problem 5-36) to show that in this problem the existence of a singular interval signals the non-uniqueness of optimal controls for certain initial states and the non-existence of optimal controls for the rest of the initial states.

Let us now consider linear fuel-optimal systems in more generality. We shall assume that the system has one control input and is described by state equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t). \tag{5.6-41}$$

The admissible controls must satisfy

$$|u(t)| \le 1.0. \tag{5.6-42}$$

The system is to be transferred from an arbitrary initial state $\mathbf{x}_0$ to a specified target set $S(t)$ by a control that minimizes the performance measure

$$J(u) = \int_0^{t_f} |u(t)|\, dt \tag{5.6-43}$$

with t_f free. The Hamiltonian is

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = |u(t)| + \mathbf{p}^T(t)\mathbf{A}\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{b}u(t). \tag{5.6-44}$$

From the minimum principle

$$\begin{aligned} |u^*(t)| + \mathbf{p}^{*T}(t)\mathbf{A}\mathbf{x}^*(t) + \mathbf{p}^{*T}(t)\,\mathbf{b}u^*(t) &\le |u(t)| \\ + \mathbf{p}^{*T}(t)\mathbf{A}\mathbf{x}^*(t) &+ \mathbf{p}^{*T}(t)\,\mathbf{b}u(t). \end{aligned} \tag{5.6-45}$$

From Eq. (5.6-45), we see that for a singular interval to exist it is necessary that either

$$\mathbf{p}^{*T}(t)\,\mathbf{b} = +1.0 \qquad \text{for all } t \in [t_1, t_2] \tag{5.6-46a}$$

or

$$\mathbf{p}^{*T}(t)\,\mathbf{b} = -1.0 \qquad \text{for all } t \in [t_1, t_2]. \tag{5.6-46b}$$

If $u^*(t)$ minimizes the Hamiltonian, and either (5.6-46a) or (5.6-46b) is satisfied, then since $\mathscr{H}$ must be identically zero,

$$\mathbf{p}^{*T}(t)\,\mathbf{A}\mathbf{x}^*(t) = 0. \tag{5.6-47}$$

If $\mathbf{p}^{*T}(t)\,\mathbf{b}$ is to be either $+1$ or -1 during the entire time interval $[t_1, t_2]$, then this implies that

$$\frac{d^k}{dt^k}[\mathbf{p}^{*T}(t)\mathbf{b}] = 0, \qquad k = 1, 2, \ldots, \qquad t \in [t_1, t_2]. \tag{5.6-48}$$

Again using the necessary condition that $\mathscr{H}(\mathbf{x}^*(t), u^*(t), \mathbf{p}^*(t)) \equiv 0$, and following the same procedure as for minimum-time problems, we eventually obtain

$$\left[\epsilon^{-\mathbf{A}^T t}\mathbf{c}\right]^T \left[\mathbf{A}\mathbf{b} \,\vdots\, \mathbf{A}^2\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^n\mathbf{b}\right] = \mathbf{0}^T \tag{5.6-49}$$

[compare this with Eq. (5.6-33a)]. This equation can also be written as

$$\left[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}\right]^T \mathbf{A}^T \epsilon^{-\mathbf{A}^T t}\mathbf{c} = \mathbf{0}. \tag{5.6-50}$$

But

$$\epsilon^{-\mathbf{A}^T t}\mathbf{c} = \mathbf{p}^*(t) \ne \mathbf{0} \qquad \text{for } t \in [t_1, t_2] \tag{5.6-51}$$

because if $\mathbf{p}^*(t) = \mathbf{0}$, this would imply that $\mathbf{p}^{*T}(t)\,\mathbf{b} = 0$, which contradicts Eq. (5.6-46). Thus, if Eq. (5.6-50) is to be satisfied the matrix

$$\left[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}\right]^T \mathbf{A}^T$$

must be singular. For this matrix to be singular either $\mathbf{A}$ or $[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}]$, or both must be singular.† Notice that even if the system is completely controllable, in which case $[\mathbf{b} \,\vdots\, \mathbf{A}\mathbf{b} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{b}]$ is nonsingular, an interval of singularity can still occur if the matrix $\mathbf{A}$ is singular. Thus a *necessary condition* for a singular interval to exist is that either the system (5.6-41) is not completely controllable, or $\mathbf{A}$ is singular.

Necessary Conditions for Singular Intervals

So far, we have concentrated on one aspect of singular solutions—necessary conditions for their existence. We have considered only linear, fixed, single-input systems, but the procedure followed applies as well to systems

† Because determinant $[\mathbf{M}_1\mathbf{M}_2]$ = determinant $\mathbf{M}_1 \cdot$ determinant $\mathbf{M}_2$ if $\mathbf{M}_1$ and $\mathbf{M}_2$ are square matrices; hence, determinant $[\mathbf{M}_1\mathbf{M}_2] = 0$ implies determinant $\mathbf{M}_1 = 0$, or determinant $\mathbf{M}_2 = 0$, or both.

that have several inputs or are nonlinear. The idea is quite straightforward: examine the Hamiltonian to determine whether there are situations in which the minimum principle does not yield sufficient information to determine the relationship between $\mathbf{u}^*(t)$, $\mathbf{x}^*(t)$, and $\mathbf{p}^*(t)$. If this situation occurs, use the fact that the Hamiltonian must be zero† (and that $\dot{\mathscr{H}}, \ddot{\mathscr{H}}, \ldots$ equal zero) to determine other necessary conditions for the existence of singular intervals.

Effects of Singular Intervals on Problem Solution

Let us now consider an example that illustrates another facet of singular problems—the effects of singular intervals on problem solution.

Example 5.6-3. Find the control law that causes the response of the system

$$\dot{x}_1(t) = x_2(t) \tag{5.6-52a}$$
$$\dot{x}_2(t) = u(t) \tag{5.6-52b}$$

to minimize the performance measure

$$J = \tfrac{1}{2}\int_0^{t_f} [x_1^2(t) + x_2^2(t)]\, dt. \tag{5.6-53}$$

The final time t_f and the final states are free, and the controls are constrained by the inequality

$$|u(t)| \le 1.0. \tag{5.6-54}$$

The Hamiltonian is given by

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}x_1^2(t) + \tfrac{1}{2}x_2^2(t) + p_1(t)x_2(t) + p_2(t)u(t). \tag{5.6-55}$$

From the minimum principle and (5.6-55)

$$p_2^*(t)u^*(t) \le p_2^*(t)u(t) \tag{5.6-56}$$

for all admissible $u(t)$ and for all $t \in [0, t_f]$. For $p_2^*(t) \neq 0$, Eq. (5.6-56) indicates that

$$u^*(t) = \begin{cases} -1.0, & \text{for } p_2^*(t) > 0 \\ +1.0, & \text{for } p_2^*(t) < 0. \end{cases} \tag{5.6-57}$$

Switchings of the optimal control occur at isolated instants when $p_2^*(t) = 0$. On the other hand, if there is a time interval $[t_1, t_2]$ during which

$$p_2^*(t) = 0 \qquad \text{for all } t \in [t_1, t_2], \tag{5.6-58}$$

† We assume free final time and $\mathscr{H}$ explicitly independent of time.

then $[t_1, t_2]$ is a singular interval; let us investigate this possibility.

Since the final time is free, and time does not appear explicitly in the Hamiltonian, it is necessary that

$$\tfrac{1}{2}x_1^{*2}(t) + \tfrac{1}{2}x_2^{*2}(t) + p_1^*(t)x_2^*(t) + p_2^*(t)u^*(t) = 0 \tag{5.6-59}$$

for $t \in [0, t_f]$. If $p_2^*(t) = 0$ for $t \in [t_1, t_2]$, then

$$p_2^*(t) = \dot{p}_2^*(t) = \ddot{p}_2^*(t) = \cdots = 0,\ t \in [t_1, t_2]. \tag{5.6-60}$$

In addition, from Eq. (5.6-59) we have

$$M \triangleq \tfrac{1}{2}x_1^{*2}(t) + \tfrac{1}{2}x_2^{*2}(t) + p_1^*(t)x_2^*(t) = 0 \tag{5.6-61}$$

for $t \in [t_1, t_2]$, and hence

$$M = \dot{M} = \ddot{M} = \cdots = 0,\ t \in [t_1, t_2], \tag{5.6-62}$$

if a singular interval is to exist.

The costate equations are

$$\dot{p}_1^*(t) = -x_1^*(t) \tag{5.6-63}$$
$$\dot{p}_2^*(t) = -x_2^*(t) - p_1^*(t). \tag{5.6-64}$$

During a singular interval, using Eqs. (5.6-60) and (5.6-64), we obtain

$$p_1^*(t) = -x_2^*(t). \tag{5.6-65}$$

Substituting this in (5.6-61) yields

$$x_1^{*2}(t) - x_2^{*2}(t) = 0 \tag{5.6-66}$$

or

$$[x_1^*(t) + x_2^*(t)][x_1^*(t) - x_2^*(t)] = 0, \qquad \text{for } t \in [t_1, t_2]. \tag{5.6-66a}$$

Equation (5.6-66) is satisfied if

$$x_1^*(t) + x_2^*(t) = 0 \tag{5.6-67a}$$

or if

$$x_1^*(t) - x_2^*(t) = 0, \qquad \text{for } t \in [t_1, t_2]. \tag{5.6-67b}$$

By differentiating Eq. (5.6-67a) and substituting in the state equation (5.6-52a) we find that

$$\dot{x}_1^*(t) = -\dot{x}_2^*(t) = x_2^*(t), \tag{5.6-68}$$

which with (5.6-52b) implies

$$u^*(t) = -x_2^*(t), \qquad \text{for } t \in [t_1, t_2]. \tag{5.6-69a}$$

Similarly, differentiating Eq. (5.6-67b) and substituting in the state equations, we obtain

$$u^*(t) = +x_2^*(t), \qquad \text{for } t \in [t_1, t_2]. \tag{5.6-69b}$$

Equations (5.6-67) define a locus of points in the state plane where singular controls may exist, and Eq. (5.6-69) gives an explicit expression for the singular control law. The singular lines, truncated at $|x_2(t)| = 1$, because $|u(t)| \leq 1$, are shown in Fig. 5-40. The arrows indicate the direction of increasing time.

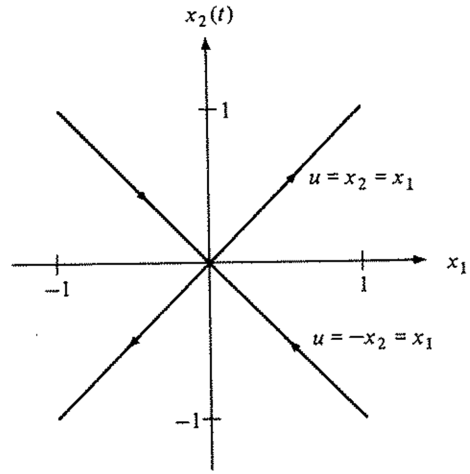


Figure 5-40 The singular lines for Example 5.6-3

We have determined two lines in the state plane where the control, states, and costates all satisfy the necessary conditions given by the minimum principle and the requirement that $\mathscr{H} \equiv 0$ on an extremal trajectory. Clearly, since the system moves away from the origin on the line $x_1 = x_2$, this segment cannot be part of an optimal trajectory. We still must determine the optimal control law for states not on the singular line, and also if the singular control law is optimal. Let us investigate some of the possibilities.

Suppose that at $t = 0$ the system is at state $\mathbf{x}_0$ shown in Fig. 5-41. The optimal control must be ± 1, because the system is not on the singular line. By examining the trajectories for this system with $u = \pm 1$, shown in Fig. 5-20, Section 5.4, it is clear that the optimal control should initially be $u^* = -1$. With this control the system trajectory is as shown in Fig. 5-41. We next ask the question: what happens when the trajectory intersects the singular line? Is the optimal control the one that keeps the system on the singular line, or should the control continue to be $u = -1$

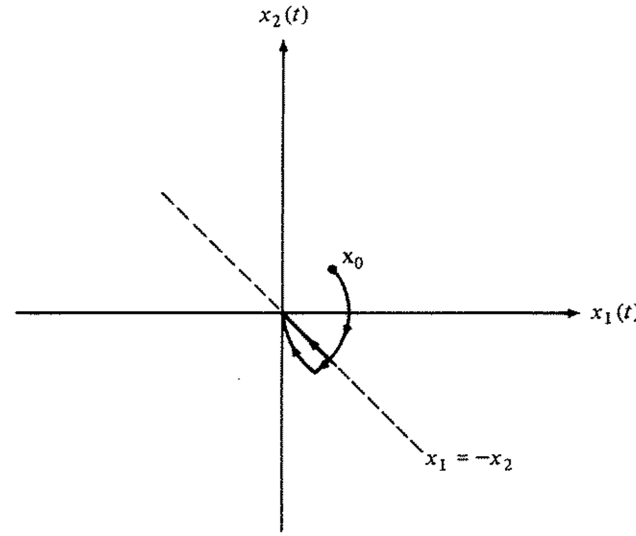


Figure 5-41 Optimal trajectory candidates for Example 5.6-3

until intersecting the curve from which the origin is reached by applying $u = +1$? To answer this question, consider what happens when a control switching is indicated. If u^* switches from $+1$ to -1 at some time t_1, then it follows that

$$\begin{aligned} p_2^*(t_1) &= 0 \\ \dot{p}_2^*(t_1) &> 0, \end{aligned} \tag{5.6-70}$$

or, if u^* switches from -1 to $+1$ at time t_1, then this implies

$$\begin{aligned} p_2^*(t_1) &= 0 \\ \dot{p}_2^*(t_1) &< 0. \end{aligned} \tag{5.6-71}$$

Now, since $\mathscr{H} \equiv 0$, $p_2^*(t_1) = 0$ implies that

$$p_1^*(t_1) = \frac{-\frac{1}{2}x_1^{*2}(t_1) - \frac{1}{2}x_2^{*2}(t_1)}{x_2^*(t_1)}. \tag{5.6-72}$$

Substituting this expression into the costate equation (5.6-64) gives

$$\dot{p}_2^*(t_1) = \frac{\frac{1}{2}[x_1^*(t_1) + x_2^*(t_1)][x_1^*(t_1) - x_2^*(t_1)]}{x_2^*(t_1)}. \tag{5.6-73}$$

By determining the sign of $\dot{p}_2^*(t_1)$ indicated by Eq. (5.6-73) for various regions in the state plane, we then know the allowable switchings that may occur. Table 5-3 shows how the sign of $\dot{p}_2^*(t_1)$ is determined for the regions of the state plane, and Fig. 5-42 illustrates these regions and the allowable switchings.

Table 5-3 DETERMINATION OF ALLOWABLE SWITCHINGS FOR REGIONS OF THE STATE PLANE

Region	Sign of $x_1(t_1)$	Sign of $x_2(t_1)$	Sign of $x_1(t_1)+x_2(t_1)$	Sign of $x_1(t_1)-x_2(t_1)$	Sign of $\dot{p}_2(t_1)$
R_1	+	+	+	−	−
R_2	+	+	+	+	+
R_3	+	−	+	+	−
R_4	+	−	−	+	+
R_5	−	−	−	+	+
R_6	−	−	−	−	−
R_7	−	+	−	−	+
R_8	−	+	+	−	−

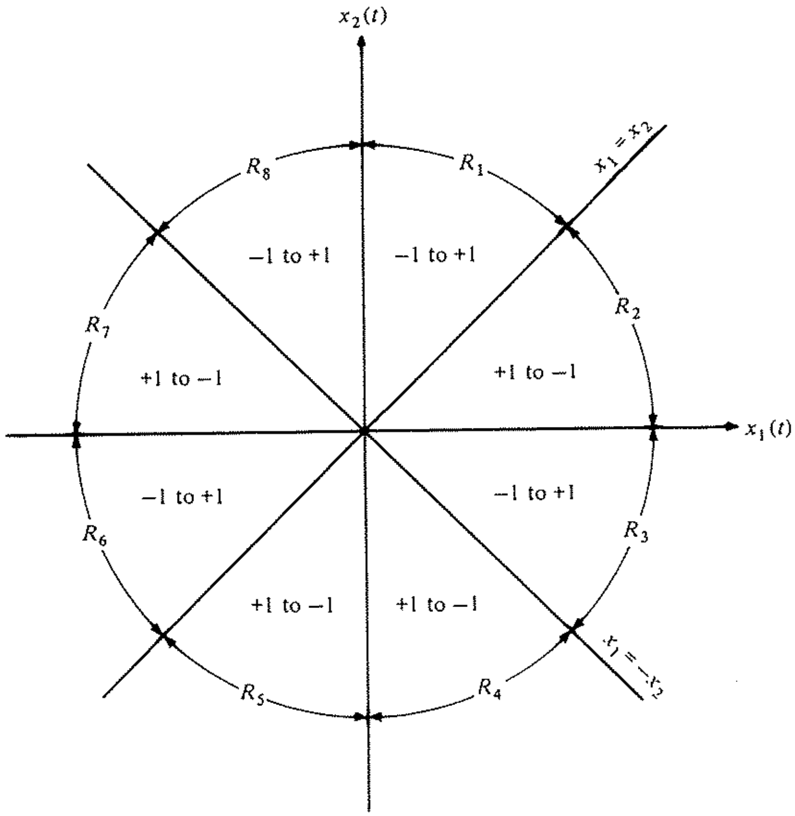


Figure 5-42 Allowable switchings in various regions of the state plane

Referring to Fig. 5-42, we see that if the trajectory in Fig. 5-41 is allowed to cross the singular line it is then in a region where switching from $u = -1$ to $u = +1$ violates the necessary condition that $\mathscr{H} \equiv 0$. We conclude then that the optimal trajectory beginning at this value of $\mathbf{x}_0$ must have its terminal segment on the singular line.

When an initial trajectory segment with $u^* = \pm 1$ does not intersect the singular line with $|x_2(t)| \leq 1$, then the optimal control will switch to $u^* = \mp 1$ and the optimal trajectory will eventually reach either the origin or the singular line. To determine where the switching occurs, let t_2 be the time when the trajectory reaches the singular line or the origin. Notice that the origin lies on the singular line, and from (5.6-60)

$$p_2(t_2) = 0. \tag{5.6-74}$$

Solving for the value of $p_1(t_2)$ on the line $x_1(t_2) = -x_2(t_2)$, which satisfies Eq. (5.6-59), gives

$$p_1(t_2) = -x_2(t_2). \tag{5.6-75}$$

Using the values of the costates given by (5.6-74) and (5.6-75) as initial conditions, and integrating the state and costate equations backward in time with $u = \pm 1$, we can determine the locations in the state plane where $p_2(t)$ again passes through zero. Doing this for several values of $x_2(t_2)$ (including zero) on the singular line, we obtain a locus of points that defines the switching curve *C-D-*0*-E-F* shown in Fig. 5-43. The optimal control law is given by

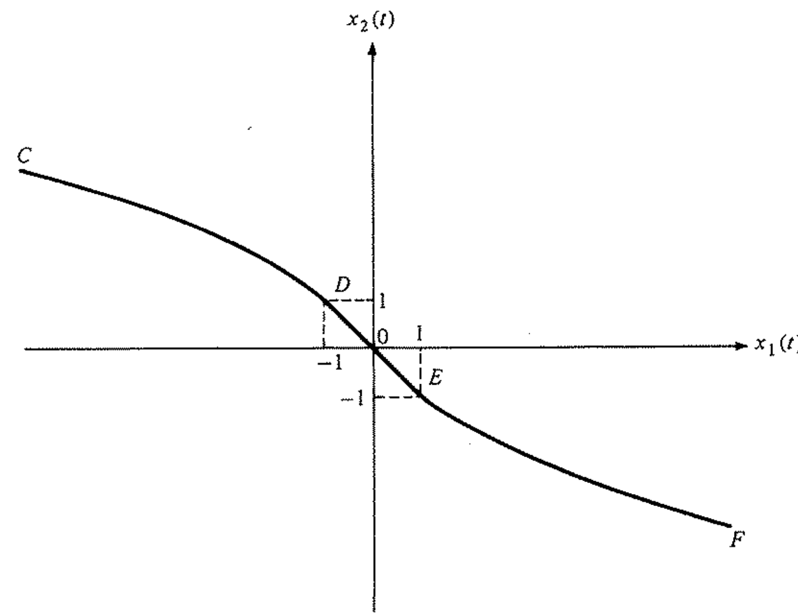


Figure 5-43 The optimal switching curve for Example 5.6-3

$$u^*(t) = \begin{cases} -1, & \text{for } \mathbf{x}(t) \text{ to the right of } C\text{-}0\text{-}F \\ +1, & \text{for } \mathbf{x}(t) \text{ to the left of } C\text{-}0\text{-}F \\ -1, & \text{for } \mathbf{x}(t) \text{ on segment } C\text{-}D \\ +1, & \text{for } \mathbf{x}(t) \text{ on segment } E\text{-}F \\ -x_2(t), & \text{for } \mathbf{x}(t) \text{ on segment } D\text{-}0\text{-}E. \end{cases} \tag{5.6-76}$$

Several optimal trajectories are pictured in Fig. 5-44; notice that *the switching curve is not a trajectory* except on the singular line D-0-E. As further illustration of this point, Fig. 5-45 shows the optimal switching curve, the curve $x_1 = \frac{1}{2}x_2^2$, which is the switching curve for bang-bang operation, the curve $x_1 = \frac{1}{2}x_2^2 + \frac{1}{2}$, which is the $u = +1$ trajectory that intersects the singular line at the point $(1, -1)$, and the line $x_1 = -x_2$. Observe that the optimal switching curve is above the line $x_1 = -x_2$ for all positive values of x_1; therefore, the switchings that occur on segment E-F do not violate the allowable switchings indicated in Fig. 5-42. Similarly, it can be verified that segment C-D of the switching curve lies entirely in region R_7 of the state plane, and so does not cause the allowable switching conditions to be violated.

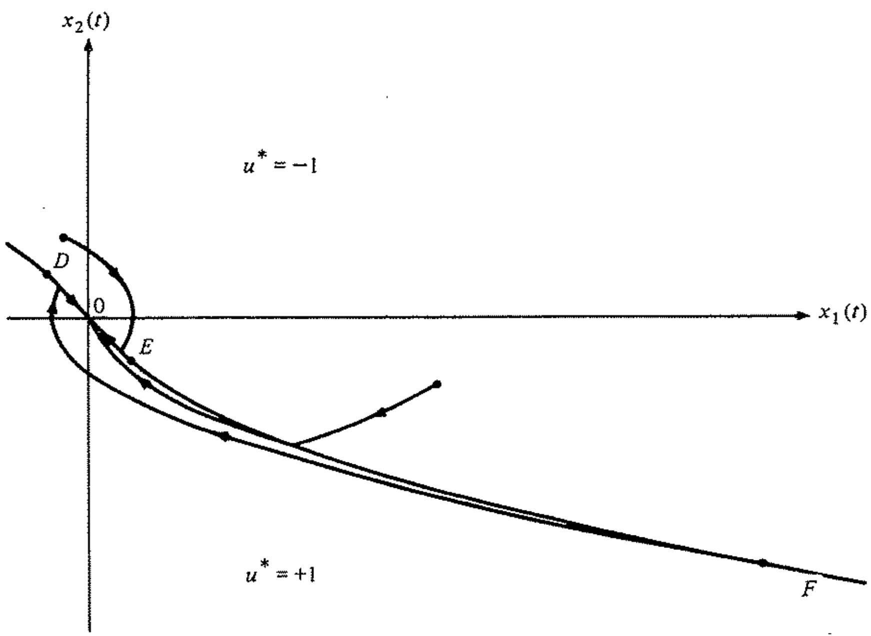


Figure 5-44 Some optimal trajectories for Example 5.6-3

Summary

The existence of singular intervals, although complicating the solution of optimal control problems, may turn out to be helpful in other respects.

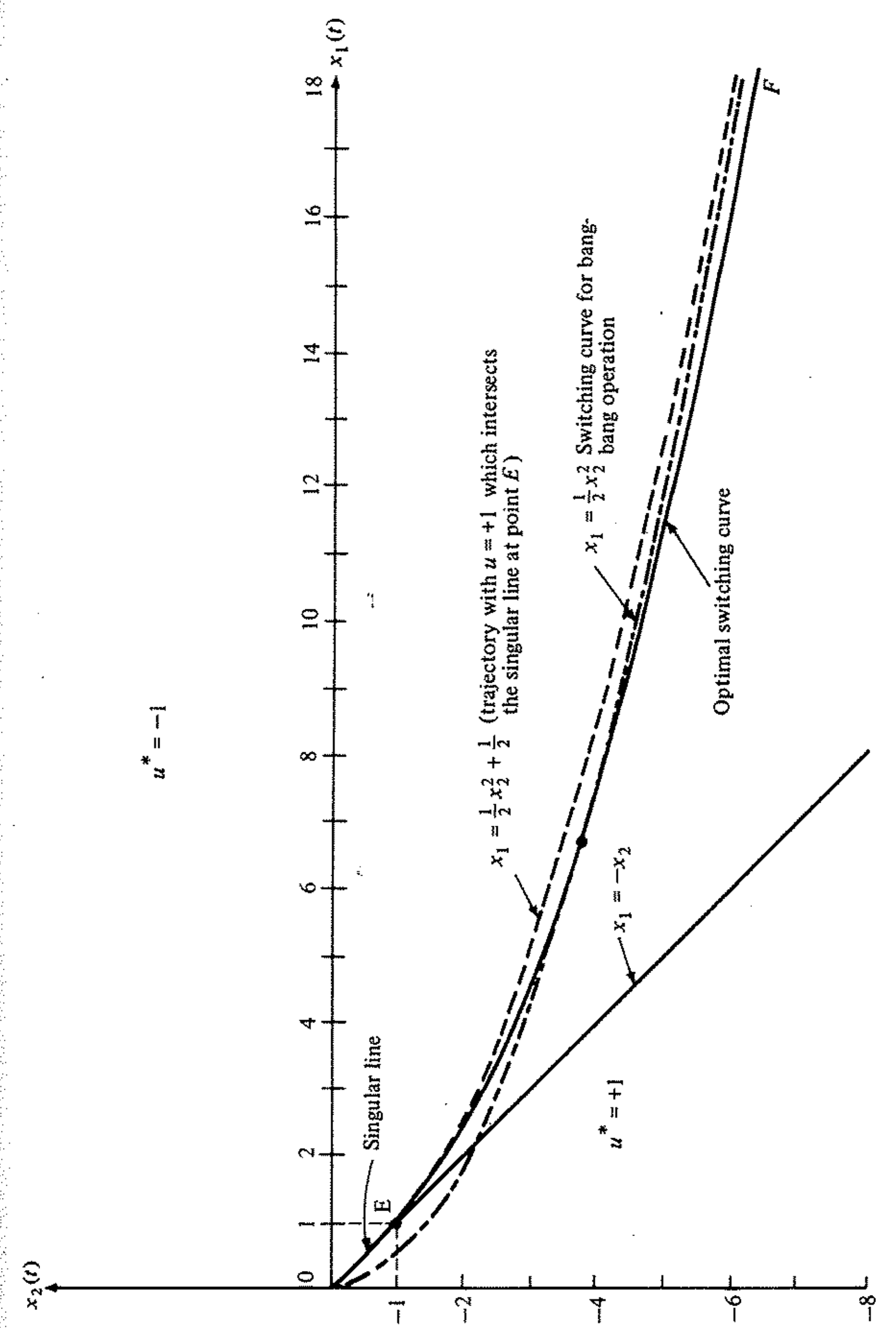


Figure 5-45 The optimal switching curve and three suboptimal alternatives

For example, a singular interval may indicate that an optimal control is nonunique; in this case we can select the optimal control which is easiest to implement, or which has other desirable features.

Our discussion has emphasized the following aspects of singular problems:

1. The determination of necessary conditions for the existence of singular intervals.
2. The use of these necessary conditions to find the regions in the state space where a singular control law exists.
3. The investigation of the singular control law to ascertain whether or not it is optimal.

The reader interested in additional material on singular intervals should refer to [A-2], [A-3], [J-1], [J-2], [R-2], [R-3], and [S-4].

5.7 SUMMARY AND CONCLUSIONS

In this chapter we have discussed the application of variational techniques to optimal control problems. The calculus of variations was used to derive a set of necessary conditions that must be satisfied by an optimal control and its associated state-costate trajectory. These necessary conditions for optimality lead to a (generally nonlinear) two-point boundary-value problem that must be solved to determine an explicit expression for the optimal control. In linear regulator problems, the resulting two-point boundary-value problem is linear and can be solved to obtain a linear, time-varying optimal control law.

Motivated by an interest in problems with bounded control or state variables, we then gave a heuristic derivation of Pontryagin's minimum principle and discussed a technique for dealing with state inequality constraints. The remainder of the chapter was concerned with applications of Pontryagin's minimum principle to problems with bounded admissible controls. Several examples of minimum-time and minimum-control-effort systems were discussed. These examples were elementary, but nonetheless indicative of procedures that are useful in obtaining optimal control laws. Finally, we investigated the occurrence of singular intervals during which the minimum principle fails to yield a relationship for the extremal control in terms of the extremal state-costate trajectory.

This chapter was not intended to be a handbook of solutions to optimal control problems. Indeed, the difficulties encountered should make the reader aware that no such handbook exists. We may regard the linear regulator problem as being solved; however, in the sections on minimum-time and minimum-control-effort problems we found that analytical solutions are generally impossible for higher-order systems ($n \geq 3$) even if the systems are linear and time-invariant. For nonlinear systems it is even more difficult to obtain closed-form expressions for the optimal control laws.

Realistically, then, we must view the minimum principle as a starting point for obtaining numerical solutions to optimal control problems. From the minimum principle we obtain knowledge of the *form* of the optimal control (if it exists) and a statement of the two-point boundary-value problem, which, when solved, yields an explicit relationship for the optimal control.

REFERENCES

A-2 Athans, M., and P. L. Falb, *Optimal Control: An Introduction to the Theory and Its Applications*. New York: McGraw-Hill Book Company, 1966.

A-3 Athans, M., "On the Uniqueness of the Extremal Controls for a Class of Minimum Fuel Problems," *IEEE Trans. Automatic Control* (1966), 660–668.

J-1 Johnson, C. D., and J. E. Gibson, "Optimal Control with Quadratic Performance Index and Fixed Terminal Time," *IEEE Trans. Automatic Control* (1964), 355–360.

J-2 Johnson, C. D., and J. E. Gibson, "Singular Solutions in Problems of Optimal Control," *IEEE Trans. Automatic Control* (1963), 4–15.

J-3 Johnson, C. D., "Singular Solutions in Problems of Optimal Control," *Advances in Control Systems: Theory and Applications*, Vol. 2, C. T. Leondes, ed. New York: Academic Press, Inc., 1965.

K-5 Kalman, R. E., "The Theory of Optimal Control and the Calculus of Variations," *Mathematical Optimization Techniques*, R. E. Bellman, ed. Santa Monica, Cal.: The RAND Corporation, 1963.

K-6 Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," *J. SIAM Control*, series A (1963), 152–192.

K-7 Kalman, R. E., "Contributions to the Theory of Optimal Control," *Bol. Soc. Mat. Mex.* (1960), 102–119.

L-3 Leitmann, G., *An Introduction to Optimal Control*. New York: McGraw-Hill Book Company, 1966.

L-4 Leitmann, G., ed., *Optimization Techniques with Applications to Aerospace Systems*. New York: Academic Press, Inc., 1962.

M-2 Meditch, J. S., "On the Problem of Optimal Thrust Programming for a Lunar Soft Landing," *IEEE Trans. Automatic Control* (1964), 477–484.

M-3 Miele, A., "The Calculus of Variations in Applied Aerodynamic and Flight Mechanics," *Optimization Techniques with Applications to Aerospace Systems*, G. Leitmann, ed. New York: Academic Press, Inc., 1962.