10

DESIGN OF LINEAR FEEDBACK SYSTEMS

10-1 INTRODUCTION

The design of linear feedback systems, also called linear control systems, is certainly an important area of linear systems since we deal with many control systems in our everyday lives such as thermostats, automobile automatic speed controls, machine tool controls, etc.

There are basically two kinds of control systems, open-loop control and feedback (or closed-loop) control. Consider the functional block diagrams of two single-input and single-output cases of Figure 10-1.

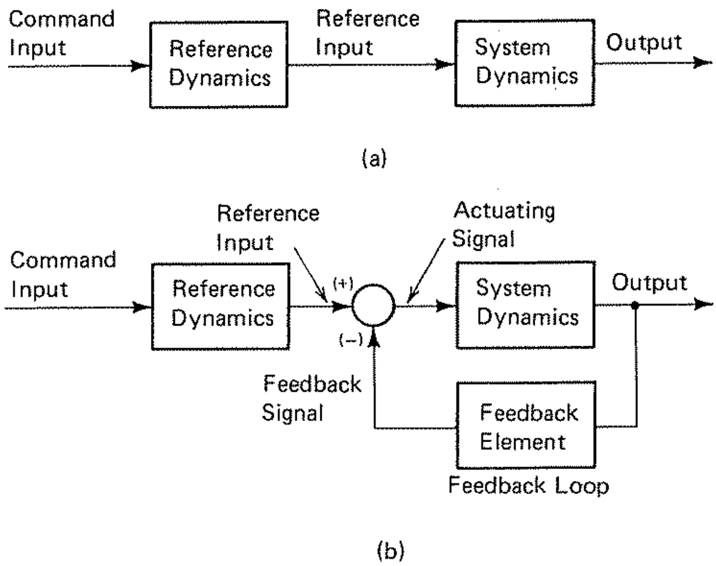

Figure 10-1. Functional Block Diagrams of Control Systems. (a) Open-loop, (b) Closed-loop

The system of Figure 10-1a is called an open-loop system since there is no feedback from the output to the input. The command input is converted to a reference input which is directly applied to the system dynamics to produce the output. If the desired output should "wander", no restoring action is possible in this configuration since there is no measurement of the output. The reader should note that the majority of systems presented in this book until this chapter have been open-loop systems. See Figure 3-28 and Figure 4-21 for block diagrams of the state-space representations of open-loop systems for continuous-time and discrete-time systems.

If the feedback loop is closed, then the actuating signal is affected by the output signal, and we call these systems feedback (or closed-loop) control systems as in Figure 10-1b. A command input is converted to a reference input and compared with the feedback signal which is a function of the output. The difference between the reference input and the feedback signal is applied to the system dynamics, or controlled element, resulting in the output. The designation feedback control implies that the action resulting from the comparison between the output and input quantities is necessary in order to maintain the output at the desired value.

The design of feedback systems consists of forcing the closed-loop pole locations to be suitably located. The proper pole locations for a "good" or acceptable design depend upon the design specifications concerning relative stability, speed of response, accuracy, and insensitivity to disturbance inputs. The details of these inter-relationships and pole locations are outside the scope of this book, and the interested reader is referred to the excellent texts (1-3).

This chapter considers the design of feedback compensators for linear, time-invariant, continuous-time systems, although many of the techniques apply to discrete systems in an analogous manner. In Section 10-2 we consider the concepts of state-variable feedback and output feedback and derive the closed-loop matrix descriptions. In Section 10-3 we study the effects of feedback on controllability, observability, and stability of the closed-loop systems. Assuming that all the state variables are accessible for feedback, in Section 10-4 state feedback is utilized to arbitrarily assign the closed-loop poles for controllable open-loop system dynamics. When the controlled element has inaccessible state variables, and a state feedback system is to be designed, we can derive an estimate of the state vector to utilize as a feedback signal rather than the true state vector. This state estimate and final system designs are discussed in Section 10-5.

10-2 STATE FEEDBACK AND OUTPUT FEEDBACK

There are two possible sources of feedback. These sources help classify control systems into the broad categories of state feedback systems and output feedback systems.

In state feedback the state is fed back into the input; in output feedback, the output is fed back into the input. Let us consider the general, time-invariant, continuous-time system described by Eqs. 10-1 and 10-2

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \tag{10-1}$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \tag{10-2}$$

where $\underline{x}$ is the n x 1 state vector, $\underline{u}$ is the p x 1 control vector, $\underline{y}$ is the q x 1 output vector, and $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are constant matrices with order n x n, n x p, q x n, and q x p, respectively.

Eqs. 10-1 and 10-2 represent the system to be controlled, sometimes referred to as the system dynamics or plant dynamics, and therefore, are unalterable. The state variable feedback system is shown in Figure 10-2, and the output feedback system is shown in Figure 10-3.

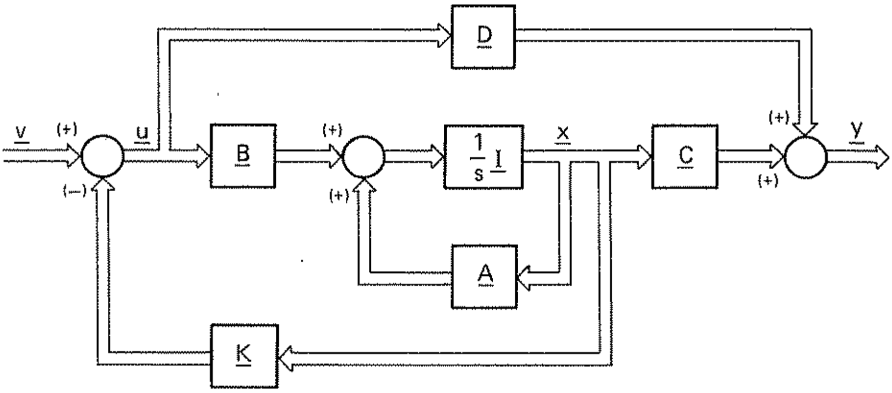

Figure 10-2. State Variable Feedback System

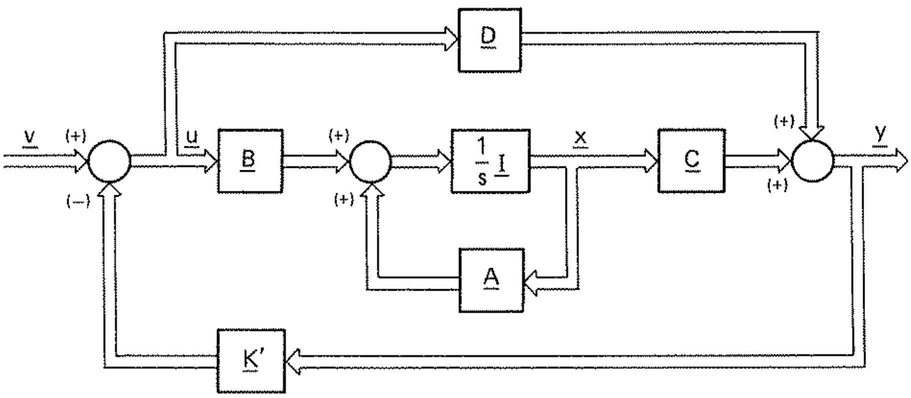

Figure 10-3. Output Feedback System

Thus, for the state feedback system, we have the defining equations

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u} \qquad (10\text{-}1)$$

$$\underline{y} = \underline{C}\ \underline{x} + \underline{D}\ \underline{u} \qquad (10\text{-}2)$$

$$\underline{u} = \underline{v} - \underline{K}\ \underline{x} \qquad (10\text{-}3)$$

where $\underline{v}$ is the p x 1 reference input vector, and $\underline{K}$ is the p x n feedback constant-gain matrix. Eq. 10-3 is called the control law. Modern-day optimal control is mainly concerned with how to physically implement it with real hardware. This solution for the "best" $\underline{K}$ is beyond the scope of this book, and the interested reader should see references (8-14) of Chapter 1.

For the output feedback case, we have the defining equations of

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \tag{10-1}$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \tag{10-2}$$

$$\underline{u} = \underline{v} - \underline{K}'\,\underline{y} \tag{10-4}$$

where $\underline{v}$ is the p x 1 reference input vector, and $\underline{K}'$ is the p x q feedback constant-gain matrix.

At this point one might ask why we use the state variable feedback when some of the state variables may not be available for measurement? There are numerous reasons some of which are:

1. Since the number of state variables is generally greater than the number of output variables, there is more room for design alternatives in state feedback than in output feedback.
2. The state $\underline{x}$ contains all the pertinent information about the system, and therefore, we wish to know what can be accomplished in this ideal case by using feedback.
3. There are cases for which all the state variables are accessible to measurement.
4. Several optimal control algorithms take the form of a state feedback control, and it is worthwhile to understand the effects of state feedback.
5. There are effective methods for estimating or reconstructing the state variables from the available inputs and outputs. (See Section 10-5.)

We can simplify the defining equations in each feedback case by eliminating $\underline{u}$. For the state feedback case, we have

$$\dot{\underline{x}} = [\underline{A} - \underline{B}\,\underline{K}]\underline{x} + \underline{B}\,\underline{v} \tag{10-5}$$

$$\underline{y} = [\underline{C} - \underline{D}\,\underline{K}]\underline{x} + \underline{D}\,\underline{v} \tag{10-6}$$

Let $\underline{A}_c \triangleq \underline{A} - \underline{B}\ \underline{K}$, and $\underline{C}_c \triangleq \underline{C} - \underline{D}\ \underline{K}$. Notice that Eqs. 10-5 and 10-6 are of the same form as Eqs. 10-1 and 10-2. If we are interested in the qualitative effects of feedback, then the important matrices are $[\underline{A} - \underline{B}\ \underline{K}]$, $\underline{B}$, $[\underline{C} - \underline{D}\ \underline{K}]$ and $\underline{D}$. Thus, stability of the state feedback system depends on the eigenvalues of $[\underline{A} - \underline{B}\ \underline{K}]$. Controllability depends on the pair of matrices $\{[\underline{A} - \underline{B}\ \underline{K}], \underline{B}\}$. Observability depends on the pair of matrices $\{[\underline{A} - \underline{B}\ \underline{K}], [\underline{C} - \underline{D}\ \underline{K}]\}$.

The output feedback system simplifies by substituting Eq. 10-4 into Eq. 10-2. Therefore,

$$\underline{y} = \underline{C}\ \underline{x} + \underline{D}\ \underline{v} - \underline{D}\ \underline{K}'\ \underline{y}$$

or

$$y = [\underline{I}_q + \underline{D}\ \underline{K}']^{-1}\{\underline{C}\ \underline{x} + \underline{D}\ \underline{v}\} \tag{10-7}$$

Substituting Eq. 10-7 into Eq. 10-1 yields

$$\dot{\underline{x}} = \{\underline{A} - \underline{BK}'[\underline{I}_q + \underline{DK}']^{-1}\underline{C}\}\underline{x} + \underline{B}\{\underline{I}_p - \underline{K}'[\underline{I}_q + \underline{DK}']^{-1}\underline{D}\}\underline{v} \tag{10-8}$$

Once again Eqs. 10-7 and 10-8 have the same general form of Eqs. 10-1 and 10-2, and we can make qualitative statements about the system with output feedback. Thus, stability depends on the eigenvalues of $\{\underline{A} - \underline{BK}'[\underline{I}_q + \underline{DK}']^{-1}\underline{C}\}$. Controllability depends on the pair of matrices $\{\underline{A} - \underline{BK}'[\underline{I}_q + \underline{DK}']^{-1}\underline{C}\}$ and $\underline{B}\{\underline{I}_p - \underline{K}'[\underline{I}_q + \underline{DK}']^{-1}\underline{D}\}$. Observability depends on the pair of matrices $\{\underline{A} - \underline{BK}'[\underline{I}_q + \underline{DK}']^{-1}\underline{C}\}$ and $[\underline{I}_q + \underline{DK}']^{-1}\underline{C}$.

Before considering, in general, the effects of feedback on system controllability, observability, and stability, let us consider an example to gain some insight.

Example 10-1 Consider a single-input, single-output, two-dimensional system described by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \quad y = [1 \quad 2]\underline{x}$$

The system without feedback is controllable, observable, and asymptotically stable since $\rho[\underline{B}, \underline{AB}] = 2$, $\rho[\underline{C}^T, \underline{A}^T\underline{C}^T] = 2$, and the characteristic equation is $\Delta(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ producing $\lambda_{1,2} = -1$ which are in the left-half complex plane. Let us now apply different feedback signals.

(a) Introduce a feedback signal of $u = v - [-2 \quad 1]\underline{x}$. Then, the system becomes using Eqs. 10-5 and 10-6

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}v, \quad y = [1 \quad 2]\underline{x}$$

The reader can verify that the new system is controllable and observable. The characteristic equation is now $\Delta(\lambda) = \lambda +$ $3\lambda - 1$ which, from the Routh criteria, has roots with positive real parts. Thus, the feedback system is unstable. To summarize, we have preserved controllability and observability but have "managed" to destabilize by using state feedback.

(b) Introduce a feedback signal of $u = v - [1 \quad 5/2]\underline{x}$. Then, the system becomes

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -9/2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}v, \quad y = [1 \quad 2]\underline{x}$$

The new system is controllable, asymptotically stable, and unobservable. In this case, we have preserved stability and controllability but have altered the system's observability by introducing state feedback.

(c) Let us try to choose a $\underline{K}$ such that controllability will be altered. Let it be $\underline{K} = [k_1 \quad k_2]^T$. Then

$$\rho[\underline{B}, (\underline{A} - \underline{B}\underline{K})\underline{B}] = \rho\begin{bmatrix} 1 & 0 \\ -k_2 - 2 & 1 \end{bmatrix} = 2$$

independent of k_1 or k_2. Thus, for this example, it is impossible to destroy the system controllability by introducing state feedback. ■

In Example 10-1 we saw how observability and stability could be altered by the use of state feedback, but we were unable to find a $\underline{K}$ to alter controllability. In Section 10-3 we discuss the more general aspects of these problems.

10-3 THE EFFECT OF FEEDBACK ON SYSTEM QUALITATIVE PROPERTIES

When designing a feedback system it is, of course, necessary to determine for the contemplated design whether it is controllable, observable, and stable. Let us assume in this section that all state variables are available (accessible) to be fed back.

10-3.1 Controllability

Theorem 10-1 The state feedback system given by Eqs. 10-5 and 10-6 is controllable for any feedback gain matrix $\underline{K}$ if and only if the system described by Eqs. 10-1 and 10-2 is controllable.

Proof: The controlled element is controllable if and only if $\rho[\underline{U}_s] = \rho[\underline{B}, \underline{AB}, \underline{A}^2\underline{B}, \ldots, \underline{A}^{n-1}\underline{B}] = n$. The state variable feedback system is controllable if and only if

$$\rho[\underline{U}_f] = \rho[\underline{B}, (\underline{A} - \underline{BK})\underline{B}, (\underline{A} - \underline{BK})^2\underline{B}, \ldots, (\underline{A} - \underline{BK})^{n-1}\underline{B}] = n \quad (10\text{-}9)$$

We need to show that these two ranks are both n. Let us expand Eq. 10-9 and examine the terms

$$\rho[\underline{B}, \underline{AB} - \underline{BKB}, \underline{A}^2\underline{B} - \underline{ABKB} - \underline{BKAB} + \underline{BKBKB}, \ldots, \text{etc.}] \overset{?}{=} n$$

The term $\underline{BKB}$ has columns which are linear combinations of $\underline{B}$ and, thus, are linearily dependent on $\underline{B}$. This adds no new information about rank. Similarily, the columns of $\underline{ABKB}$ are linearily dependent on the columns of $\underline{AB}$. All other terms, except $\underline{B}$ and terms of the type $\underline{A}^k\underline{B}$ where $k = 1, \ldots, n-1$, also do not contribute to the rank since their columns are linear combinations of terms of the type $\underline{A}^k\underline{B}$. Thus,

$$\rho[\underline{B}, \underline{AB}, \ldots, \underline{A}^{n-1}\underline{B}] = \rho[\underline{B}, (\underline{A} - \underline{BK})\underline{B}, \ldots, (\underline{A} - \underline{BK})^{n-1}\underline{B}] \quad (10\text{-}10)$$

Eq. 10-10 completes the proof since these ranks are equal.

Q.E.D.

Theorem 10-1 also holds for time-varying systems when state feedback is given as $\underline{u}(t) = \underline{v}(t) - \underline{K}(t)\underline{x}(t)$. For the system given by

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \\ \underline{y}(t) &= \underline{C}(t)\underline{x}(t) + \underline{D}(t)\underline{u}(t) \end{aligned} \quad (10\text{-}11)$$

this is summarized in Theorem 10-2.

Theorem 10-2 The controllability of the multivariable, time-varying system of Eq. 10-11 is invariant under any state feedback of the form $\underline{u}(t) = \underline{v}(t) - \underline{K}(t)\underline{x}(t)$.

The proof is left as an exercise for the reader.

For controllability of the output feedback system, we must check the rank of $\underline{U}_o = n$ where $\underline{U}_o$ is defined as

$$\underline{U}_o \triangleq [\underline{B}_o, \ \underline{A}_o\underline{B}_o \ \underline{A}_o{}^2\underline{B}_o, \ \ldots, \ \underline{A}_o{}^{n-1}\underline{B}_o]$$

and

$$\underline{A}_o \triangleq \underline{A} - \underline{B}\underline{K}'[\underline{I}_q + \underline{D}\underline{K}']^{-1}\underline{C}$$

$$\underline{B}_o \triangleq B\{I_p - \underline{K}'[\underline{I}_q + \underline{D}\underline{K}']^{-1}\underline{D}\}$$

If the rank of $\{\underline{I}_p - \underline{K}'[\underline{I}_q + \underline{D}\underline{K}]^{-1}\underline{D}\}$ is p, then the output feedback system is controllable since we can let $\underline{K}$ in Theorem 10-1 be equal to $\underline{K}'[\underline{I}_q + \underline{D}\underline{K}']^{-1}\underline{C}$.

Example 10-2. Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ e^{-t} & 2 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u, \quad y = [1 \quad 0]\underline{x}$$

(a) Let $u(t) = v(t) - [1 \quad t]\underline{x}$. Let $\underline{R}_o = \underline{B} = [1 \quad 1]^T$ and $\underline{R}_1 = -\underline{A}\ \underline{R}_o + d\ \underline{R}_o/dt = [-1 \quad -2 - e^{-t}]^T$. Then

$$\rho[\underline{R}_o, \ \underline{R}_1] = \rho\begin{bmatrix} 1 & -1 \\ 1 & -2 - e^{-t} \end{bmatrix} = 2 \text{ for all } t$$

In Theorem 8-8a the open-loop system is controllable for all t. The state feedback system reduces to

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 1 - t \\ e^{-t} - 1 & 2 - t \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}v, \quad y = [1 \quad 0]\underline{x}$$

Theorem 10-2 states that this new system is also controllable.

Let us check it by Theorem 8-8a where $\underline{R}_o = \underline{B} = [1 \quad 1]^T$ and

$$\rho[\underline{R}_o, \underline{R}_1] = \rho\begin{bmatrix} 1 & t \\ 1 & -e^{-t} - 1 + t \end{bmatrix} = 2 \text{ for all } t$$

(b) Let $u = v - ky$. By using Eq. 10-8 this output feedback system reduces to

$$\dot{\underline{x}} = \begin{bmatrix} -k & 1 \\ e^{-k} - k & 2 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}v, \quad y = [1 \quad 0]\underline{x}$$

It is controllable since $\underline{R}_o = [1 \quad 1]^T$, and $\underline{R}_1 = [k - 1 \quad -2 + k - e^{-t}]^T$. Thus, $\det[\underline{R}_o, \underline{R}_1] = -1 - e^{-t}$ which is never zero. By Theorem 8-8a the output feedback system is controllable. ■

10-3.2 Observability

We have already shown in Example 10-1 that observability is not preserved under state feedback. Thus, adding state feedback can cause a loss of observability. However, output feedback does preserve observability. The proof is similar to but more complex than that of Theorem 10-1. (See reference (4)).

10-3.3 Stability

In Example 10-1a we generated an unstable system by using state feedback on an asymptotically stable system. In many design problems we are interested in the opposite effect. We want to stabilize an unstable system by the use of state or output feedback. In fact, one of the most important properties of state feedback is that it can be used to control the eigenvalues of the closed-loop control system. This control of the eigenvalues is called <u>pole placement</u> or <u>pole assignment</u>.

Many of the above discussions and theorems apply equally well to discrete-time systems. Theorems 10-1 and 10-2 can be applied to discrete-time systems in a straightforward way because of the similarities in their state-space descriptions. Observability is not preserved under state feedback but is under output feedback. For stabilization we are interested in moving the poles of the transfer function (the zeros of the characteristic equation) inside the unit circle in the complex plane.

In summary, state and output feedback both preserve controllability for time-varying, continuous-time and discrete-time systems. Output feedback preserved observability for all four types of systems previously mentioned. Stability is affected by both types of feedback.

10-4 POLE PLACEMENT USING STATE FEEDBACK

The problem of assigning arbitrary closed-loop poles (zeros of $\det[\lambda \underline{I} - (\underline{A} - \underline{BK})]$ by means of state variable feedback has been the subject of considerable research since 1965. It was first demonstrated by Brockett (5) that, for a single input, controllable system, there is a simple and unique $\underline{K}$ matrix. A controllable, multiple input system, however, has no unique solution, but arbitrary pole placement is possible if and only if the system is completely controllable (6). Since then, many investigators have proposed algorithms for realizing this closed-loop pole assignment. Typically, these algorithms assume that the open-loop system has been transformed to a special form, usually the phase-variable form also called the controllable-canonical form, by use of a similarity transformation, $\hat{\underline{A}} = \underline{P}^{-1}\underline{AP}$. Techniques of this type or slight variations are found in references (7-11)

Recently, techniques have been developed for multi-input, multi-output systems which do not require this similarity transformation. References (12, 13, and 4 pages 311-313) are three of these techniques. We have selected the general technique of Brogan (4) for presentation since it seems straightforward and computationally efficient. But before discussing the general technique, let us consider an obvious "brute force" technique as follows:

1. First compute the characteristic polynomial of $\{\underline{A} - \underline{BK}\}$ in terms of the unknown np components of the $\underline{K}$ matrix.
2. If the matrix $\{\underline{A} - \underline{BK}\}$ has a set of eigenvalues $\{\hat{\lambda}_i$ for $i = 1, 2, \ldots, n\}$, its characteristic equation is equal to

$$\Delta(s) = \prod_{i=1}^{n} (s - \hat{\lambda}_i) = 0$$

3. Equate the coefficients of the powers of s to generate a set of n equations to solve for the np components of $\underline{K}$.

For multiple-inputs this technique does not yield a unique solution since there are n equations in np unknown components in the $\underline{K}$ matrix. In addition, this set of n equations is usually not linear in the components of $\underline{K}$. Let us illustrate these features with two examples.

Example 10-3 Consider the position control of a rotating part of inertia J with viscous friction B as shown in Figure 10-4.

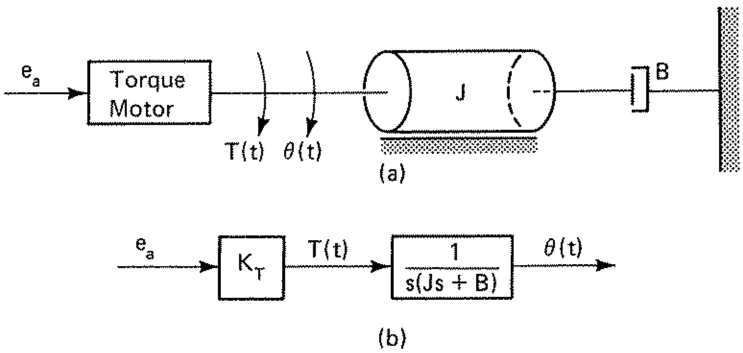

Figure 10-4. Inertia and Friction Load

Let us assume that the torque, T(t), is applied by a DC motor which converts an electrical input voltage signal, e_a, into an output torque by the relationship $T(t) = K_T e_a$. The differential equation of motion of the part is $J\ddot{\theta} + B\dot{\theta} = T(t)$, or if we use e_a as the input, $J\ddot{\theta} + B\dot{\theta} = K_T e_a$. The transfer function is then

$$\frac{\theta}{e_a}(s) = \frac{K_T}{s(Js + B)}$$

If we let $\theta \triangleq x_1$, $\dot{\theta} \triangleq x_2$ and $e_a \triangleq u$, we can obtain a state-space description of the open-loop control of θ by e_a as

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -B/J \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K_T/J \end{bmatrix} u, \quad y = [1 \quad 0]\underline{x}$$

The reader can easily verify that the open-loop system is controllable but unobservable. Notice that $\underline{A}$ has eigenvalues of 0 and B/J. The system is, therefore, stable but not asymptotically stable. It is also not BIBO stable since a step input in voltage would produce a ramp-type output $\theta(t)$. Thus, it is not totally stable. Let us assume that we can utilize state-variable feedback to place the closed-loop poles in the left-half plane and attempt a stabilization. Figure 10-5 shows a block diagram of this proposed state-feedback solution. Physically, a tachometer could provide the speed feedback for k_2 while a potentiometer might provide the k_1 position feedback.

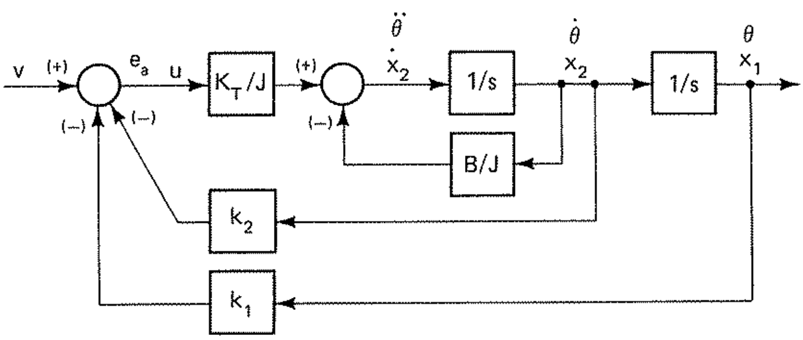

Figure 10-5. Position State Variable Feedback System

The state feedback system results in a state-space description by using Eqs. 10-5 and 10-6 of

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_1 K_T}{J} & \dfrac{-B}{J} - \dfrac{k_2 K_T}{J} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \dfrac{K_T}{J} \end{bmatrix} v$$

$$y = [1 \quad 0]\underline{x}$$

If $k_1 = 0$ (i.e., no position feedback), then the system is still unobservable and not totally stable. If $k_1 \neq 0$, it is controllable and observable. The closed loop transfer is then, calculated from $\frac{Y}{V}(s) = \underline{C}_c(s\underline{I} - \underline{A}_c)\underline{B}$ as

$$\frac{Y}{V}(s) = [1 \quad 0]\begin{bmatrix} s & 1 \\ \frac{k_1 K_T}{J} & s + \frac{B}{J} + \frac{k_2 K_T}{J} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{K_T}{J} \end{bmatrix}$$

$$= \frac{\frac{K_T}{J}}{s^2 + s\left[\frac{B}{J} + k_2 \frac{K_T}{J}\right] + \frac{k_1 K_T}{J}}$$

By using the Routh-Hurwitz test, this system will be stable if and only if $k_1 K_T/J > 0$ and $(B/J + k_2 K_T/J) > 0$. Note also that we can select the poles of the transfer anywhere we desire. Let $B = 0.2$, $J = 2$, and $K_T = 0.25$. Let us place the poles at $-1 - j$ and $-1 + j$. Thus, the characteristic equation is $s^2 + 2s + 2$. Matching coefficients, we have $k_1 = 16.0$ and $k_2 = 15.2$.

Example 10-4 Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\underline{u}, \quad \underline{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\underline{x}$$

$$\underline{u} = \underline{v} - \underline{K}\underline{x} = \underline{v} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}\underline{x}$$

It is controllable, observable, irreducible, but unstable since the eigenvalues of $\underline{A}$ are $\lambda = 1, -2$. Let us position the closed-loop poles at $\lambda = -1, -2$. The closed-loop characteristic equation is then

$$\Delta(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 =$$

$$\det[\lambda\underline{I} - (\underline{A} - \underline{B}\underline{K})] = \det\begin{bmatrix} \lambda + k_3 & k_4 + 1 \\ 2 + 2k_1 + k_3 & \lambda + (1 + 2k_2 + k_4) \end{bmatrix}$$

There is no unique solution since there are only two coefficients of s to match and four $\underline{K}$ components. If the reader expands the determinant and matches coefficients, he(she) can see nonlinear equations. One solution is $k_1 = -3/2$, $k_2 = 1/2$, $k_3 = 0$, and $k_4 = 1$. Another is $k_1 = -2$, $k_2 = 1$, $k_3 = 0$, and $k_4 = 0$. ■

The remainder of this section follows Brogan (4) in presenting the general, state-feedback, pole assignment problem. The eigenvalues of the closed-loop state feedback system are roots of

$$\Delta'(\lambda) \triangleq |\lambda \underline{I}_n - (\underline{A} - \underline{BK})| = 0 \qquad (10\text{-}12)$$

which can be rewritten as

$$\Delta'(\lambda) = |(\lambda \underline{I}_n - \underline{A})[\underline{I}_n + (\lambda \underline{I}_n - \underline{A})^{-1}\underline{BK}]|$$

$$= |\lambda \underline{I}_n - \underline{A}| \cdot |\underline{I}_n + (\lambda \underline{I}_n - \underline{A})^{-1}\underline{BK}| = \Delta(\lambda) \cdot |\underline{I}_n + (\lambda \underline{I}_n - \underline{A})^{-1}\underline{BK}| \qquad (10\text{-}13)$$

since we have previously defined $\Delta(\lambda)$ as $|\lambda \underline{I}_n - \underline{A}|$. Note also that $(\lambda \underline{I} - \underline{A})^{-1}$ has the same form as the state transition matrix $\underline{\Phi}(s)$. Thus, let us rewrite Eq. 10-13 as

$$\Delta'(\lambda) = \Delta(\lambda) \cdot |\underline{I}_n + \underline{\Phi}(\lambda)\underline{BK}| = \Delta(\lambda) \cdot |\underline{I}_p + \underline{K\Phi}(\lambda)\underline{B}| \qquad (10\text{-}14)$$

The second form of Eq. 10-14 is possible from the determinant identity

$$|\underline{I}_n + \underline{FH}| = |\underline{I}_p + \underline{HF}| \qquad (10\text{-}15)$$

where $\underline{F} \triangleq \underline{\Phi}(\lambda)\underline{B}$, and $\underline{H} \triangleq \underline{K}$. The matrix $\underline{K}$ must be selected so that $\Delta'(\hat{\lambda}_i) = 0$ for each $\hat{\lambda}_i$ representing a desired closed-loop pole. This will be accomplished by forcing the p x p determinant $|\underline{I}_p + \underline{K\Phi}(\lambda)\underline{B}|$ to vanish. If any desired $\hat{\lambda}_i$ is also a root of $\Delta(\lambda)$, the same procedure holds but a limiting process is necessary. (See Example 10-8). The determinant of $\underline{I}_p + \underline{K\Phi}(\lambda)\underline{B}$ will be zero if any column or row is zero. Let us force a column to zero here. Define the j-th column of $\underline{I}_p$ as $\underline{e}_j$ and define $\underline{\Omega}(\lambda_i) = \underline{\Phi}(\lambda_i)\underline{B}$ with the j-th column being $\underline{\Omega}_j$. Then $\hat{\lambda}_i$ is a root of $\Delta'(\lambda)$ if $\underline{K}$ is selected to satisfy $\underline{e}_j + \underline{K\Omega}_j(\hat{\lambda}_i) = \underline{0}$, since this forces column j to vanish.

Thus

$$-\underline{K}\underline{\Omega}_j(\hat{\lambda}_i) = \underline{e}_j \qquad (10\text{-}16)$$

Eq. 10-16 by itself is not sufficient to determine $\underline{K}$. However, if an independent equation of this type can be found for each desired root $\hat{\lambda}_i$, then $\underline{K}$ can be determined. Controllability of ($\underline{A}$, $\underline{B}$) is sufficient to guarantee that rank $\underline{\Omega}(\hat{\lambda}_i) = p$ for each $\hat{\lambda}_i$ reference (11, p. 278). If all the desired $\hat{\lambda}_i$'s are distinct, it will always be possible to find n linearily independent columns $\{\underline{\Omega}_{j_1}(\hat{\lambda}_1), \underline{\Omega}_{j_2}(\hat{\lambda}_2), \ldots, \underline{\Omega}_{j_n}(\hat{\lambda}_n)\}$ from the n x np matrix $[\underline{\Omega}(\hat{\lambda}_1), \underline{\Omega}(\hat{\lambda}_2), \ldots, \underline{\Omega}(\hat{\lambda}_n)]$. Then,

$$-K[\underline{\Omega}_{j_1}(\hat{\lambda}_1), \underline{\Omega}_{j_2}(\hat{\lambda}_2), \ldots, \underline{\Omega}_{j_n}(\hat{\lambda}_n)] = [\underline{e}_{j_1}, \underline{e}_{j_2}, \ldots, \underline{e}_{j_n}]$$

or

$$\underline{K} = -[\underline{e}_{j_1}, \underline{e}_{j_2}, \ldots, \underline{e}_{j_n}][\underline{\Omega}_{j_1}(\hat{\lambda}_1), \underline{\Omega}_{j_2}(\hat{\lambda}_2), \ldots, \underline{\Omega}_{j_n}(\hat{\lambda}_n)]^{-1} \qquad (10\text{-}17)$$

Example 10-5 Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \quad y = [1 \quad 1]\underline{x}, \quad u = v - [k_1 \quad k_2]\underline{x}$$

and place the closed-loop poles at $\lambda = -4$ and -2.

Solution: The system is controllable since

$$\rho[\underline{B}, \underline{AB}] = \rho\begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix} = 2$$

Thus, we are guaranteed that the feedback gain matrix $\underline{K}$ exists and is unique. Then,

$$\underline{\Phi}(\lambda) = (\lambda\underline{I} - \underline{A})^{-1} = \frac{\begin{bmatrix} \lambda + 6 & 1 \\ -5 & \lambda \end{bmatrix}}{\lambda^2 + 6\lambda + 5}$$

and

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\underline{B} = \frac{\begin{bmatrix} 1 \\ \lambda \end{bmatrix}}{\lambda^2 + 6\lambda + 5}$$

Then $\underline{\Omega}(-2) = [-1/3 \quad 2/3]^T$, and $\underline{\Omega}(-4) = [-1/3 \quad 4/3]^T$. Note that $\underline{\Omega}(-2)$ and $\underline{\Omega}(-4)$ are linearily independent. Then Eq. 10-17 yields

$$\underline{K} = -[1 \quad 1] \begin{bmatrix} -1/3 & -1/3 \\ 2/3 & 4/3 \end{bmatrix}^{-1} = [3 \quad 0]$$

<u>Example 10-6</u> Let us reconsider Example 10-3 where

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad \underline{B} = [0 \quad 0.125]^T$$

in light of Eq. 10-17.

<u>Solution</u>:

$$\underline{\Phi}(\lambda) = [\lambda\underline{I} - \underline{A}]^{-1} = \frac{\begin{bmatrix} \lambda + 0.1 & 1 \\ 0 & \lambda \end{bmatrix}}{\lambda(\lambda + 0.1)}$$

Since there is only one input (p = 1), $\underline{e}_{j_1} = \underline{e}_{j_2} = 1$. The desired roots were $\hat{\lambda}_1 = -1 - j$ and $\hat{\lambda}_2 = -1 + j$. Then

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\underline{B} = \frac{[0.125 \quad 0.125\lambda]^T}{\lambda(\lambda + 0.1)}$$

Substituting in $\hat{\lambda}_1$ and $\hat{\lambda}_2$ yields

$$\underline{\Omega}(\hat{\lambda}_1) = \frac{0.125}{2(1.81)} \quad [-0.1 - 1.9j \qquad -1.8 + 2j]^T$$

$$\underline{\Omega}(\hat{\lambda}_2) = \frac{0.125}{2(1.81)} \quad [-0.1 + 1.9j \qquad -1.8 - 2j]^T$$

Applying Eq. 10-17 yields

$$\underline{K} = -[1 \quad 1][\underline{\Omega}(\hat{\lambda}_1),\ \underline{\Omega}(\hat{\lambda}_2)]^{-1} = [16.0 \quad 15.2]$$

There should be no doubt that in a single-input, single-output case with desired complex poles, this technique is far more tedious than the "brute" force technique.

Example 10-7 Consider the system described by

$$\underline{A} = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \underline{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the desired roots of $\hat{\lambda}_1 = -1$ and $\hat{\lambda}_2 = -2$.

Solution:

$$\underline{\Phi}(\lambda) = (\lambda\underline{I} - \underline{A})^{-1} = \frac{\begin{bmatrix} \lambda - 3 & 1 \\ 2 & \lambda + 1 \end{bmatrix}}{\lambda^2 - 2\lambda - 5}$$

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\underline{B} = \frac{\begin{bmatrix} 1 & \lambda - 3 \\ \lambda + 1 & 2 \end{bmatrix}}{\lambda^2 - 2\lambda - 5} = [\underline{\Omega}_1(\lambda),\ \underline{\Omega}_2(\lambda)]$$

and

$$\underline{\Omega}(\hat{\lambda}_1) = \begin{bmatrix} -1/2 & 2 \\ 0 & 1 \end{bmatrix}, \quad \underline{\Omega}(\hat{\lambda}_2) = \begin{bmatrix} 1/3 & -5/3 \\ -1/3 & 2/3 \end{bmatrix}$$

We can choose any two linearily independent columns such as $\underline{\Omega}_1(\hat{\lambda}_1) = [-1/2 \quad 0]^T$ and $\underline{\Omega}_2(\hat{\lambda}_2) = [-5/3 \quad 2/3]^T$ or $\underline{\Omega}_1(\hat{\lambda}_1)$ and $\underline{\Omega}_1(\hat{\lambda}_2) = [1/3 \quad -1/3]^T$ to substitute into Eq. 10-17. Let us first choose $\underline{\Omega}_1(\hat{\lambda}_1)$ and $\underline{\Omega}_2(\hat{\lambda}_2)$. Then, since we have chosen the first and second columns of $\underline{\Omega}(\lambda)$, $\underline{e}_{1_1} = [1 \quad 0]^T$ and $\underline{e}_{1_2} = [0 \quad 1]^T$. $\underline{K}$ is then calculated from Eq. 10-17 as

$$\underline{K} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & -5/3 \\ 0 & 2/3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 5 \\ 0 & -3/2 \end{bmatrix}$$

If we now choose $\underline{\Omega}_2(\hat{\lambda}_1)$ and $\underline{\Omega}_1(\hat{\lambda}_2)$, then

$$\underline{K} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1/3 \\ -1 & -1/3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 6 \\ -1 & -1 \end{bmatrix}$$

Both feedback gains $\underline{K}$ give systems with closed-loop poles at $\hat{\lambda}_1 = -1$ and $\hat{\lambda}_2 = -2$.

Example 10-8 This is an example to illustrate the limiting process when one or more of the open-loop poles correspond to one or more of the desired closed-loop poles. Consider

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u, \quad y = [1 \quad 0]\underline{x}$$

which is controllable and observable. Find a state feedback gain $\underline{K} = [k_1 \quad k_2]$ such that $\hat{\lambda}_1 = -1$ and $\hat{\lambda}_2 = -8$.

Solution: Inverting $(\lambda\underline{I} - \underline{A})$ yields

$$\underline{\Phi}(\lambda) = \frac{\begin{bmatrix} \lambda + 4 & 1 \\ -3 & \lambda \end{bmatrix}}{(\lambda + 1)(\lambda + 3)}$$

Then

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\underline{B} = \frac{\begin{bmatrix} \lambda + 6 \\ 2\lambda - 3 \end{bmatrix}}{(\lambda + 1)(\lambda + 3)}$$

Since $\hat{\lambda}_1 = -1$ is a pole of the open loop, $\underline{\Omega}(-1)$ yields an infinite column.. Let us use a limiting process, and assume that $\beta = 1/(1 + \lambda)$. Then by Eq. 10-17

$$\underline{K} = -[1 \quad 1][\underline{\Omega}(-1), \ \underline{\Omega}(-8)]^{-1} = -[1 \quad 1]\ \frac{1}{105\beta}\begin{bmatrix} 38 & -4 \\ -175\beta & -175\beta \end{bmatrix} =$$

$$\frac{1}{105\beta}\ [-38 + 175\beta \qquad 4 + 175\beta]$$

Then, if we take the limit as $\lambda = -1$, β approaches infinity, and $\underline{K}$ becomes

$$\underline{K} = [5/3 \quad 5/3]$$

■

When repeated closed-loop poles are desired, the pole assignment technique may not yield n linearily independent columns of $\underline{\Omega}_j(\lambda_j)$ for Eq. 10-17. A derivative-type modification is required. Assume that $\hat{\lambda}_i$ is a desired pole with multiplicity k where k < n. Then

$$\left.\frac{d^{\gamma}\Delta'(\lambda)}{d\lambda^{\gamma}}\right|_{\lambda = \hat{\lambda}_i} = 0 \text{ for } \gamma = 0, 1, 2, \ldots, k - 1$$

which can be used to generate additional independent columns. This procedure is analogous to the Cayley-Hamilton technique for multiple eigenvalued matrices of Chapter 5. If we differentiate Eq. 10-16 with respect to λ and evaluate at $\lambda = \hat{\lambda}_i$, then we have

$$\underline{K} \left.\frac{d\,\underline{\Omega}_j}{d\lambda}\right|_{\lambda = \hat{\lambda}_i} = \underline{0} \qquad (10\text{-}18)$$

Successive differentiation will produce

$$\underline{K} \left.\frac{d^{\gamma}\,\underline{\Omega}_j}{d\lambda^{\gamma}}\right|_{\lambda = \hat{\lambda}_i} = \underline{0} \text{ for } \gamma = 0, 1, 2, \ldots, k - 1 \qquad (10\text{-}19)$$

Thus, for repeated closed-loop poles, we may need equations of the type Eq. 10-17 and derivative equations of the type Eq. 10-19.

Example 10-9 For the system of Example 10-8, design a state variable feedback system to place the closed-loop poles at $\hat{\lambda}_1 = \hat{\lambda}_2 = -2$.

Solution: We can find only one linearily independent column from $\Omega(-2)$ which is $\underline{\Omega}(-2) = [-4 \quad 7]^T$. We can generate an equation from Eq. 10-18 as

$$\left.\frac{d\Omega(\lambda)}{d\lambda}\right|_{\lambda=-2} = \left.\frac{d}{d\lambda}\left\{\frac{1}{\lambda^2 + 4\lambda + 3}\begin{bmatrix}\lambda + 6\\ 2\lambda - 3\end{bmatrix}\right\}\right|_{\lambda=-2} = \begin{bmatrix}-1\\ -2\end{bmatrix}$$

Then, $\underline{K}$ can be calculated from Eq. 10-17 and Eq. 10-18 since

$$\underline{K}\,[-4 \quad 7]^T = -1 \quad \text{and} \quad \underline{K}\,[-1 \quad -2]^T = 0$$

After combining, we can write the two equations as

$$\underline{K}\begin{bmatrix}-4 & -1\\ 7 & -2\end{bmatrix} = -[1 \quad 0]$$

Solving for $\underline{K}$ yields

$$\underline{K} = [2/15 \quad -1/15]$$

■

State feedback pole placement has a number of possible defects: (1) The compensation is in the feedback loop rather than in the direct path (cascade mode), and experience has shown that cascade compensation is usually better. (2) If a similarity transformation technique is used, the solution allows little engineering feeling for the system. (3) The closed- loop system may be quite sensitive to small variations in plant (controlled system) parameters. (4) All the state variables must be available for measurement. This last restriction is commonly encountered in practice because many systems occur in which only the output or only some components of the state vector are accessible for feedback purposes. This presents two possible questions: (1) Can we not develop techniques similar to those of this section for pole assignment using output feedback? and (2) Can we reconstruct or obtain a "good" estimate of the state vector from the available outputs? The answer to question (1) is yes, and Problem 15 gives an outline of these output feedback results. Question (2) is dealt with in Section 10-5.

10-5 STATE ESTIMATORS AND OBSERVER SYSTEMS

In the previous sections we introduced state feedback assuming that all the state variables were available to be fed back. In practice this assumption is not always met either because all the state variables are not accessible to direct measurement or because the number of measurement devices is limited, possibly due to total cost considerations. Thus, in order to utilize state feedback, we must obtain a "reasonable estimate" of the state vector $\underline{x}(t)$. In this section we will discuss various techniques to reconstruct the state vector by using the available inputs and outputs of the system dynamics (plant) to drive a device called a state estimator.

It is desired to obtain a "good" estimate of the state $\underline{x}(t)$ given a knowledge of the output $\underline{y}(t)$, the input $\underline{u}(t)$, and the system matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ from Eqs. 10-1 and 10-2. Of course, the system must be observable. Without loss of generality, let us assume that $\underline{D} = \underline{0}$. If $\underline{D} \neq \underline{0}$, an equivalent output $\underline{y}' = \underline{y} - \underline{D}\underline{u}$ can be used for $\underline{y}$ in the following discussions. Since we know the matrices $\underline{A}$, $\underline{B}$, and $\underline{C}$, we can simulate a model that has an accessible state vector and the same dynamics as the original system. Letting $\hat{\underline{x}}^{+}$ denote the state vector of the model and $\underline{y}$ the model output, we drive it with the same input $\underline{u}$. Then,

$$\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{B}\,\underline{u}, \quad \hat{\underline{y}} = \underline{C}\,\hat{\underline{x}}$$

Figure 10-6 illustrates this model usually referred to as the open-loop estimator.

Let $\underline{x}_e$ denote the error in our estimate so that

$$\underline{x}_e \triangleq \underline{x} - \hat{\underline{x}} \tag{10-20}$$

+The circumflex will be used in the section to indicate an estimate of a quantity, e.g., $\hat{\underline{x}}$ is an estimate of $\underline{x}$. The reader should not confuse this with the notation used for equivalent systems in Sections 3-13 and 4-17.

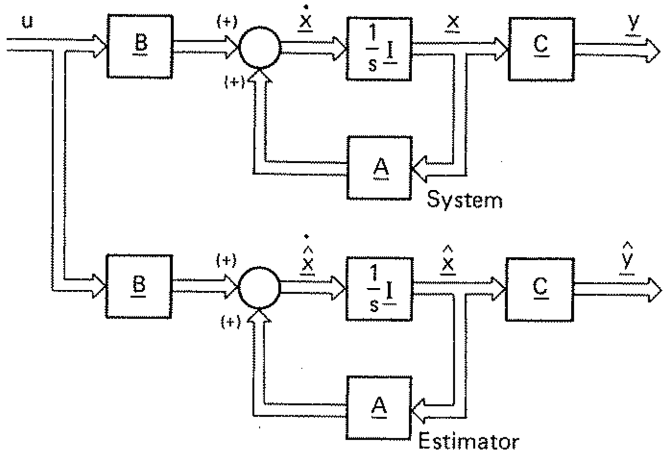

Figure 10-6. An Open-Loop State Estimator

Then

$$\dot{\underline{x}}_e = \dot{\underline{x}} - \dot{\hat{\underline{x}}} = (\underline{B}\underline{u} + \underline{A}\underline{x}) - (\underline{B}\underline{u} + \underline{A}\hat{\underline{x}}) = \underline{A}(\underline{x} - \hat{\underline{x}}) = \underline{A}\,\underline{x}_e \qquad (10\text{-}21)$$

and the error dynamics are determined by $\underline{A}$ over which we have no control. The solution of Eq. 10-21 is

$$\underline{x}_e(t) = e^{\underline{A}(t - t_o)}\,\underline{x}_e(t_o)$$

If we set the initial condition of the model exactly so that $\underline{x}_e(t_o) = \underline{0}$, the model will estimate the state exactly with no error. If, however, $\underline{x}_e(t_o) \neq \underline{0}$, then the error varies as $e^{\underline{A}(t - t_o)}$. If $\underline{A}$ has eigenvalues with negative real parts, then Eq. 10-21 is asymptotically stable, and the error dies out exponentially making the estimate better as time progresses. If $\underline{A}$ has eigenvalues with positive real parts, then any small error in our initial guess of $\hat{\underline{x}}(t_o)$ will grow rapidly and produce an unstable model. This open-loop dependence on the matrix $\underline{A}$ generally makes this an unacceptable model.

Let us "close" the loop around our model for a new improved estimator. Since both $\underline{y}$ and $\hat{\underline{y}}$ are available in the open-loop estimator, we may compare them and use this difference as a corrective term.

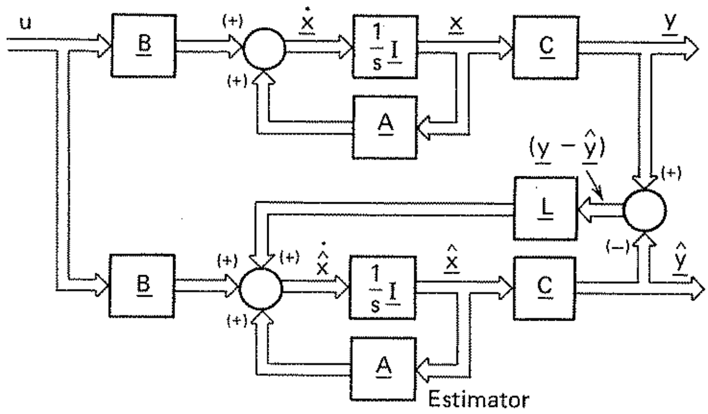

Figure 10-7. Asymptotic State Estimator

Figure 10-7 shows such a model usually referred to as the <u>asymptotic state estimator</u>. The n x q matrix $\underline{L}$ is necessary to convert the difference $(\underline{y} - \hat{\underline{y}})$ which is q x 1 to an n x 1 matrix in order to sum it with $\underline{B}\ \underline{u}$ and $\underline{A}\ \underline{x}$. Let us analyze this new system for errors. The closed loop estimator model has the equations

$$\dot{\hat{\underline{x}}} = \underline{A}\ \hat{\underline{x}} + \underline{B}\ \underline{u} + \underline{L}(\underline{y} - \hat{\underline{y}}),\ \hat{\underline{y}} = \underline{C}\ \hat{\underline{x}}$$

which can be combined to yield

$$\dot{\hat{\underline{x}}} = [\underline{A} - \underline{L}\ \underline{C}]\hat{\underline{x}} + \underline{B}\ \underline{u} + \underline{L}\ \underline{y}$$

If the error is defined by $\underline{x}_e \triangleq \underline{x} - \hat{\underline{x}}$, we have

$$\dot{\underline{x}}_e = \dot{\underline{x}} - \dot{\hat{\underline{x}}} = [\underline{A}\ \underline{x} + \underline{B}\ \underline{u}] - [\underline{A} - \underline{L}\ \underline{C}]\hat{\underline{x}} - [\underline{B}\ \underline{u} + \underline{L}\ \underline{y}]$$

$$= \underline{A}[\underline{x} - \hat{\underline{x}}] + \underline{L}\ \underline{C}\ \hat{\underline{x}} - \underline{L}\ \underline{y} = \underline{A}[\underline{x} - \hat{\underline{x}}] + \underline{L}\ \underline{C}\ \hat{\underline{x}} - \underline{L}\ \underline{C}\ \underline{x} = [\underline{A} - \underline{L}\ \underline{C}]\underline{x}_e \tag{10-22}$$

If the eigenvalues of $\underline{A} - \underline{LC}$ can be chosen arbitrarily, then we can control the behavior of $\underline{x}_e$. For example, if all the eigenvalues of $\underline{A} - \underline{LC}$ have negative real parts that are smaller than $-\sigma$, then all the components of $\underline{x}_e$ will decay to zero at the rate $e^{-\sigma t}$. Even if there is a large error between $\hat{\underline{x}}(t_o)$ and $\underline{x}(t_o)$, $\hat{\underline{x}}$ will approach $\underline{x}$ rapidly. Theorem 10-3 shows that observability of the system dynamics is a necessary and sufficient condition for the arbitrary placement of the eigenvalues of the asymptotic estimator.

Theorem 10-3 An asymptotic state estimator, described by Figure 10-7, can be constructed with arbitrary eigenvalues if and only if the controlled element (plant) is observable.

Proof: In Section 10-4 we saw that we could arbitrarily place the poles of the closed-loop system using state feedback if and only if the controlled element was controllable. This required finding a $\underline{K}$ such that the roots of $\det[\lambda\underline{I} - (\underline{A} - \underline{BK})]$ were anywhere that we desired. Here we are trying to place the roots of $\det[\lambda\underline{I} - (\underline{A} - \underline{LC})]$. Let us try an analogous solution by transposing to $[\underline{A} - \underline{LC}] = (\underline{A}^T - \underline{C}^T\underline{L}^T)^T$ which is of the form $\tilde{\underline{A}} - \tilde{\underline{B}}\,\tilde{\underline{K}}$ where

$$\tilde{\underline{A}} = \underline{A}^T, \quad \tilde{\underline{B}} = \underline{C}^T, \quad \text{and} \quad \tilde{\underline{K}} = \underline{L}^T \tag{10-23}$$

Since the determinant of any matrix is equal to the determinant of its transpose, then

$$\det[\lambda\underline{I} - (\underline{A} - \underline{L}\,\underline{C})] = \det[\lambda\underline{I} - (\underline{A} - \underline{L}\,\underline{C})]^T$$

$$= \det[(\lambda\underline{I})^T - (\underline{A} - \underline{L}\,\underline{C})^T] = \det[\lambda\underline{I} - (\tilde{\underline{A}} - \tilde{\underline{B}}\,\tilde{\underline{K}})]$$

Thus, the eigenvalues of $\underline{A} - \underline{L}\,\underline{C}$ are exactly those of $\tilde{\underline{A}} - \tilde{\underline{B}}\,\tilde{\underline{K}}$. From Section 10-4 we can solve for $\tilde{K}$ and, therefore, $\underline{L}^T$ if and only if

$$\rho[\underline{U}_s] = \rho[\tilde{\underline{B}}, \tilde{\underline{A}}\tilde{\underline{B}}, \tilde{\underline{A}}^2\tilde{\underline{B}}, \ldots, \tilde{\underline{A}}^{n-1}\tilde{\underline{B}}] = n$$

But

$$\underline{U}_s = [\underline{C}^T, \underline{A}^T\underline{C}^T, (\underline{A}^T)^2\underline{C}^T, \ldots, (\underline{A}^T)^{n-1}\underline{C}^T]$$

which is our observability matrix $\underline{V}_s$. Thus, we can solve for $\underline{L}$ to arbitrarily place the estimator poles if and only if the controlled system is observable. Q.E.D.

Because of the similarities between estimator pole placement and state-variable feedback pole placement as pointed out in the proof of Theorem 10-3, we can apply the techniques of Section 10-4, particularily Eqs. 10-17 and 10-19, to $\tilde{\underline{A}}$, $\tilde{\underline{B}}$, and

$\tilde{\underline{K}}$ as defined in Eq. 10-23 to place the observed poles. As in pole placement, the solution for $\underline{L}$ in the single-input, single-output case is unique but not so in the multiple-input situation.

Example 10-10 Consider a system described by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u, \quad y = [1 \quad 0]\underline{x}$$

Assume that the state vector $\underline{x}$ is inaccessible and design an asymptotic estimator which will decay in 4 seconds or less.

Solution: The system is observable, and, therefore, a solution for $\underline{L}$ is possible. Here $\underline{L}$ is 2 x 1. Let us assume that the error has decayed to about 5 percent of its original value after 4 seconds. Then $e^{-\sigma t}\Big|_{t=4} = 0.05$ yields a σ of about 0.75. Therefore, let us choose $\hat{\lambda}_1 = -0.75$ and $\hat{\lambda}_2$ arbitrarily as -2.0 since is will decay more quickly. And then

$$\tilde{\underline{A}} = \underline{A}^T = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \quad \tilde{\underline{B}} = \underline{C}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{\underline{K}} = \underline{L}^T$$

Then

$$\underline{\Phi}(\lambda) = (\lambda\underline{I} - \tilde{\underline{A}})^{-1} = \frac{\begin{bmatrix} \lambda + 2 & -1 \\ 1 & \lambda \end{bmatrix}}{\lambda^2 + 2\lambda + 1}$$

and

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\tilde{\underline{B}} = \frac{[\lambda + 2 \quad 1]^T}{\lambda^2 + 2\lambda + 1}$$

yielding from Eq. 10-17

$$\tilde{\underline{K}} = -[1 \quad 1][\underline{\Omega}(-0.75), \underline{\Omega}(-2)]^{-1} = -[1 \quad 1]\begin{bmatrix} 20 & 0 \\ 16 & 1 \end{bmatrix}^{-1} = [3/4 \quad -1]$$

Thus $\underline{L} = \underline{\tilde{K}}^T = [3/4 \quad -1]^T$. The system equations now become

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0]\underline{x}$$

$$\dot{\hat{\underline{x}}} = \begin{bmatrix} -3/4 & 0 \\ -1 & -2 \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 3/4 \\ -1 \end{bmatrix} y$$

Let us now consider the closed-loop control problem for an observable system described by $\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u}$, $\underline{y} = \underline{C}\ \underline{x}$ with a state estimator described by $\dot{\hat{\underline{x}}} = [\underline{A} - \underline{L}\ \underline{C}]\hat{\underline{x}} + \underline{B}\ \underline{u} + \underline{L}\ \underline{y}$ and feedback $\underline{u} = \underline{v} - \underline{K}\ \hat{\underline{x}}$. Note that our feedback equation uses the estimate $\hat{\underline{x}}$ rather than $\underline{x}$ which we are assuming is inaccessible. Figure 10-8 shows a block diagram of this configuration.

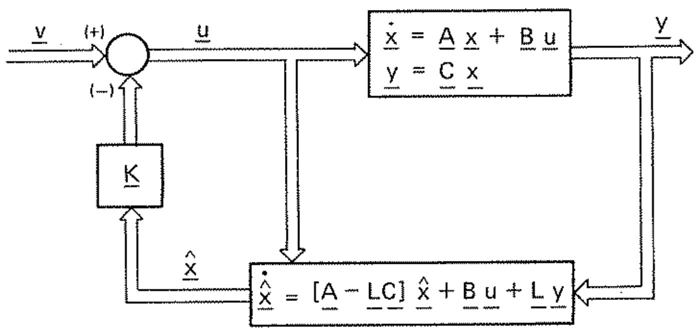

Figure 10-8. Closed-Loop Control With Asymptotic State Estimation

The equations can be combined to yield

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\ \underline{x} - \underline{B}\ \underline{K}\ \hat{\underline{x}} + \underline{B}\ \underline{v} \\ \dot{\hat{\underline{x}}} &= \underline{L}\ \underline{C}\ \underline{x} + [\underline{A} - \underline{L}\ \underline{C} - \underline{B}\ \underline{K}]\hat{\underline{x}} + \underline{B}\ \underline{v} \qquad (10\text{-}24) \\ \underline{y} &= \underline{C}\ \underline{x} \end{aligned}$$

which can be written in partitioned form as

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} \underline{A} & -\underline{B}\,\underline{K} \\ \underline{L}\,\underline{C} & \underline{A} - \underline{L}\,\underline{C} - \underline{B}\,\underline{K} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \underline{B} \end{bmatrix} \underline{v} \tag{10-25}$$

$$\underline{y} = [\underline{C} \quad \underline{0}] \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix}$$

The 2n closed-loop poles for Eq. 10-25 are roots of

$$\Delta_c(\lambda) \triangleq \begin{vmatrix} \lambda \underline{I}_n - \underline{A} & \underline{BK} \\ -\underline{LC} & \lambda \underline{I}_n - \underline{A} + \underline{LC} + \underline{BK} \end{vmatrix} = 0$$

There are now 2n state variables in the feedback system, n due to the system and n due to the estimator.

Two possible questions arise due to the use of this asymptotic estimator to obtain the signal $\hat{\underline{x}}$ which is fed back to achieve the closed-loop control system. (1) What is the effect of introducing the state estimator? Will the estimator eigenvalues appear in the result without change? (2) Does the estimator change the desired feedback pole locations of the system if we design $\underline{K}$ using the real state $\underline{x}$? Let us consider these questions by applying a linear transformation of variables $\underline{P}$ where $\underline{P}$ is 2n x 2n such that

$$\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} = \underline{P} \begin{bmatrix} \underline{x} \\ \underline{x} - \hat{\underline{x}} \end{bmatrix} = \left[\begin{array}{c|c} \underline{I}_n & \underline{0}_{n\mathrm{x}n} \\ \hline \underline{I}_n & -\underline{I}_n \end{array}\right] \begin{bmatrix} \underline{x} \\ \underline{x} - \hat{\underline{x}} \end{bmatrix} \tag{10-26}$$

Theorem 8-13 guarantees us that this linear transformation will not affect the controllability or observability of Eq. 10-25. The choice of the $\underline{P}$ matrix or the new variables probably does not appear obvious. Remember that $\underline{x}_e = \underline{x} - \hat{\underline{x}}$

is the estimator error. Eq. 10-25 reduces to Eq. 10-27 by the use of Eq. 3-122.

$$\begin{bmatrix} \dot{\underline{x}} \\ \hline \dot{\underline{x}}_e \end{bmatrix} = \left[\begin{array}{c|c} \underline{A} - \underline{BK} & -\underline{BK} \\ \hline 0 & \underline{A} - \underline{LC} \end{array}\right] \begin{bmatrix} \underline{x} \\ \hline \underline{x}_e \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \hline \underline{0} \end{bmatrix} \underline{v}$$

$$\underline{y} = [\underline{C} \quad \underline{0}] \begin{bmatrix} \underline{x} \\ \hline \underline{x}_e \end{bmatrix} \tag{10-27}$$

Since Eq. 10-25 and Eq. 10-27 represent equivalent representations of the same system, we can see that the eigenvalues of $\Delta_c(\lambda)$ and those of Eq. 10-27 are equal. Let $\Delta'(\lambda) = |\lambda \underline{I}_n - (\underline{A} - \underline{BK})|$ and $\Delta_a(\lambda) = |\lambda \underline{I}_n - (\underline{A} - \underline{LC})|$. Then

$$\Delta_c(\lambda) = \Delta'(\lambda)\Delta_a(\lambda) \tag{10-28}$$

Equation 10-28 is called the <u>separation property</u> and allows us to calculate the feedback gain $\underline{K}$ and the asymptotic estimator matrix $\underline{L}$ independently using the techniques of Section 10-4. We are assured that these poles will be unchanged in the final design.

<u>Example 10-11</u> Let us design a feedback system for a system described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{s + 2}{(s + 1)(s - 1)}$$

assuming that only the input u(t) and output y(t) are accessible.

<u>Solution</u>: The plant is clearly unstable due to the pole at s = 1. Let us use a Jordan form state-space representation since

$$\frac{Y(s)}{U(s)} = \frac{-1/2}{s + 1} + \frac{3/2}{s - 1}$$

Figure 10-9 is the simulation diagram. From the diagram we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [-1/2 \quad 3/2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

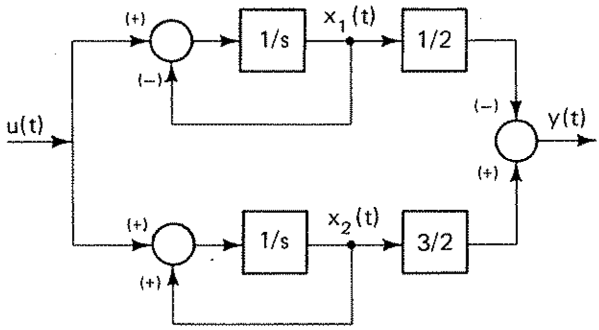

Figure 10-9. Example 10-11 Simulation Diagram

A quick calculation will show that the plant is controllable and observable. Let us place the closed-loop poles of the feedback system at $-1 + j$ and $-1 - j$. Due to the separation property we can calculate $\underline{K}$ independent of the asymptotic estimator matrix $\underline{L}$. Then

$$\underline{\Omega}(\lambda) = \underline{\Phi}(\lambda)\underline{B} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s-1} \end{bmatrix}$$

and

$$\underline{K} = [k_1 \quad k_2] = -[1 \quad 1] \begin{bmatrix} \dfrac{1}{j} & \dfrac{1}{-j} \\ \dfrac{1}{-2+j} & \dfrac{1}{-2-j} \end{bmatrix}^{-1} = [-1/2 \quad 5/2]$$

Letting $u = v - \underline{K}x$, the feedback system without the estimator is

$$\dot{\underline{x}} = [\underline{A} - \underline{B}\underline{K}]\underline{x} + \underline{B}v \,, \qquad y = \underline{C}\underline{x}$$

Substituting we obtain a closed-loop law of

$$\dot{\underline{x}} = \begin{bmatrix} -1/2 & -5/2 \\ 1/2 & -3/2 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \qquad y = [-1/2 \quad 3/2]\underline{x}$$

But $\underline{x}$ is inaccessible, and an asymptotic estimator is necessary which also makes $u = v - \underline{K}\hat{\underline{x}}$. Let us place the poles of the estimator to the left of $-1 + j$ and $-1 - j$ in the complex plane, say $\hat{\lambda}_a = -3$ and -5. Thus, the error $\underline{x}_e = \underline{x} - \hat{\underline{x}}$ will decay at a rate e^{-3t} or e^{-5t}, and in 1 time unit will be about 5% of its original value since $e^{-3} \approx 0.0498$. We can calculate $\underline{L}$ from the system described by

$$\tilde{\underline{A}} = \underline{A}^T, \ \tilde{\underline{B}} = \underline{C}^T, \ \text{and} \ \tilde{\underline{K}} = \underline{L}^T \tag{10-23}$$

Then

$$\underline{\Omega}_a(\lambda) = \underline{\Phi}_a(\lambda)\tilde{\underline{B}} = \left[\lambda\underline{I} - \underline{A}^T\right]^{-1}\underline{C}^T = \begin{bmatrix} \dfrac{-1/2}{s + 1} \\ \\ \dfrac{3/2}{s - 1} \end{bmatrix}$$

Then

$$\tilde{\underline{K}} = \underline{L}^T = -[1 \quad 1][\underline{\Omega}_a(-3), \ \underline{\Omega}_a(-5)]^{-1} = [8 \quad 8]$$

Thus, $\underline{L} = [8 \quad 8]^T$, and the estimator equation becomes

$$\dot{\hat{\underline{x}}} = \begin{bmatrix} 3 & -12 \\ 4 & -11 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 8 \\ 8 \end{bmatrix} y$$

The final system description with state estimation is from Eq. 10-25

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 0 & 1/2 & -5/2 \\ 0 & 1 & 1/2 & -5/2 \\ \hline -4 & 12 & 7/2 & -29/2 \\ -4 & 12 & 9/2 & -27/2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \hline 1 \\ 1 \end{bmatrix} v$$

$$y = [-1/2 \quad 3/2 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \hline \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

■

It is always possible to design a lower order observer since information about some of the states is directly obtainable from the output $\underline{y}$. Reference (11, page 289) discusses the (n - 1) estimator which eliminates the redundancy due to information obtained from $\underline{y}$. References (14) and (15) discuss the possibility of even lower order systems for some systems.

10-6 SUMMARY – CHAPTER 10

In this chapter we studied the design of linear feedback systems and the practical implications of controllability and observability. We showed that if the controlled element is controllable, we can arbitrarily assign the closed-loop poles by introducing state variable feedback or output feedback. When using state variable feedback, all state variables must be accessible to measurement. If they are not available, then a state estimator (observer) must be constructed. If the plant is observable, the poles of the estimator may be arbitrarily placed. The separation property guarantees that the estimator matrix, $\underline{L}$, and the feedback matrix, $\underline{K}$, can be designed independently.

10-7 PROBLEMS – CHAPTER 10

1. Given the matrix feedback system of Figure P10-1a, show that it reduces to Figure P10-1b.