6 Discrete variational calculus and the discrete maximum principle

Perhaps the most useful single technique in modern control system theory is that branch of mathematics known as the calculus of variations. Variational principles have been applied to physical problems, such as wave propagation, since the time of Huygens. The Hamiltonian formulation of the variational problem has existed since the early nineteenth century in the works of Hamilton, Jacobi, and others. The most significant contribution in recent times was made by L. S. Pontryagin. The work of Pontryagin [1, 2] extended the variational method to include problems wherein the available control and state vector is bounded, as we have seen in the previous two chapters.

Recently, the maximum principle has been applied to problems involving discrete-data systems [3, 4]. In reality, the maximum principle is not universally valid for the case of discrete systems [1]. Due to restrictions on possible variations of the control signal, the maximum principle must be modified for the general discrete case. Jordan and Polak [5] discuss the limitations and derive a modified form of the maximum principle, which is applicable to the general discrete problem. Pearson and Sridhar [6] investigate the discrete maximum principle using the framework of nonlinear programming [7]. Further discussion can be found in [8]. The results in [9] represent a particularly general development of necessary conditions for the discrete time case.

We begin this chapter by determining a discrete version of the Euler-Lagrange equations and transversality conditions. The discrete maximum principle is then stated and compared with the continuous maximum principle. The final section explores the relationship between discrete optimal control and mathematical programming.

6.1 Derivation of the discrete Euler-Lagrange equations

In our previous work we minimized cost functions which were integrals of scalar functions. Here we are interested in minimization of cost functions which are summations of scalar functions. Thus we are concerned with minimizing (or maximizing) functions such as

$$J = \sum_{k=k_o}^{k_f-1} \Phi(\mathbf{x}_k, \mathbf{x}_{k+1}, k) = \sum_{k=k_o}^{k_f-1} \Phi_k \tag{6.1-1}$$

where $\mathbf{x}_k = \mathbf{x}(t_k)$. For the case of synchronous sampling, or sampling with an equal time interval between samples, $\mathbf{x}_k = \mathbf{x}(kT)$, where T is the sampling period.†

It should also be noted that the function Φ_k represents the incremental cost for one stage of the discrete process. For a cost function equivalent to that of an integral for a continuous process, Φ_k will contain, as a multiplying factor, the sampling period T. We let $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ take on variations

$$\mathbf{x}_k = \hat{\mathbf{x}}_k + \epsilon\boldsymbol{\eta}_{x_k}, \qquad \mathbf{x}_{k+1} = \hat{\mathbf{x}}_{k+1} + \epsilon\boldsymbol{\eta}_{x_{k+1}} \tag{6.1-2}$$

where $\hat{\mathbf{x}}$ denotes the solution to the optimization problem. We use the same procedure as used previously for continuous systems. We substitute the foregoing values assumed for $\mathbf{x}(t_k)$ and $\mathbf{x}(t_{k+1})$ into the given cost function J. We compute $\partial J/\partial\epsilon$ and set it equal to zero at $\epsilon = 0$ independent of the variation $\boldsymbol{\eta}_{x_k}$ and $\boldsymbol{\eta}_{x_{k+1}}$. Thus we obtain

$$\sum_{k=k_o}^{k_f-1} \left\{\left\langle\frac{\partial\Phi_k}{\partial\hat{\mathbf{x}}_k}, \boldsymbol{\eta}_{x_k}\right\rangle + \left\langle\frac{\partial\Phi_k}{\partial\hat{\mathbf{x}}_{k+1}}, \boldsymbol{\eta}_{x_{k+1}}\right\rangle\right\} = 0 \tag{6.1-3}$$

where the notation $\langle\mathbf{x}, \mathbf{y}\rangle$ is used to indicate the inner product, or $\mathbf{x}^T\mathbf{y}$.

If we use variational notation, Eq. (6.1-3) can be written in a simpler form as follows. (We remember that the first variation δJ is set equal to 0 to extremize J.)

$$\delta J = \sum_{k=k_o}^{k_f-1} \left\{\delta\mathbf{x}_k^T \frac{\partial\Phi_k}{\partial\hat{\mathbf{x}}_k} + \delta\mathbf{x}_{k+1}^T \frac{\partial\Phi_k}{\partial\hat{\mathbf{x}}_{k+1}}\right\} = 0. \tag{6.1-4}$$

When we manipulate the last term of Eq. (6.1-4) into a more convenient form by exchanging summation indices (replacing k by $m - 1$), and when we drop the ^ superscript, we obtain

$$\sum_{k=k_o}^{k_f-1} \delta\mathbf{x}_{k+1}^T \frac{\partial\Phi_k}{\partial\mathbf{x}_{k+1}} = \sum_{m=k_o+1}^{k_f} \delta\mathbf{x}_m^T \frac{\partial\Phi[\mathbf{x}_{m-1}, \mathbf{x}_m, m-1]}{\partial\mathbf{x}_m}. \tag{6.1-5}$$

If we rewrite this equation letting $k = m$ and starting the summation at $k = k_o$ and ending at $k_f - 1$, we get the result

$$\sum_{k=k_o}^{k_f-1} \delta\mathbf{x}_{k+1}^T \frac{\partial\Phi_k}{\partial\mathbf{x}_{k+1}} = \sum_{k=k_o}^{k_f-1} \delta\mathbf{x}_k^T \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k} + \delta\mathbf{x}_k^T \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k}\bigg|_{k=k_o}^{k=k_f}. \tag{6.1-6}$$

Therefore, Eq. (6.1-4) becomes

$$\sum_{k=k_o}^{k_f-1} \delta\mathbf{x}_k^T \left\{\frac{\partial\Phi[\mathbf{x}_k, \mathbf{x}_{k+1}, k]}{\partial\mathbf{x}_k} + \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k}\right\} + \delta\mathbf{x}_k^T \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k}\bigg|_{k=k_o}^{k=k_f} = 0. \tag{6.1.7}$$

For Eq. (6.1-7) to be equal to zero for arbitrary variations, the following vector difference equation, which is necessary for an extremum of the cost function, Eq. (6.1-1), must hold:

$$\frac{\partial\Phi[\mathbf{x}_k, \mathbf{x}_{k+1}, k]}{\partial\mathbf{x}_k} + \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k} = \mathbf{0}. \tag{6.1-8}$$

This may be spoken of as the *discrete Euler-Lagrange equation*. The transversality condition is obtained when we set the last term in Eq. (6.1-7) equal to zero:

$$\delta\mathbf{x}_k^T \frac{\partial\Phi[\mathbf{x}_{k-1}, \mathbf{x}_k, k-1]}{\partial\mathbf{x}_k} = 0 \qquad \text{for} \quad k = k_o, k_f. \tag{6.1-9}$$

The discussions in Chapters 3 and 4 regarding application of the transversality conditions apply well here. Also, the discussion of the Lagrange multiplier method to treat equality constraints applies almost without modification. This will be illustrated by an example.

Example 6.1-1. In this example, we will consider a simple scalar problem and solve it by very elementary techniques. The cost function to be minimized is

$$J = \tfrac{1}{2} \sum_{k=0}^{9} u^2(k).$$

The cost is minimized subject to the equality constraints

$$x(k+1) = x(k) + \alpha u(k), \qquad x(0) = 1, \qquad x(10) = 0.$$

We adjoin to the original cost function the given constraint via a Lagrange multiplier. This yields

$$J' = \sum_{k=0}^{9} [\tfrac{1}{2}u^2(k) + \lambda(k+1)\{-x(k+1) + x(k) + \alpha u(k)\}].$$

†We will also use T to represent the transpose of a vector or matrix. Clearly, no confusion should result from this.

The reader may well question why the stage $k+1$ is associated with the Lagrange multiplier. The reason is simplicity of the final result, as will be apparent in the next section. For this example, we have

$$\Phi_k = \tfrac{1}{2}u^2(k) + \lambda(k+1)\{-x(k+1) + x(k) + \alpha u(k)\}$$

$$\frac{\partial \Phi[x_k, x_{k+1}, k]}{\partial x_k} = +\lambda(k+1), \qquad \frac{\partial \Phi[x_{k-1}, x_k, k-1]}{\partial x_k} = -\lambda(k)$$

$$\frac{\partial \Phi[x_k, x_{k+1}, k]}{\partial u_k} = \alpha\lambda(k+1) + u(k), \qquad \frac{\partial \Phi[x_{k-1}, x_k, k-1]}{\partial u_k} = 0.$$

Thus the discrete Euler-Lagrange equation (6.1-8) yields

$$\lambda(k) - \lambda(k+1) = 0, \qquad u(k) + \alpha\lambda(k+1) = 0.$$

Also, the original equation must hold, subject to the stated boundary conditions

$$x(k+1) = x(k) + \alpha u(k), \qquad x(0) = 1, \qquad x(10) = 0.$$

Solving the last two equations, we obtain

$$\lambda(k) = \text{constant} = c, \qquad u(k) = -\alpha c, \qquad x(k+1) = x(k) - \alpha^2 c.$$

This is the final difference equation to be solved. By solving it stage by stage, we obtain

$$\begin{aligned} x(1) &= x(0) - \alpha^2 c \\ x(2) &= x(1) - \alpha^2 c = x(0) - 2\alpha^2 c \\ x(3) &= x(2) - \alpha^2 c = x(0) - 3\alpha^2 c \\ &\vdots \\ x(k) &= x(0) - k\alpha^2 c. \end{aligned}$$

Therefore, to satisfy the boundary conditions, we must have

$$x(10) = 0 = x(0) - 10\alpha^2 c, \qquad c = \frac{+x(0)}{10\alpha^2} = \frac{+1}{10\alpha^2}.$$

Hence the control to be applied to this discrete system is $u(k) = -1/10\alpha$. The resulting trajectory is $x(k) = 1 - k/10$.

Example 6.1-2. We now return to the distributed parameter system discussed in Example 5.1-3 where we now discretize both in time and space. We wish to reduce the problem to the form

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

where

$$\mathbf{x}(k) = \begin{bmatrix} x_{o,k} \\ x_{1,k} \\ \vdots \\ x_{m,k} \end{bmatrix}, \qquad \mathbf{u}(k) = \begin{bmatrix} u_{o,k} \\ u_{1,k} \\ \vdots \\ u_{m,k} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1-r & r & 0 & 0 & \cdots & 0 & 0 \\ r & 1-2r & r & 0 & \cdots & 0 & 0 \\ 0 & r & 1-2r & r & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & r & 1-r \end{bmatrix}$$

$$\mathbf{B} = \Delta t \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad r = \frac{\Delta t}{(\Delta y)^2}.$$

The discrete version of the performance index is approximated as

$$J = \tfrac{1}{2}\int_0^{t_f}\int_0^{y_f} \{Q' x^2(y,t) + R' u^2(y,t)\}\, dy\, dt$$

$$\approx \tfrac{1}{2}\,\Delta y\,\Delta t \sum_{k=0}^{K-1} \{\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k)\}.$$

We shall now use the discrete maximum principle to obtain the optimal control $\mathbf{u}(k)$. The Hamiltonian is

$$H[\mathbf{x}(k), \mathbf{u}(k), \boldsymbol{\lambda}(k+1), k] = \tfrac{1}{2}\,\Delta y\,\Delta t\, \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \tfrac{1}{2}\,\Delta y\,\Delta t\, \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) + \boldsymbol{\lambda}^T(k+1)[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)].$$

If we apply the discrete maximum principle, we need to solve

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \qquad x_i(0) = 1 + \alpha i y_f/n, \qquad i = 1, 2, \ldots, n$$

$$\boldsymbol{\lambda}(k) = \mathbf{A}^T\boldsymbol{\lambda}(k+1) + \Delta y\,\Delta t\,\mathbf{Q}\mathbf{x}(k), \qquad \boldsymbol{\lambda}(k) = \mathbf{0}$$

$$\mathbf{u}(k) = -\frac{1}{\Delta y\,\Delta t}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(k+1).$$

We shall solve this problem by generating the closed-loop control and the discrete matrix Riccati equation, where we assume $\boldsymbol{\lambda}(k) = \mathbf{P}(k)\mathbf{x}(k)$. The optimal control is discerned by solution of the difference equations

$$\mathbf{P}(k) = \Delta y\,\Delta t\,\mathbf{Q} + \mathbf{A}^T\mathbf{P}(k+1)\left[\mathbf{A}^{-1} + \frac{1}{\Delta y\,\Delta t}\mathbf{A}^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)\right]^{-1},$$

$$\mathbf{P}(K) = \mathbf{0}$$

$$\mathbf{x}(k) = \left[\mathbf{I} + \frac{1}{\Delta y\,\Delta t}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k)\right]^{-1}\mathbf{A}\mathbf{x}(k-1), \qquad x_i(0) = 1 + \alpha i y_f/n$$

$$\begin{aligned} \mathbf{u}(k) &= -\frac{1}{\Delta y\,\Delta t}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(k+1)\mathbf{x}(k+1) \\ &= -\frac{1}{\Delta y\,\Delta t}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{A}^{-T}[\mathbf{P}(k) - \mathbf{Q}]\mathbf{x}(k). \end{aligned}$$

Let us consider the following two cases.

Case A $t_f = 1.0, \quad \Delta t = 0.01$
$y_f = 4.0, \quad \Delta y = 1.0$
$\alpha = 1, \quad Q' = 1, \quad R' = 1$
$r = \Delta t/(\Delta y)^2 = 0.01, \quad \mathbf{B} = 0.01\mathbf{I}.$

$$\mathbf{A} = \begin{bmatrix} 0.99 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.98 & 0.01 & 0 & 0 \\ 0 & 0.01 & 0.98 & 0.01 & 0 \\ 0 & 0 & 0.01 & 0.98 & 0.01 \\ 0 & 0 & 0 & 0.01 & 0.99 \end{bmatrix}, \qquad \mathbf{R} = \mathbf{Q} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Case B $t_f = 1.0, \quad \Delta t = 0.01$
$y_f = 4.0, \quad \Delta y = 0.5$
$\alpha = 1.0, \quad Q' = R' = 1$
$r = 0.04, \quad \mathbf{B} = 0.01\mathbf{I}.$

$$\mathbf{A} = \begin{bmatrix} 0.96 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04 & 0.92 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.04 & 0.92 & 0.04 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.04 & 0.92 & 0.04 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.04 & 0.92 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.04 & 0.92 & 0.04 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.04 & 0.92 & 0.04 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0.92 & 0.04 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0.96 \end{bmatrix},$$

$$\mathbf{Q} = \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

As indicated in Fig. 6.1-1, there is a slight difference in the computed controls for these two cases. A third trial with $\Delta t = 0.01$, $\Delta y = 0.25$, and $r = 0.16$ indicates that the result for $r = 0.04$ is acceptable in that there is no noticeable change in the controls computed for the two values of r. Again the number of spatial coordinates required for an accurate model is a function of α with only a single coordinate required for $\alpha = 0$. Figures 6.1-2 and 6.1-3 illustrate optimum system behavior for $\alpha = 20$.

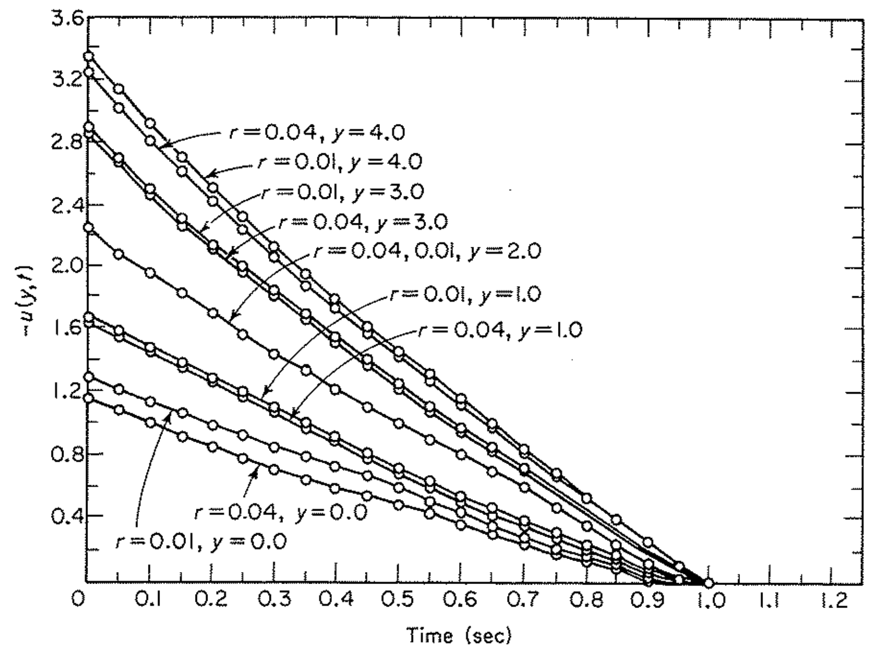

Fig. 6.1-1 Optimal control versus spatial coordinate and time, Example 6.1-2.

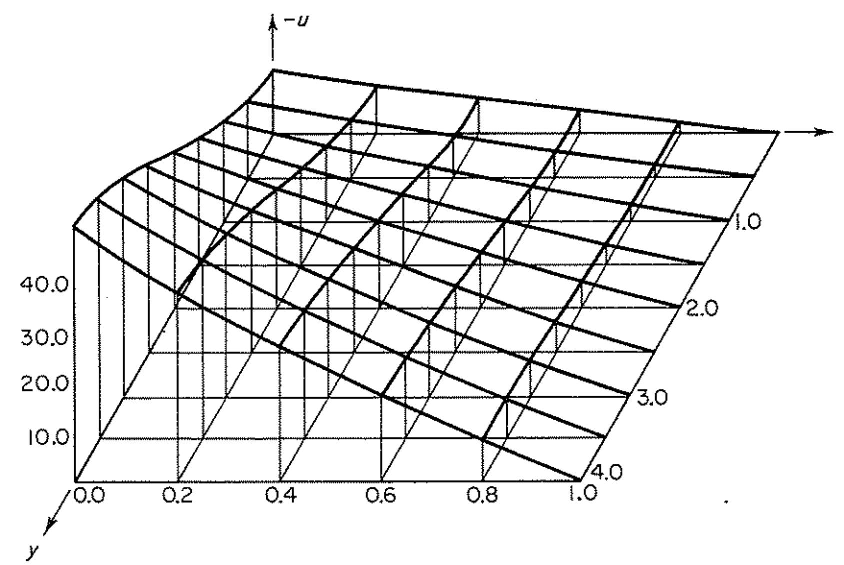

Fig. 6.1-2 Optimal control versus spatial coordinate and time, Example 6.1-2, $\alpha = 20$.

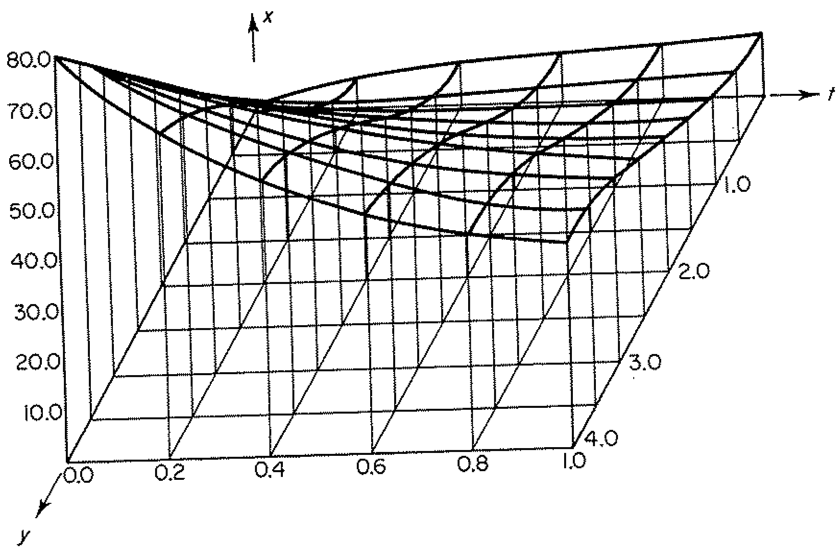

Fig. 6.1-3 Optimal state versus spatial coordinate and time, Example 6.1-2, $\alpha = 20$.

6.2 The discrete maximum principle

Analogous to the continuous time case, use of the discrete Euler-Lagrange equations for problems having equality and inequality constraints can become quite cumbersome. We now consider the development of necessary conditions using a Hamiltonian approach for discrete problems having equality constraints of the form

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k), \qquad k = k_o, \ldots, k_f - 1 \tag{6.2-1}$$

and inequality constraints of the form

$$\mathbf{u}_k \in \mathbf{\mho}, \tag{6.2-2}$$

where $\mathbf{\mho}$ is a given set in R^r and k_o and k_f are fixed integers. The problem considered is to find an admissible sequence $\mathbf{u}_k$, $k = k_o, \ldots, k_f - 1$, i.e., $\mathbf{u}_k \in \mathbf{\mho}$ for all k, in order to minimize the criterion

$$J = \Theta(\mathbf{x}_k, k)\Big|_{k=k_o}^{k=k_f} + \sum_{k=k_o}^{k_f-1} \phi(\mathbf{x}_k, \mathbf{u}_k, k), \tag{6.2-3}$$

subject to Eq. (6.2-1), with respect to the set of all admissible control sequences. We now state a maximum principle for this problem.

Let $\hat{\mathbf{u}}_k$, $k = k_o, \ldots, k_f - 1$, be an optimal sequence, and let $\hat{\mathbf{x}}_k$, $k = k_o, \ldots, k_f$, be the state response of $\hat{\mathbf{u}}$ uniquely defined by Eq. (6.2-1).

Then, under reasonable assumptions†, there exists a nontrivial function $\hat{\boldsymbol{\lambda}}$ satisfying

$$\hat{\boldsymbol{\lambda}}_k = \frac{\partial H(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \hat{\boldsymbol{\lambda}}_{k+1}, k)}{\partial \hat{\mathbf{x}}_k}, \tag{6.2-4}$$

$$\boldsymbol{\eta}_{k_o}^T = \left[\hat{\boldsymbol{\lambda}}_{k_o} - \left(\frac{\partial \theta_{k_o}}{\partial \hat{\mathbf{x}}_{k_o}}\right)\right] = \mathbf{0}, \tag{6.2-5}$$

and

$$\boldsymbol{\eta}_{k_f}^T = \left[\hat{\boldsymbol{\lambda}}_{k_f} - \left(\frac{\partial \theta_{k_f}}{\partial \hat{\mathbf{x}}_{k_f}}\right)\right] = \mathbf{0}, \tag{6.2-6}$$

where

$$H(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\lambda}_{k+1}, k) = \phi(\mathbf{x}_k, \mathbf{u}_k, k) + \boldsymbol{\lambda}_{k+1}^T \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k) \tag{6.2-7}$$

such that for all $k = k_o, \ldots, k_f - 1$,

$$H(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \hat{\boldsymbol{\lambda}}_{k+1}, k) = \min_{\mathbf{v} \in \mathbf{\mho}} H(\hat{\mathbf{x}}_k, \hat{\mathbf{v}}, \hat{\boldsymbol{\lambda}}_{k+1}, k). \tag{6.2-8}$$

A proof of these necessary conditions can follow analogously to the proof of the continuous time maximum principle presented in Chapter 4. A more general discrete problem statement, involving state-space constraints, and proof of the discrete necessary conditions can be found in [9]. Sufficient conditions for a related problem are presented in [10].

We note that for the unconstrained case where $\mathbf{\mho} = R^r$, Eq. (6.2-8) implies the necessary condition

$$\frac{\partial H(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \hat{\boldsymbol{\lambda}}_{k+1}, k)}{\partial \mathbf{u}_k} = \mathbf{0}, \qquad k = k_o, \ldots, k_f - 1. \tag{6.2-9}$$

A perturbation method can be used to develop necessary conditions for this case as follows. By means of the Lagrange multiplier $\boldsymbol{\lambda}_k$, an equivalent cost function can be written as

$$J' = \theta(\mathbf{x}_k, k)\Big|_{k=k_o}^{k=k_f} + \sum_{k=k_o}^{k_f-1} \{\phi(\mathbf{x}_k, \mathbf{u}_k, k) - \boldsymbol{\lambda}_{k+1}^T[\mathbf{x}_{k+1} - \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k)]\} \tag{6.2-10}$$

which becomes, upon introduction of the Hamiltonian Eq. (6.2-7),

$$J' = \theta(\mathbf{x}_k, k)\Big|_{k=k_o}^{k=k_f} + \sum_{k=k_o}^{k_f-1} [H_k - \boldsymbol{\lambda}_{k+1}^T \mathbf{x}_{k+1}]. \tag{6.2-11}$$

Let

$$\mathbf{x}_k = \hat{\mathbf{x}}_k + \epsilon \boldsymbol{\eta}_k$$
$$\mathbf{x}_{k+1} = \hat{\mathbf{x}}_{k+1} + \epsilon \boldsymbol{\eta}_{k+1}$$
$$\mathbf{u}_k = \hat{\mathbf{u}}_k + \epsilon \mathbf{v}_k.$$

†These conditions are for each $k = k_o, \ldots, k_f - 1$, $\phi(\cdot, \cdot, k)\colon R^n x R^m \longrightarrow R$ is continuously differentiable (*c.d.*) in both arguments, $\theta(\cdot, k)\colon R^n \longrightarrow R$ is *c.d.* for $k = k_o, k_f$, $f(\cdot, \mathbf{v}, k)\colon R^n \longrightarrow R^n$ is *c.d.* for $k = k_o, \ldots, k_f - 1$ and $\mathbf{v} \in \mathbf{\mho}$, and the set $\{f(\mathbf{x}, \mathbf{v}, k)\colon \mathbf{v} \in \mathbf{\mho}\}$ is convex for $k = k_o, \ldots, k_f - 1$ and $\mathbf{x} \in R^n$. These assumptions are presented in a substantially weakened form in [9].

We note that the perturbations at different stages are independent; hence, $\boldsymbol{\eta}_k$, $\boldsymbol{\eta}_{k+1}$, and $\mathbf{v}_k$ are all mutually independent.

Introducing the perturbations into Eq. (6.2-11), we obtain

$$J' = \theta(\hat{\mathbf{x}}_{k_f} + \epsilon\boldsymbol{\eta}_{k_f}, k_f) - \theta(\hat{\mathbf{x}}_{k_o} + \epsilon\boldsymbol{\eta}_{k_o}, k_o) + \sum_{k=k_o}^{k_f-1}[H(\hat{\mathbf{x}}_k + \epsilon\boldsymbol{\eta}_k, \hat{\mathbf{u}}_k + \epsilon\mathbf{v}_k, \boldsymbol{\lambda}_{k+1}, k) - \boldsymbol{\lambda}_{k+1}^T[\hat{\mathbf{x}}_{k+1} + \epsilon\boldsymbol{\eta}_{k+1}]].$$

From our previous work, we know that a minimum of J' requires

$$\frac{\partial J'}{\partial \epsilon} = 0, \qquad \frac{\partial^2 J'}{\partial \epsilon^2} > 0 \tag{6.2-12}$$

for $\epsilon = 0$, independent of the variations. In this development, we will assume that the second-derivative requirement is satisfied for all cost functions and systems of interest. Equating to zero the first derivative in Eq. (6.2-12) requires that

$$\left(\frac{\partial \theta_{k_f}}{\partial \hat{\mathbf{x}}_{k_f}}\right)^T \boldsymbol{\eta}_{k_f} - \left(\frac{\partial \theta_{k_o}}{\partial \hat{\mathbf{x}}_{k_o}}\right)^T \boldsymbol{\eta}_{k_o} + \sum_{k=k_o}^{k_f-1}\left(\frac{\partial H_k}{\partial \hat{\mathbf{x}}_k}\right)^T \boldsymbol{\eta}_k - \sum_{k=k_o}^{k_f-1}\boldsymbol{\lambda}_{k+1}^T\boldsymbol{\eta}_{k+1} + \sum_{k=k_o}^{k_f-1}\left(\frac{\partial H_k}{\partial \hat{\mathbf{u}}_k}\right)^T \mathbf{v}_k = 0. \tag{6.2-13}$$

Employing the discrete version of integration by parts, we can write the fourth term of Eq. (6.2-13) as:

$$-\sum_{k=k_o}^{k_f-1}\boldsymbol{\lambda}_{k+1}^T\boldsymbol{\eta}_{k+1} = -\sum_{k=k_o+1}^{k_f}\boldsymbol{\lambda}_k^T\boldsymbol{\eta}_k = -\sum_{k=k_o}^{k_f-1}[\boldsymbol{\lambda}_k^T\boldsymbol{\eta}_k] - \boldsymbol{\lambda}_{k_f}^T\boldsymbol{\eta}_{k_f} + \boldsymbol{\lambda}_{k_o}^T\boldsymbol{\eta}_{k_o}. \tag{6.2-14}$$

Using Eq. (6.2-14) in Eq. (6.2-13), combining terms, and dropping the ^ notation, we obtain

$$\left[\left(\frac{\partial \theta_{k_f}}{\partial \mathbf{x}_{k_f}}\right)^T - \boldsymbol{\lambda}_{k_f}^T\right]\boldsymbol{\eta}_{k_f} - \left[\left(\frac{\partial \theta_{k_o}}{\partial \mathbf{x}_{k_o}}\right)^T - \boldsymbol{\lambda}_{k_o}^T\right]\boldsymbol{\eta}_{k_o} + \sum_{k=k_o}^{k_f-1}\left[\left(\frac{\partial H_k}{\partial \mathbf{x}_k}\right)^T - \boldsymbol{\lambda}_k^T\right]\boldsymbol{\eta}_k + \sum_{k=k_o}^{k_f-1}\left(\frac{\partial H_k}{\partial \mathbf{u}_k}\right)^T \mathbf{v}_k = 0. \tag{6.2-15}$$

The necessary conditions Eqs. (6.2-4), (6.2-5), (6.2-6), and (6.2-9) hold due to the mutual independence of the appropriate variations in Eq. (6.2-15). This development implies a discrete, two-point boundary value problem of the form

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \boldsymbol{\lambda}_{k+1}, k), \qquad \boldsymbol{\lambda}_{k+1} = \mathbf{g}(\mathbf{x}_k, \boldsymbol{\lambda}_k, k). \tag{6.2-16}$$

If desired, Eq. (6.2-4) can be used to eliminate $\boldsymbol{\lambda}_{k+1}$ from the first part of Eq. (6.2-10). The equations which result from the foregoing are the canonical equations of the required optimal system. The nonlinear, two-point boundary value problem represented by these equations must be solved, in general, by reiterative techniques. Since these reiterative techniques will be used on a digital computer, the discrete maximum principle provides an optimization method which is the natural one to use for many continuous problems

after an accurate discrete model has been chosen. We will present several methods for the solution of the resulting two-point boundary value problems in later chapters. The discrete linear regulator will now be considered, since (with a quadratic cost function) the two-point boundary value problem can be easily overcome.

Example 6.2-1. Discrete Linear Regulator We consider a general discrete system represented by

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \qquad \mathbf{x}(0) = x_0, \qquad k = 0, 1, 2, \ldots, k_f$$

where $\mathbf{A}$ and $\mathbf{B}$ may be functions of k. The cost function is

$$J = \tfrac{1}{2}\|\mathbf{x}_{k_f}\|_{\mathbf{S}}^2 + \tfrac{1}{2}\sum_{k=0}^{k_f-1}\{\|\mathbf{x}_k\|_{\mathbf{Q}}^2 + \|\mathbf{u}_k\|_{\mathbf{R}}^2\}$$

where the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ may be functions of the stage, k. We thus form the Hamiltonian given by

$$H_k = \tfrac{1}{2}\mathbf{x}_k^T\mathbf{Q}\mathbf{x}_k + \tfrac{1}{2}\mathbf{u}_k^T\mathbf{R}\mathbf{u}_k + \boldsymbol{\lambda}_{k+1}^T[\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k].$$

From Eq. (6.2-4) the adjoint vector equation is given by

$$\boldsymbol{\lambda}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T\boldsymbol{\lambda}_{k+1}.$$

We see that this equation cannot be solved for $\boldsymbol{\lambda}_{k+1}$ in terms of $\boldsymbol{\lambda}_k$ unless $\mathbf{A}^{-1}$ exists. Since $\mathbf{A}$ is a state transition matrix,† it will always have an inverse. And since the terminal state is unspecified, the boundary condition is obtained from Eq. (6.2-6) as

$$\boldsymbol{\lambda}(k_f) = \mathbf{S}\mathbf{x}(k_f).$$

From Eq. (6.2-9) we have

$$\frac{\partial H}{\partial \mathbf{u}_k} = \mathbf{0} = \mathbf{R}\mathbf{u}_k + \mathbf{B}^T\boldsymbol{\lambda}_{k+1}.$$

Therefore we have linear difference equations to solve, the solution of which will yield an optimum open-loop control. These equations are

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}_{k+1}, \qquad \mathbf{x}(k_0) = \mathbf{x}_0$$
$$\boldsymbol{\lambda}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T\boldsymbol{\lambda}_{k+1}, \qquad \boldsymbol{\lambda}(k_f) = \mathbf{S}\mathbf{x}(k_f).$$

We now guess a solution for these equations of the form

$$\boldsymbol{\lambda}_k = \mathbf{P}_k\mathbf{x}_k$$

and substitute in order to eliminate $\boldsymbol{\lambda}$. This yields

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{x}_{k+1}$$
$$\mathbf{P}_k\mathbf{x}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T\mathbf{P}_{k+1}\mathbf{x}_{k+1}.$$

†We have a linear difference equation, the homogeneous part of which is

$$\mathbf{x}(t_{k+1}) = \mathbf{A}(t_{k+1}, t_k)\mathbf{x}(t_k).$$

Thus $\mathbf{A}$ is clearly a state transition matrix.

By solving for $\mathbf{x}_{k+1}$ and eliminating it, we obtain

$$\mathbf{P}_k\mathbf{x}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T\mathbf{P}_{k+1}[\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}]^{-1}\mathbf{A}\mathbf{x}_k$$

which will hold for arbitrary $\mathbf{x}_k$ only if

$$\mathbf{P}_k = \mathbf{Q} + \mathbf{A}^T\mathbf{P}_{k+1}[\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}]^{-1}\mathbf{A} = \mathbf{Q} + \mathbf{A}^T[\mathbf{P}_{k+1}^{-1} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T]^{-1}\mathbf{A}$$

with the condition at the final stage obtained as

$$\mathbf{P}_{k_f} = \mathbf{S}.$$

If the matrix Riccati difference equation is solved backward in time from $k = k_f$ to $k = 0$, certain "gain" functions are obtained which are stored after they are precomputed and applied to the physical system as it runs forward in real time. Thus we have designed a closed-loop optimal discrete system. Most of the remarks in the last chapters on the continuous linear regulator apply here. It is necessary that **Q**, **R**, and **S** be nonnegative definite in order for the second variation to be positive. Also, **R** must be positive definite since its inverse must exist to compute $\mathbf{u}_k$. The "gains" precomputed by this method are called "Kalman gains" and are instrumented as shown in Fig. 6.2-1. The closed-loop control is obtained from the prestored memory as

$$\mathbf{u}_k = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{A}^{-T}[\mathbf{P}_k - \mathbf{Q}]\mathbf{x}_k = -\mathbf{R}^{-1}\mathbf{B}^T[\mathbf{P}_{k+1}^{-1} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T]^{-1}\mathbf{A}\mathbf{x}_k$$

which is, of course, very similar to the way in which the closed-loop control is obtained for the continuous linear regulator.

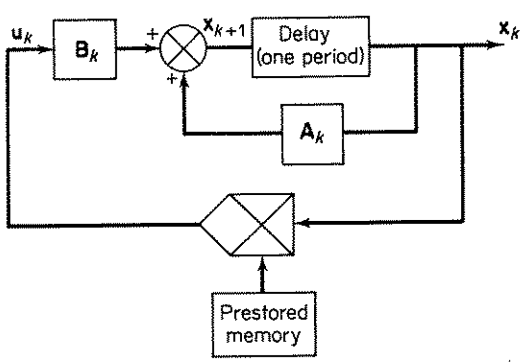

Fig. 6.2-1 Block diagram of closed-loop controller for discrete linear regulator, Example 6.2-1.

6.3 Comparison between the discrete and continuous maximum principle

Having discussed both the continuous and discrete maximum principle, we shall now inquire about the comparison and interconnections between the two. It is only natural to expect that both the continuous and discrete maximum principle will yield very similar (and perhaps the same) solutions

to a given problem. We shall see in this section that the two-point discrete boundary value problems which we solve are different in each case. For reasonable sample periods, however, the computational solutions for the two approaches will be essentially the same. Consider the Lagrange problem of the variational calculus. We desire to minimize

$$J = \int_{t_o}^{t_f} \phi(\mathbf{x}, \mathbf{u}, t)\, dt \qquad (6.3\text{-}1)$$

subject to the equality (vector) constraint

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \qquad (6.3\text{-}2)$$

$$\mathbf{x}(t_o) = \mathbf{x}_o. \qquad (6.3\text{-}3)$$

The TPBVP is obtained from the maximum principle as follows. We define the Hamiltonian

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \phi(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}, \mathbf{u}, t). \qquad (6.3\text{-}4)$$

The optimum control is determined by

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} = \frac{\partial \phi(\mathbf{x}, \mathbf{u}, t)}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{f}^T(\mathbf{x}, \mathbf{u}, t)}{\partial \mathbf{u}}\right]\boldsymbol{\lambda}(t). \qquad (6.3\text{-}5)$$

The adjoint equations and associated boundary conditions are

$$-\dot{\boldsymbol{\lambda}} = \frac{\partial H}{\partial \mathbf{x}} = \frac{\partial \phi(\mathbf{x}, \mathbf{u}, t)}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{f}^T(\mathbf{x}, \mathbf{u}, t)}{\partial \mathbf{x}}\right]\boldsymbol{\lambda}(t) \qquad (6.3\text{-}6)$$

$$\boldsymbol{\lambda}(t_f) = \mathbf{0}. \qquad (6.3\text{-}7)$$

Thus the continuous TPBVP to be solved is given by Eqs. (6.3-2) and (6.3-6), with the boundary conditions of Eqs. (6.3-3) and (6.3-7), and the coupling equation, Eq. (6.3-5). If a digital computer is used to solve this nonlinear TPBVP, with the first difference expression being used for $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\lambda}}$, we use the first difference approximations

$$\dot{\mathbf{x}}\big|_{t=kT} = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{T} = \frac{\mathbf{x}(\overline{k+1}T) - \mathbf{x}(kT)}{T} \qquad (6.3\text{-}8)$$

$$\dot{\boldsymbol{\lambda}}\big|_{t=kT} = \frac{\boldsymbol{\lambda}_{k+1} - \boldsymbol{\lambda}_k}{T} = \frac{\boldsymbol{\lambda}(\overline{k+1}T) - \boldsymbol{\lambda}(kT)}{T}. \qquad (6.3\text{-}9)$$

The resulting discrete TPBVP becomes

$$\mathbf{x}_{k+1} = \mathbf{x}_k + T\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k) \qquad (6.3\text{-}10)$$

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k - T\frac{\partial \phi(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k} - T\left[\frac{\partial \mathbf{f}^T(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k}\right]\boldsymbol{\lambda}_k \qquad (6.3\text{-}11)$$

$$\mathbf{x}_{k_o} = \mathbf{x}_o \qquad (6.3\text{-}12)$$

$$\boldsymbol{\lambda}_{k_f} = \mathbf{0} \qquad (6.3\text{-}13)$$

$$\frac{\partial \phi(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{u}_k} + \left[\frac{\partial \mathbf{f}^T(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{u}_k}\right]\boldsymbol{\lambda}_k = \mathbf{0}. \qquad (6.3\text{-}14)$$