d) $|u| \leq 1$

e) $\int_0^{t_f} u^2 \, dt = 1.$

13. Find the Hamilton-Jacobi equation for the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - x_1^2 + u$$

if the performance index is

$$J = \int_0^{t_f} (x_1^2 + u^2) \, dt.$$

14. Show that the solution of the Hamilton-Jacobi equation for the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u}, \qquad \mathbf{A}^T + \mathbf{A} = \mathbf{0}, \qquad \|\mathbf{u}\| \leq 1$$

and the cost function

$$J = \int_0^{t_f} dt = t_f$$

is

$$V(\mathbf{x}) = \|\mathbf{x}\|.$$

What is the optimal control?

15. Find the optimal control to minimize

$$J = \int_0^{t_f} dt$$

for the system

$$\dot{x} = -x + u,$$

when

$$x(0) = 1, \qquad x(t_f) = 0$$

$$|u| \leq 1 + |x|.$$

Optimum systems control examples 5

In this chapter, we will illustrate some, but certainly not all or even most, of the optimal control problems for which closed-form analytic solutions have been obtained. The problems we will solve in this chapter are very important in their own right and illustrate the use of the maximum principle for problems in which closed-form analytic solutions may be obtained. Specifically, we will discuss the linear regulator problem, the first solution of which was due to Kalman [1, 2, 3, 4]. We then discuss the minimum time problem which has been considered by Pontryagin [5], Bellman [6], LaSalle [7], and many others [8 through 13].

A characteristic of some minimum time problems is the possibility of a singular solution. The possibility of singular solutions is well-recognized in the variational calculus literature and has been extensively discussed for control problems by Johnson [14, 15, 16] and others. Minimum fuel problems for linear differential systems are then discussed. A variety of authors, but notably Athans, have discussed various aspects of minimum fuel problems including the possibility of singular solutions [17 through 20]. Although we will not consider the minimum time–fuel–energy control of self–adjoint systems [21] due to its limited practical usefulness, we do note that such systems admit a particularly thorough analysis. For a survey of many other problems plus a lengthy bibliography, we refer to the survey papers of Paiewonsky [22] and Athans [23].

5.1 The linear regulator

We will now study a particular control problem which has as its solution a linear feedback control law. It occurs where we have a linear differential system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{5.1-1}$$

and wish to find the control which minimizes the cost function (for t_f fixed)

$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt. \tag{5.1-2}$$

Clearly, there is no loss of generality in assuming **Q**, **R**, and **S** to be symmetric. We may obtain the solution to this problem via the maximum principle or the Hamilton-Jacobi equation. Here, we will use the former method. The Hamiltonian is

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{u}. \tag{5.1-3}$$

Application of the maximum principle requires that, for an optimum control,

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} = \mathbf{R}(t)\mathbf{u}(t) + \mathbf{B}^T(t)\boldsymbol{\lambda}(t) \tag{5.1-4}$$

and

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \tag{5.1-5}$$

with the terminal condition

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \theta}{\partial \mathbf{x}(t_f)} = \mathbf{S}\mathbf{x}(t_f). \tag{5.1-6}$$

Thus we require that

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t), \tag{5.1-7}$$

and we shall inquire whether we may convert this to a closed-loop control by assuming that the solution for the adjoint is similar to Eq. (5.1-6)

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t). \tag{5.1-8}$$

If we substitute this relation into Eqs. (5.1-1) and (5.1-7), we see that we must require

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t). \tag{5.1-9}$$

Also, from Eqs. (5.1-8) and (5.1-5) we require

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}\mathbf{x}(t) + \mathbf{P}(t)\dot{\mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{P}(t)\mathbf{x}(t). \tag{5.1-10}$$

By combining Eqs. (5.1-9) and (5.1-10) we have

$$[\dot{\mathbf{P}} + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t) + \mathbf{Q}(t)]\mathbf{x}(t) = \mathbf{0}. \tag{5.1-11}$$

Since this must hold for all nonzero $\mathbf{x}(t)$, the term premultiplying $\mathbf{x}(t)$ must be zero. Thus the **P** matrix, which we see is an $n \times n$ symmetric matrix and which has $n(n + 1)/2$ different terms, must satisfy the matrix Riccati equation—which, as we shall see later, must be positive definite—

$$\dot{\mathbf{P}} = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t) - \mathbf{Q}(t) \tag{5.1-12}$$

with a terminal condition given by Eqs. (5.1-6) and (5.1-8)

$$\mathbf{P}(t_f) = \mathbf{S}. \tag{5.1-13}$$

Thus we may solve the matrix Riccati equation backward in time from t_f to t_o, store the matrix

$$\mathbf{K}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t), \tag{5.1-14}$$

and then obtain a closed-loop control from

$$\mathbf{u}(t) = +\mathbf{K}(t)\mathbf{x}(t). \tag{5.1-15}$$

It is important to note that all components of the state vector must be accessible. We will remove this restriction in Chapter 8 when we discuss the ideal observer. A block diagram for accomplishing this solution to the regulator problem is shown in Fig. 5.1-1. If we compute the second variation, we find that

$$\delta^2 J = \tfrac{1}{2}\,\delta\mathbf{x}^T(t_f)\mathbf{S}\,\delta\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\delta\mathbf{x}^T(t)\mathbf{Q}(t)\,\delta\mathbf{x}(t) + \delta\mathbf{u}^T(t)\mathbf{R}(t)\,\delta\mathbf{u}(t)]\, dt. \tag{5.1-16}$$

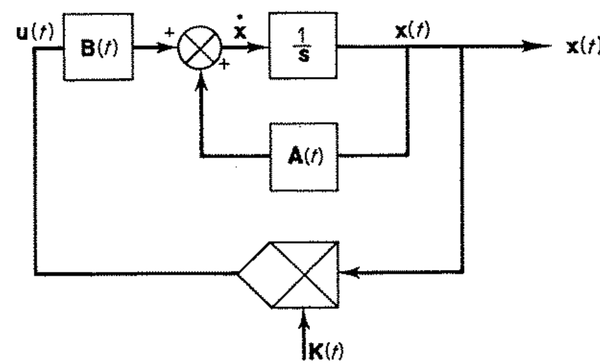

Fig. 5.1-1 Optimum linear closed-loop regulator.

Thus, **Q**, **R**, and **S** must be at least positive semidefinite in order to establish the sufficient condition for a minimum. In addition, we know from Eq. (5.1-7) that **R** must have an inverse;† therefore, it is sufficient that **R** be positive definite and the **Q** and **S** be at least positive semidefinite.

†Approaches that allow this assumption to be relaxed can be found in [24] and [25].

In some cases it may turn out that certain elements of the S matrix are large enough to give computational difficulties. In this case, it is possible and perhaps desirable to obtain an inverse Riccati differential equation; we let

$$\mathbf{P}(t)\mathbf{P}^{-1}(t) = \mathbf{I}, \tag{5.1-17}$$

and, by differentiating, we obtain

$$\dot{\mathbf{P}}\mathbf{P}^{-1}(t) + \mathbf{P}(t)\dot{\mathbf{P}}^{-1} = \mathbf{0} \tag{5.1-18}$$

such that we obtain an "inverse" matrix Riccati equation

$$\dot{\mathbf{P}}^{-1} = \mathbf{A}(t)\mathbf{P}^{-1}(t) + \mathbf{P}^{-1}(t)\mathbf{A}^T(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) + \mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) \tag{5.1-19}$$

with

$$\mathbf{P}^{-1}(t_f) = \mathbf{S}^{-1}. \tag{5.1-20}$$

In this way, for example, it is possible to solve the Riccati equation such that $\mathbf{S}^{-1} = [\mathbf{0}]$, the null matrix, which will require that each and every component of the state vector approach the origin as the time approaches the terminal time. The "gains" $\mathbf{K}(t)$, or at least some components of them, become infinite at the terminal time in this case. It is also necessary to assume certain controllability requirements here, as we shall see in Chapter 7.

It is possible to write the nonlinear $n \times n$ matrix Riccati equation with a terminal condition as a $2n$ vector linear differential equation with two-point boundary conditions. We will use this approach, in part, to solve a Riccati equation associated with a filtering problem in Chapter 8. Our discussion of the second variation method in Chapter 10 will also make use of a Riccati transformation.

Example 5.1-1. Consider the scalar system

$$\dot{x} = -\tfrac{1}{2}x(t) + u(t), \qquad x(t_o) = x_o$$

with the cost function

$$J = \tfrac{1}{2}sx^2(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [2x^2(t) + u^2(t)]\, dt.$$

The Riccati equation, Eq. (5.1-12), becomes

$$p = p + p^2 - 2, \qquad p(t_f) = s$$

which has a solution we may write as either

$$p(t) = -0.5 + 1.5 \tanh(-1.5t + \xi_1)$$

or

$$p(t) = -0.5 + 1.5 \coth(-1.5t + \xi_2)$$

where ξ_1 and ξ_2 are adjusted such that $p(t_f) = s$.

For example, if

(a) $s = 0$, $t_f = 1$, then $\xi_1 = 1.845$ radians, which gives

$$K(t) = -R^{-1}B^TP = 0.5 - 1.5 \tanh(-1.5t + 1.845).$$

Since $s = 0$, we are not particularly weighting the state at the final time, and the "gain" (and control) goes to zero at the final time.

(b) $s = 10$, $t_f = 10$, then $\xi_2 = 15.1425$ radians. In this case we are applying a great weight to the error at $t = t_f$, and the gain becomes large (-10) at the terminal time.

(c) $s = \infty$, the Riccati equation cannot be solved directly since it has an infinite initial condition. The inverse Riccati equation can be solved with zero terminal condition to give

$$K^{-1}(t) = 0.25 + 0.75 \tanh(-1.5t + 1.5t_f - 0.346).$$

As t_f becomes infinite, it is easy to show that $K(t)$ becomes unity and, as is expected, the feedback gain becomes constant. Figure 5.1-2 illustrates $K(t)$, the "Kalman gains" as they are sometimes called, for these three cases for this particular problem.

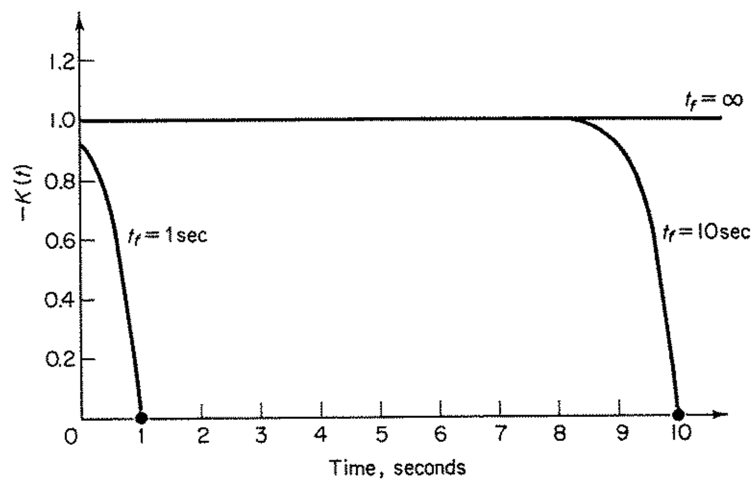

Fig. 5.1-2a (−1) times Kalman gain for controller, $s = 0$.

Example 5.1-2. Let us consider the optimum closed-loop control for a nuclear reactor system. Specifically, we wish to consider a very simple reactor model with zero temperature feedback. Only one group of delayed neutrons will be used.

The reactor kinetics are described by the equations

$$\dot{n} = \frac{(\rho - \beta)n}{\Lambda} + \lambda c, \qquad \dot{c} = \frac{\beta n}{\Lambda} - \lambda c$$

where the neutron density, n, and the precursor concentration, c, are the state variables, and the reactivity ρ is the control variable. The system has the initial conditions $n(0) = n_o$ and $c(0) = c_o$. β, Λ and λ are constants, the average fraction of precursors formed, effective neutron lifetime, and precursor decay constant.

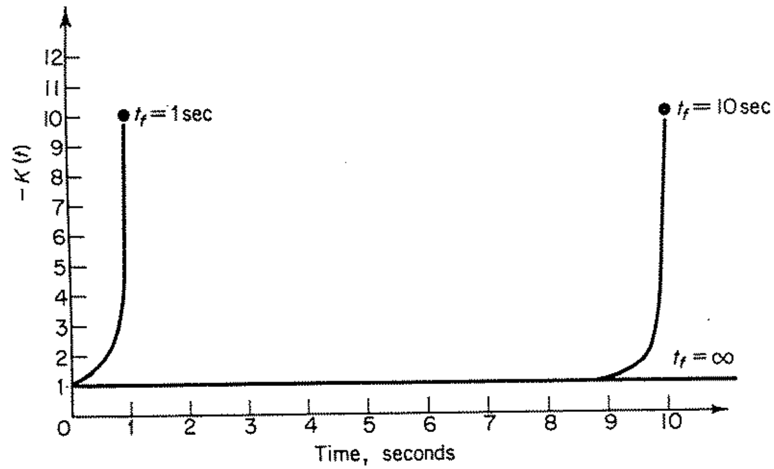

Fig. 5.1-2b (−1) times Kalman gain for controller, $s = 10$.

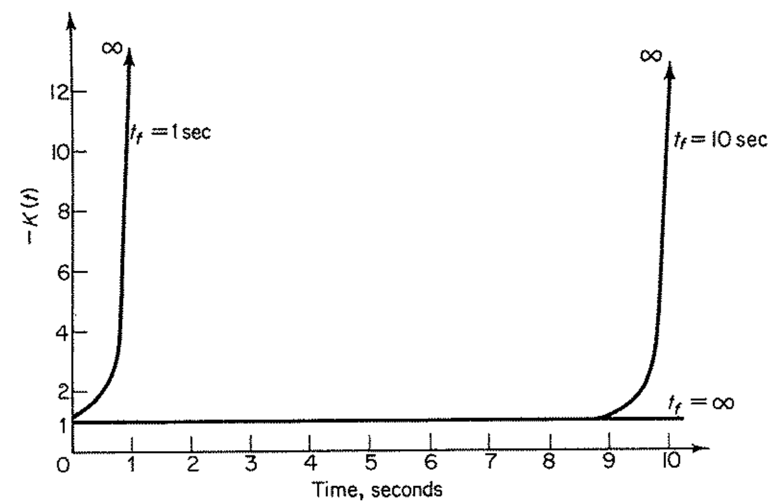

Fig. 5.1-2c (−1) times Kalman gain for controller, $s = \infty$.

The problem is to increase the power from the initial state n_o to a terminal state dn_o, where d is some constant greater than 1.0. The performance index for the system is

$$J_1 = \frac{1}{2}\int_o^{t_f} \dot{\rho}^2 \, dt.$$

The control variable therefore becomes $\dot{\rho}$, and ρ, in effect, thus becomes a state variable. The kinetics equations may then be rewritten as

$$\dot{n} = \frac{(\rho - \beta)n}{\Lambda} + \lambda c$$

$$\dot{c} = \frac{\beta n}{\Lambda} - \lambda c$$

$$\dot{\rho} = u$$

where u is the control variable. Chapter 10 on quasilinearization indicates how the nonlinear two-point boundary value problem resulting from the use of optimal control theory may be used to obtain the optimum control and trajectory, which are shown in Fig. 5.1-3, for the following system parameters

$$\lambda = 0.1 \text{ sec}^{-1} \qquad n_o = 10 \text{ kW}$$
$$d = 5$$
$$\Lambda = 10^{-3} \text{ sec} \qquad \beta = 0.0064$$
$$t_f = 0.5 \text{ sec.}$$

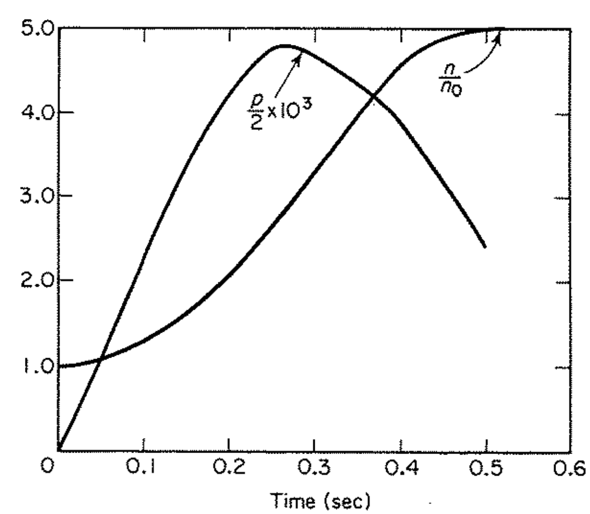

Fig. 5.1-3 Optimal control (reactivity) and trajectory (flux density) for Example (5.1-2).

We will now develop a method of feedback control about the optimal trajectory which minimizes a cost function J_2; it will be quadratic in deviation from the nominal (optimal for J_1) trajectory and control.

Having formulated a model for the nuclear reactor system and determined the optimal trajectories, we now desire to determine the linearized system coefficient matrix about the optimal trajectory. The deviations of the state and control variables about the optimal or nominal trajectories are expressed by

$$n = n_n(t) + \Delta n(t), \qquad c = c_n(t) + \Delta c(t)$$
$$\rho = \rho_n(t) + \Delta\rho(t), \qquad u = u_n(t) + \Delta u(t).$$

The state vector is

$$\Delta \mathbf{x}^T(t) = [\Delta n(t), \Delta c(t), \Delta \rho(t)].$$

The linearized model becomes

$$\Delta \dot{\mathbf{x}} = \begin{bmatrix} a_{11}(t) & \lambda & a_{13}(t) \\ \dfrac{\beta}{\Lambda} & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta u$$

$$= \mathbf{A}(t)\,\Delta \mathbf{x}(t) + \mathbf{b}(t)\,\Delta u(t)$$

where

$$a_{11}(t) = \frac{\rho_n(t) - \beta}{\Lambda}, \qquad a_{13}(t) = \frac{n_n(t)}{\Lambda}.$$

To complete our design of the closed-loop controller, we must evaluate $\mathbf{A}(t)$ and $\mathbf{b}(t)$ about the optimum or nominal trajectories, select the **R**, **Q**, and **S** matrices, and solve the associated Riccati equation. The nominal trajectory, control, and time-varying gains are then stored and used to complete the closed-loop controller design.

The choice of the **R**, **Q**, and **S** matrices to minimize

$$J_2 = \tfrac{1}{2}\,\Delta \mathbf{x}^T(t_f)\mathbf{S}\Delta \mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} [\Delta \mathbf{x}^T(t)\mathbf{Q}(t)\Delta \mathbf{x}(t) + r(t)\Delta u^2(t)]\,dt$$

is somewhat arbitrary and can perhaps best be done here by experimentation. We can accomplish this only after we have obtained a knowledge of possible disturbances which may drive the system off the nominal trajectory. Let us assume that we will use

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10^4 \end{bmatrix}, \qquad \mathbf{S} = \mathbf{0}, \qquad r = 1.$$

In Chapter 10 the second variation and neighboring optimal methods of control-law computation will lead us to a method for choosing the proper weighting matrices for a variety of cases, in particular, for relating J_1 and J_2.

The control, $\Delta u(t)$, is computed from

$$\Delta u(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\Delta \mathbf{x}(t)$$

$$= -[p_{31}(t)\,\Delta n(t) + p_{32}(t)\,\Delta c(t) + p_{33}(t)\,\Delta \rho(t)]$$

where it is necessary to solve the 3×3 matrix Riccati equation, having six different first-order differential equations, to obtain $\mathbf{P}(t)$. Figure 5.1-4 illustrates the Kalman gains, $-\mathbf{K}^T(t) = [p_{31}(t), p_{32}(t), p_{33}(t)]$, for this example. Figure 5.1-5 indicates how the complete closed-loop controller is obtained. It is interesting to note that, in an actual physical problem, the precursor concentration is not measurable, and therefore we need to add an "observer" of this particular state variable. We also need to discuss many more aspects of this problem such as disturbances and parameter variations. We will postpone further consideration of these important questions until we establish some foundation in state and parameter estimation and optimal adaptive

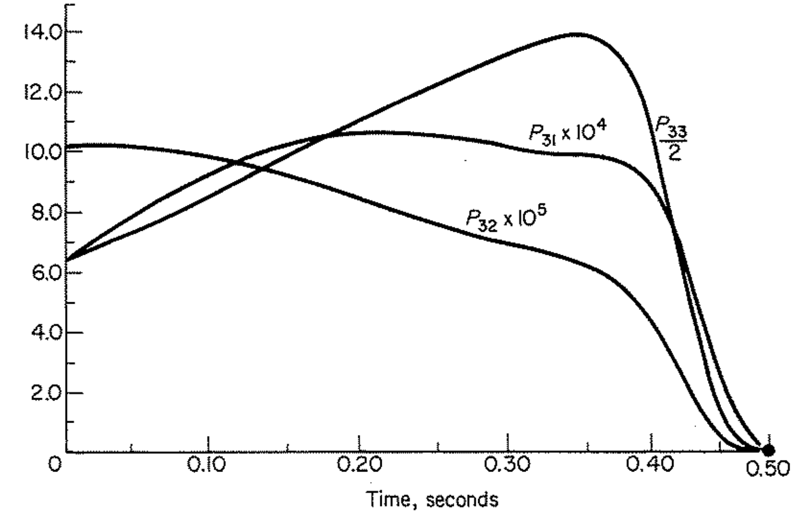

Fig. 5.1-4 Kalman gains for Example (5.1-2).

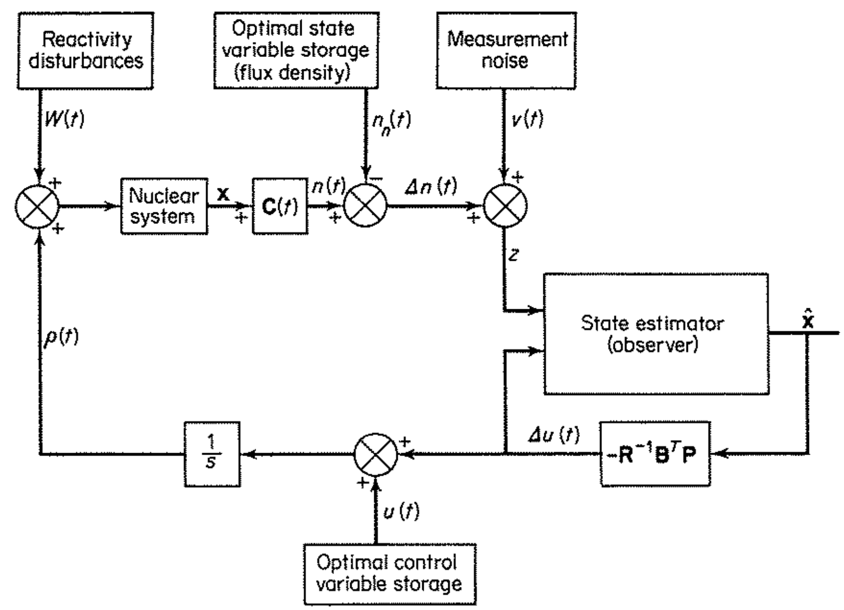

Fig. 5.1-5 Structure of controller for Example (5.1-2).

control. We have, in this example, illustrated how a basically nonlinear problem may be linearized, and a linear time-varying closed-loop controller obtained, if a nominal trajectory is known. Since this can be accomplished for a variety of problems, we see that the linear regulator problem is indeed an important one.

Example 5.1-3. We now consider the optimal control of a distributed parameter system. By a spatial discretization technique, we will reduce the distributed parameter optimal control problem to a form of the linear regulator problem.

Consider the one-dimensional diffusion equation

$$\frac{\partial x(y, t)}{\partial t} = \frac{\partial^2 x(y, t)}{\partial y^2} + u(y, t) \tag{5.1-21}$$

with initial condition $x(y, t_o = 0) = x_o(y)$,

and $\frac{\partial x(y, t)}{\partial y} = 0$ at $y = 0$, $\frac{\partial x(y, t)}{\partial y} = 0$ at $y = y_f$.

We desire to find the control $u(y, t)$ which minimizes the cost function

$$J = \tfrac{1}{2} \int_o^{t_f} \int_o^{y_f} [Q'x^2(y, t) + R'u^2(y, t)]\, dy\, dt.$$

We wish to obtain an approximate solution of Eq. (5.1-21) where $u(y, t)$ is assumed to be available. We shall establish a spatially discretized model in which the size of the space increment is $\Delta y = y_f/n$, where n is an integer. Physically, this corresponds to cutting a slab of length y_f into n slices. We shall use central difference formulas and obtain a spatially discrete model that can be described by vector differential equations. Let us use the notation

$$\frac{\partial x(y, t)}{\partial t} = \dot{x}_i(t) \quad \text{where} \quad i = 1, 2, \ldots, n \tag{5.1-22}$$

and then use central difference formulas to obtain

$$\frac{\partial^2 x(y, t)}{\partial y^2} \cong \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2} \tag{5.1-23}$$

where

$$x_{i+1}(t) = x(y + \Delta y, t), \qquad x_i(t) = x(y, t), \qquad x_{i-1}(t) = x(y - \Delta y, t).$$

Therefore, using Eqs. (5.1-22) and (5.1-23) in Eq. (5.1-21), we obtain

$$\dot{x}_i(t) = \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2} + u_i(t) \tag{5.1-24}$$

where $i = 1, 2, \ldots, n$.

By considering different values of $i, i = 1, 2, \ldots, n$, and using Eq. (5.1-24), we obtain n first-order linear differential equations which approximate Eq. (5.1-21). These are

$$\begin{aligned}
\dot{x}_1(t) &= \frac{1}{(\Delta y)^2}[x_2(t) - 2x_1(t) + x_0(t)] + u_1(t) \\
\dot{x}_2(t) &= \frac{1}{(\Delta y)^2}[x_3(t) - 2x_2(t) + x_1(t)] + u_2(t) \\
&\ \vdots \\
\dot{x}_{n-1}(t) &= \frac{1}{(\Delta y)^2}[x_n(t) - 2x_{n-1}(t) + x_{n-2}(t)] + u_{n-1}(t) \\
\dot{x}_n(t) &= \frac{1}{(\Delta y)^2}[x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)] + u_n(t).
\end{aligned} \tag{5.1-25}$$

We may use the boundary conditions to obtain $x_0(t)$ and $x_{n+1}(t)$. Then, by using a first difference approximation to the initial boundary condition, we obtain

$$\frac{x_{i+1}(t) - x_i(t)}{(\Delta y)} = 0 \quad \text{for} \quad y = 0 \quad \text{or, equivalently,} \quad i = 0. \tag{5.1-26}$$

We have therefore established the boundary condition

$$x_0(t) = x_1(t). \tag{5.1-27}$$

In a similar fashion, we may easily show that

$$x_n(t) = x_{n+1}(t). \tag{5.1-28}$$

If use is made of Eqs. (5.1-27) and (5.1-28) in the set of ordinary differential equations given in Eq. (4.3-28), we obtain

$$\begin{aligned}
\dot{x}_1(t) &= \frac{1}{(\Delta y)^2}[x_2(t) - x_1(t)] + u_1(t) \\
\dot{x}_2(t) &= \frac{1}{(\Delta y)^2}[x_3(t) - 2x_2(t) + x_1(t)] + u_2(t) \\
\dot{x}_3(t) &= \frac{1}{(\Delta y)^2}[x_4(t) - 2x_3(t) + x_2(t)] + u_3(t) \\
&\ \vdots \\
\dot{x}_{n-1}(t) &= \frac{1}{(\Delta y)^2}[x_n(t) - 2x_{n-1}(t) + x_{n-2}(t)] + u_{n-1}(t) \\
\dot{x}_n(t) &= \frac{1}{(\Delta y)^2}[-x_n(t) + x_{n-1}(t)] + u_n(t).
\end{aligned} \tag{5.1-29}$$

We will now represent this set of ordinary linear differential equations by the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.1-30}$$

where: $\mathbf{x}$ is an n-dimensional state vector; $\mathbf{u}$ is an n-dimensional control vector; $\mathbf{A}$ is the $n \times n$ tridiagonal matrix,

$$\mathbf{A} = \frac{1}{(\Delta y)^2}\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 & -1
\end{bmatrix}; \tag{5.1-31}$$

and where $\mathbf{B}$ is the identity matrix of order n, $\mathbf{B} = \mathbf{I}$. It is an easy task to verify that this linear system is always stable.

A discrete approximate form of the performance function is

$$J = \tfrac{1}{2}\Delta y \int_0^{t_f} \{\sum_{i=1}^{n-1} [Q' x_i^2(t) + R' u_i^2(t)] + \tfrac{1}{2}[Q' x_o(t) + Q' x_n(t) + R' u_o(t) + R' u_n(t)]\}\, dt$$

where n is the last discretized spatial stage. We may rewrite this as

$$J = \tfrac{1}{2}\Delta y \int_0^{t_f} [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\, dt.$$

For this problem, the Hamiltonian is

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\Delta y \mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}\Delta y \mathbf{u}^T\mathbf{R}\mathbf{u} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{u}.$$

Application of the maximum principle to this problem immediately yields the two-point boundary value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad x_i(0) = 1 + \alpha i y_f/n, \qquad i = 1, 2, \ldots, n$$

$$-\dot{\boldsymbol{\lambda}} = \Delta y \mathbf{Q}\mathbf{x}(t) + \mathbf{A}^T\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda}(t_f) = 0$$

$$\mathbf{u} = -\frac{1}{\Delta y}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}.$$

We shall solve this problem by generating the Riccati equation where, as before in Sec. 5.1, we assume $\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)$. Thus, the optimal control is a linear feedback control determined by solving

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} - \frac{1}{\Delta y}\mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \Delta y\mathbf{Q} + \mathbf{A}^T\mathbf{P}(t) = \mathbf{0}, \qquad \mathbf{P}(t_f) = \mathbf{0}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \frac{1}{\Delta y}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}(t), \qquad x_i(0) = 1 + \alpha i y_f/n, \qquad i = 1, 2, \ldots, n$$

$$\mathbf{u}(t) = -\frac{1}{\Delta y}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)\mathbf{x}(t).$$

Let us consider the following two cases:

Case A $t_f = 1.0$, $y_f = 4.0$, $\mathbf{B} = \mathbf{I}$, $Q' = R' = 1$

$\Delta t = 0.01$, $\Delta y = 1.0$, $\alpha = 1$

$$\mathbf{A} = \frac{1}{1}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{Q} = \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Case B $t_f = 1.0$, $\Delta t = 0.01$, $\mathbf{B} = \mathbf{I}$, $Q' = R' = 1$

$y_f = 4.0$, $\Delta y = 0.5$, $\alpha = 1$

$$\mathbf{A} = 4\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathbf{Q} = \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

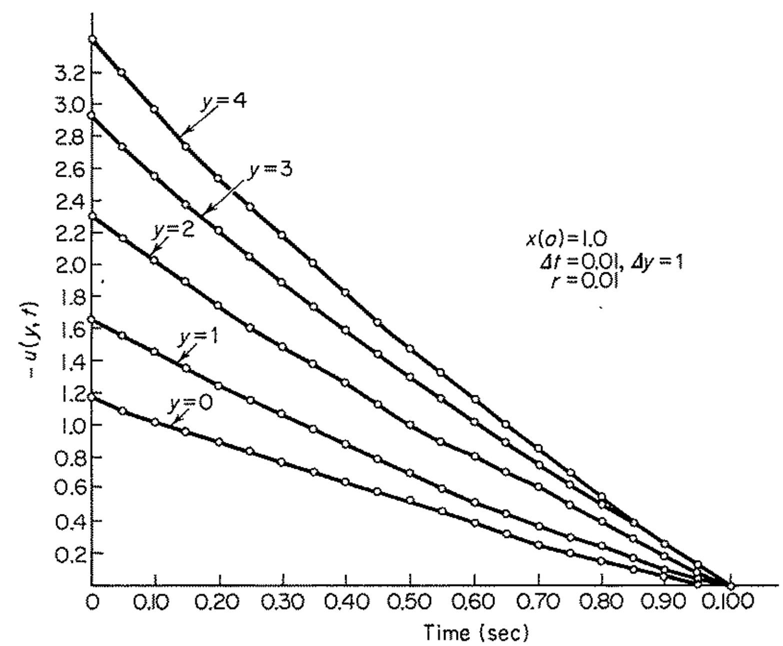

Fig. 5.1-6 Optimal control versus spatial coordinate and time, Example 5.1-3.

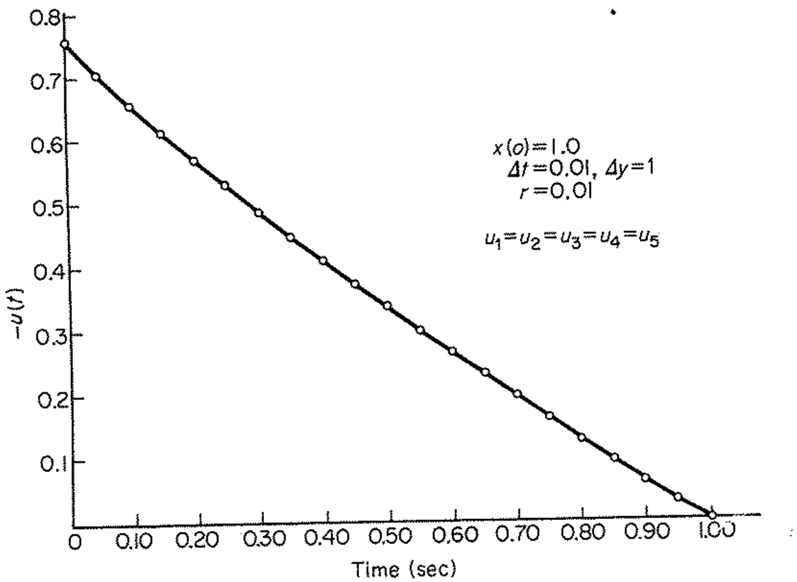

Fig. 5.1-7 Optimal control versus time for $\alpha = 0$, Example 5.1-3.

Solution of the two cases considered yields essentially the same result. This indicates that, for the particular initial condition $x(y, 0) = 1 + y$, a model with five coordinates yields as good a solution as a model with ten coordinates. Thus we may safely assume that, for this particular initial condition, lumping the distributed system into five states is a satisfactory thing to do. As we change the initial distribution, $x(y, 0) = 1 + \alpha y$, by changing α, the number of necessary states to provide a good lumped model changes. For $\alpha = 0$, where the initial condition is uniform throughout y, a single state suffices for an exact model since $\partial^2 x(y, t)/\partial y^2$ is then always zero, and the distributed system degenerates to a lumped system for this particular case.

Figure 5.1-6 illustrates a plot of the optimal control versus time and distance y for $\alpha = 1$. A plot of the optimal control versus time is shown for the spatially independent case when $\alpha = 0$ in Fig. 5.1-7.

5.2 The linear servomechanism

The linear regulator problem considered in the preceding section can be generalized in several ways. We can assume that we desire to find the control in such a way as to cause the output to track or follow a desired output state, $\boldsymbol{\eta}(t)$. We may also assume that there is a forcing function (not the control) for the system differential equations. Therefore, we will consider the minimization of

$$J = \tfrac{1}{2}\|\boldsymbol{\eta}(t_f) - \mathbf{z}(t_f)\|_{\mathbf{S}}^2 + \tfrac{1}{2}\int_{t_0}^{t_f} [\|\boldsymbol{\eta}(t) - \mathbf{z}(t)\|_{\mathbf{Q}(t)}^2 + \|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2]\, dt \quad (5.2\text{-}1)$$

for the system which contains a deterministic input or plant "noise" vector $\mathbf{w}(t)$

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{w}(t), \qquad \mathbf{x}(t_o) = \mathbf{x}_o \quad (5.2\text{-}2)$$

$$\mathbf{z}(t) = \mathbf{C}(t)\mathbf{x}(t). \quad (5.2\text{-}3)$$

The requirements on the various matrices are the same as in the preceding section. We proceed in exactly the same fashion as for the regulator problem. The Hamiltonian is, from Eq. (4.3-34),

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\|\boldsymbol{\eta}(t) - \mathbf{C}(t)\mathbf{x}(t)\|_{\mathbf{Q}(t)}^2 + \tfrac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2 + \boldsymbol{\lambda}^T(t)[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{w}(t)]. \quad (5.2\text{-}4)$$

We employ the maximum principle and set $\partial H/\partial \mathbf{u} = \mathbf{0}$ to obtain

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t) \quad (5.2\text{-}5)$$

and

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \mathbf{C}^T(t)\mathbf{Q}(t)[\mathbf{C}(t)\mathbf{x}(t) - \boldsymbol{\eta}(t)] + \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \quad (5.2\text{-}6)$$

with the terminal condition

$$\boldsymbol{\lambda}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}[\mathbf{C}(t_f)\mathbf{x}(t_f) - \boldsymbol{\eta}(t_f)]. \quad (5.2\text{-}7)$$

In order to attempt to determine a closed-loop control, we assume

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t) - \boldsymbol{\xi}(t). \quad (5.2\text{-}8)$$

We substitute this relation into the canonic equations and determine the requirements for a solution. By a procedure analogous to that of the preceding section, we easily obtain the following requirements:

$$\dot{\mathbf{P}} = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{1-}(t)\mathbf{B}^T(t)\mathbf{P}(t) - \mathbf{C}^T(t)\mathbf{Q}(t)\mathbf{C}(t) \quad (5.2\text{-}9)$$

$$\mathbf{P}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}\mathbf{C}(t_f), \quad (5.2\text{-}10)$$

and

$$\dot{\boldsymbol{\xi}} = -[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)]^T\boldsymbol{\xi} + \mathbf{P}(t)\mathbf{w}(t) - \mathbf{C}^T(t)\mathbf{Q}(t)\boldsymbol{\eta}(t) \quad (5.2\text{-}11)$$

$$\boldsymbol{\xi}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}\boldsymbol{\eta}(t_f). \quad (5.2\text{-}12)$$

Thus we see that the linear servomechanism problem is composed of two parts: a linear regulator part, plus a prefilter to determine the optimal driving function from the desired value, $\boldsymbol{\eta}(t)$, of the system output. The optimum control law is linear and is obtained from Eq. (5.2-5) as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)[\mathbf{P}(t)\mathbf{x}(t) - \boldsymbol{\xi}(t)]. \quad (5.2\text{-}13)$$

Unfortunately, the optimal control is, in practice, often computationally unrealizable because it involves $\boldsymbol{\xi}(t)$ which must be solved backward from t_f to t_o and, therefore, requires a knowledge of $\boldsymbol{\eta}(t)$ and $\mathbf{w}(t)$ for all time $t \in [t_o, t_f]$. This is quite often not known at the initial time t_o.

Example 5.2-1. Let us consider the minimization of the cost function

$$J = \frac{1}{2}\int_0^{t_f} [(x_1 - \eta_1)^2 + u^2]\, dt$$

for the system described by

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}.$$

We first use Eqs. (5.2-9) and (5.2-10) to obtain the Riccati equation for this example

$$\dot{p}_{11} = p_{12}^2 - 1, \qquad p_{11}(t_f) = 0$$
$$\dot{p}_{12} = -p_{11} + p_{12}p_{22}, \qquad p_{12}(t_f) = 0$$
$$\dot{p}_{22} = -2p_{12} + p_{22}^2, \qquad p_{22}(t_f) = 0.$$

If we allow t_f to become infinite, we obtain the solution $p_{11} = p_{22} = \sqrt{2}$, $p_{12} = 1$. Thus we have for the closed-loop control

$$u = -\mathbf{R}^{-1}\mathbf{B}^T[\mathbf{P}\mathbf{x} - \boldsymbol{\xi}] = -x_1 - \sqrt{2}x_2 + \xi_2$$

where we must determine $\boldsymbol{\xi}$ by solving Eqs. (5.2-11) and (5.2-12) which become for this example

$$\dot{\xi}_1 = \xi_1 - \eta_1, \qquad \xi_1(t_f) = 0$$
$$\dot{\xi}_2 = -\xi_1 + \sqrt{2}\xi_2, \qquad \xi_2(t_f) = 0.$$

If $\eta_1 = \alpha$, a constant, for t greater than zero, we are justified in obtaining the equilibrium solution for the $\boldsymbol{\xi}$ equation if $t_f = \infty$ by setting $\dot{\boldsymbol{\xi}} = \mathbf{0}$ to obtain $\xi_2 = 0.707\xi_1 = \eta_1 = \alpha$. If $\eta_1 = 1 - e^{-t}$, we will then find by a simple limiting process that for $t_f = \infty$,

$$\xi_2(t) = 1 + \frac{1}{2 + \sqrt{2}}e^{-t}, \qquad t \geq 0.$$

We may realize this solution as shown in Fig. 5.2-1.

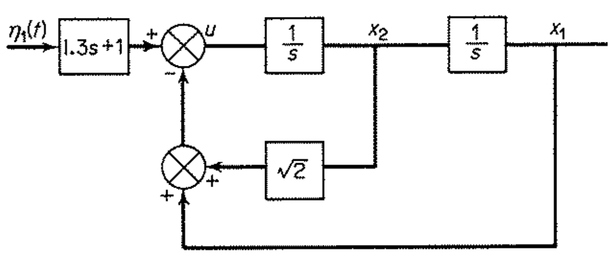

Fig. 5.2-1 Block diagram of optimum servomechanism for Example 5.2-1.

We note that if $\mathbf{w}(t) = \boldsymbol{\eta}(t) = \mathbf{0}$, or for that matter, any vector constant in time, the servomechanism problem reduces to a regulator problem except that it is an "output" regulator problem rather than a "state" regulator problem because of the presence of the output matrix $\mathbf{C}(t)$. It is not necessary for the system to be controllable in order to find a solution to the regulator problem. The only exception to this is in the limiting cases where $\mathbf{S}$ becomes infinite or where t_f becomes infinite. It is, however, necessary that the system be observable in order for a solution to the output regulator problem to exist. We will expand considerably on these ideas when we consider controllability, observability, and the reachable zone problem in Chapter 7.

5.3 Bang bang control and minimum time problems

Maximum effort control problems have become increasingly important in a variety of applications. It is natural that we ask under what circumstances optimal controls will always be maximum effort, or *bang bang*. To do this, we will restrict each component of the control vector, $\mathbf{u}(t)$, to some bounded interval. Let us consider the nonlinear differential system where the control enters in a linear fashion

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t), \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{5.3-1}$$
$$a_i \leq u_i \leq b_i, \qquad \forall\, i \tag{5.3-2}$$

and assume a performance index which likewise contains only linear terms in the control variable, such that the Hamiltonian will also be linear in $\mathbf{u}(t)$.

$$J = \theta[\mathbf{x}(t_f), t_f] + \int_{t_o}^{t_f} \{\phi[\mathbf{x}(t), t] + \mathbf{h}^T[\mathbf{x}(t), t]\mathbf{u}(t)\}\, dt \tag{5.3-3}$$

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), t] + \mathbf{h}^T[\mathbf{x}(t), t]\mathbf{u}(t) + \boldsymbol{\lambda}^T(t)\{\mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t)\}. \tag{5.3-4}$$

Since the Hamiltonian is linear in the control vector, $\mathbf{u}(t)$, minimization of the Hamiltonian with respect to $\mathbf{u}(t)$ requires that

$$u_i = \begin{cases} a_i & \text{if } \{\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t]\}_i > 0 \\ b_i & \text{if } \{\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t]\}_i < 0. \end{cases} \tag{5.3-5}$$

Thus we see that when the control vector appears linearly in both the equation of motion of the differential system and the performance index, and if, in addition, each component of the control vector is bounded, the optimal control is bang bang. The only exception to this occurs in cases where

$$\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t] = \mathbf{0}, \tag{5.3-6}$$