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The Maximal Controlled Invariant Set of Switched Linear Systems

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Abstract

We study the construction of the maximal controlled invariant set of switched linear systems. We consider two cases, based on whether fast switches are allowed or not. For both cases, iterative algorithms are given to construct such sets.

1 Introduction

Control of dynamic systems by switchings has been studied intensively. Typical results include stability properties [5], sliding mode analysis [4] and reachability and controllability properties [6].

In this paper, controlled invariant sets for switched linear systems are considered. In particular, we study the construction of the maximal controlled invariant set (MCIS) contained in a linear space. We consider two types of switched systems, namely, those in which infinitely fast switches are allowed and those in which such switches are not allowed. Throughout the text, we shall refer to the infinitely fast switches as *fast switches*. The case where fast switches are not allowed is referred to as the case with *regular switches*.

The organization of the text can be described as follows. Section 2 introduces the notation used in this text and describes the dynamics of the systems under consideration. Section 3 treats the case where only regular switches are allowed. There, an iterative algorithm is given for the construction of the MCIS. Analogously to Section 3, Section 4 deals with the case where fast switches are allowed. It is also shown that the algorithm given in this section is a generalization of that given in the previous. Finally, the last section gives an example of constructing the MCIS of a given switched system using the provided algorithm.

2 Preliminaries

By the controlled invariant set of a system, we mean the set of states such that given an initial condition in that set, it is possible (by means of control) to stay within the set [3, 8].

Throughout this text the following definitions and notations will be used.

We denote a linear system described by the input-state representation

$$\dot{x} = Ax + Bu,$$
$$x \in \mathcal{X}, u \in \mathcal{U},$$

for some finite dimensional linear spaces $\mathcal{X}$ and $\mathcal{U}$ by $\Xi(A,B)$.

The maximal controlled invariant set (space) of $\Xi(A,B)$ contained in the linear space $\mathcal{K}$ is denoted as $\mathcal{V}^*(A,B,\mathcal{K})$ is the maximal linear subspace of $\mathcal{K}$ satisfying the following property

$$x \in \mathcal{V}^* \Rightarrow Ax \in \mathcal{V}^* + \operatorname{Im} B. \quad (1)$$

It is relatively well known that $\mathcal{V}^*(A,B,\mathcal{V})$ can be computed using the following iteration [8].

$$\mathcal{V}_0 = \mathcal{K}, \quad (2a)$$
$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A^{-1}(\mathcal{V}_i + \operatorname{Im} B). \quad (2b)$$

It can be proven that after a finite number of steps this iteration will converge to a fixed point, and that the fixed point is $\mathcal{V}^*(A,B,\mathcal{K})$.

The switched linear system described by

$$\dot{x} = A_i x + B_i u,$$
$$i \in I,$$

where I is an index set, is denoted by $\Sigma(A,B,I)$. Typically $I = \{1,2,3,\ldots,m\}$.

The switched linear system $\Sigma(A,B,I)$ admits two kind of inputs, discrete and continuous. A continuous input $u(t)$ is a locally integrable time function. A discrete input Δ_k is denoted as a (possibly infinite) sequence of pairs (t_i, δ_i), with $i \in \mathbb{Z}^+$, $t_i \in \mathbb{R}^+$ and $\delta_i \in I$. The time sequence t_k is monotonically increasing, with $t_0 = 0$.

2.1 Regular switches

If fast switches are not allowed, we assume that for any discrete input Δ_k there exists a positive real ε such that for all

$k \in \mathbb{Z}^+$ the following relation holds.

$$|t_{k+1} - t_k| > \varepsilon \tag{3}$$

We denote the inputs to the switched linear system as input pairs $(u(t), \Delta_k)$.

The (state) trajectories of the switched linear system $\Sigma(A,B,I)$ under the discrete input Δ_k and continuous input $u(t)$ are defined as those for each of which there exists an $x_0 \in \mathcal{X}$ such that the following relation holds true for all $t \geq 0, k \geq 0$

$$x(t_{k+1}) = e^{A_{\delta_k}(t_{k+1}-t_k)} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A_{\delta_k}(t_{k+1}-\tau)} B_{\delta_k} u(\tau) d\tau, \tag{4a}$$

$$x(t) = e^{A_{\delta_{n_t}}(t-t_{n_t})} x(t_{n_t}) + \int_{t_{n_t}}^{t} e^{A_{\delta_{n_t}}(t-\tau)} B_{\delta_{n_t}} u(\tau) d\tau, \tag{4b}$$

where

$$n_t = \max\{i \mid t_i \leq t\}. \tag{5}$$

This seemingly complex relation can be verbally explained as follows. We think of the system as linear time-varying, where locally (in time) it behaves just like an ordinary linear system $\Sigma(A_i, B_i)$, $i \in I$ with input $u(t)$. The time-varying dynamics is due to the discrete input that switches the mode of operation (i.e. changing the pair $(A_{\delta_{i-1}}, B_{\delta_{i-1}})$ into $(A_{\delta_i}, B_{\delta_i})$) at time instance t_i. It can be seen that the inputs characterize the (state) trajectories uniquely (in $\mathcal{L}_1^{loc}$ sense) given the initial condition $x(0) = x_0$.

2.2 Fast switches

Control by incorporating fast switches involves switchings *a la* pulse width modulation control [7]. Let the system be $\Sigma(A, B, \{1, \cdots, m\})$. Let $x(0) = x_0$ and we apply the input pair $(u(t), \Delta_k)$ to the system. The input pair $(u(t), \Delta_k)$ is described as follows. The input signal $u(t)$ is a piecewise constant function, where

$$u(t) = u_i,\ t \in [t_{i-1}, t_i).$$

The discrete input Δ_k is such that

$$\begin{aligned} \Delta_k &= (t_k, \delta_k), \\ \delta_k &= \delta_{k-1} + 1,\ k < m, \\ \delta_0 &= \delta_m = 1. \end{aligned}$$

We also define the sequence $(\alpha_i)_{i \in \{1, \cdots, m\}}$ to be

$$\alpha_i := \frac{t_i - t_{i-1}}{t_m}.$$

Thus we start with the system in mode 1. At time $t = t_1$ we switch to mode 2. After that, at time $t = t_2$ we switch to mode 3. We continue doing this until all modes have been reached and then at time $t = t_m$ we switch back to mode 1. The variable α_i is the proportion of time spent in mode i compared to that spent in all modes. Hence

$$\begin{aligned} \alpha_i &\geq 0, i \in \{1, \cdots, m\}, \\ \sum_{i=1}^{m} \alpha_i &= 1. \end{aligned}$$

The following relation can be verified

$$x(t_m) = x_0 + t_m \sum_{i=1}^{m} (A_i x_0 + B_i u_i)\, \alpha_i + O(t_m^2). \tag{6}$$

Further, it is not difficult to derive

$$\lim_{t_m \downarrow 0} \frac{x(t_m) - x(0)}{t_m} = \sum_{i=1}^{m} (A_i x_0 + B_i u_i)\, \alpha_i. \tag{7}$$

We can see in (7) that by allowing fast switches the system can be steered to a direction given by a *convex combination* of the directions given by the individual modes.

3 Construction of the maximal controlled invariant sets

3.1 Systems with only regular switches

Consider the switched linear systems $\Sigma(A,B,I)$. Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. We define the controlled invariant set of $\Sigma(A,B,I)$ in $\mathcal{K}$ as the set V (not necessarily a linear space) satisfying the following conditions.

- $V \subset \mathcal{K}$.
- For any $x_0 \in V$, there exists an input pair $(u(t; x_0), \Delta_k(x_0))$ such that the trajectory of the system remains in V for all $t \geq 0$.

We can also introduce the notion of maximal controlled invariant set (MCIS) of $\Sigma(A,B,I)$ in the space $\mathcal{K}$ as $V^*(\Sigma(A,B,I), \mathcal{K})$ or just $V^*(\Sigma, \mathcal{K})$ for brevity. The MCIS $V^*(\Sigma, \mathcal{K})$ has the additional property that it contains every controlled invariant set of $\Sigma(A,B,I)$ in $\mathcal{K}$.

In the following, we present two properties of the controlled invariant sets.

- If V_1 and V_2 are both controlled invariant sets of $\Sigma(A,B,I)$ in $\mathcal{K}$, then so is $V_1 \cup V_2$.
- Let V be a controlled invariant set of $\Sigma(A,B,I)$ in $\mathcal{K}$. Define $ext(V)$ as the scalar extension of V, where

  $$ext(V) = \{x \mid \exists \lambda \neq 0, \lambda \in \mathbb{R}\ s.t.\ \lambda x \in V\}.$$

  $ext(V)$ is also a controlled invariant set of $\Sigma(A,B,I)$ in $\mathcal{K}$.

Both properties imply that $V^*(\Sigma,\mathcal{K})$ is a union of linear spaces. We therefore define the concepts of *space arrangements* and *tangent space arrangements* as follows.

Definition 1 *A space arrangement V is defined as the union of some (finitely many) linear spaces $\mathcal{V}_1 \dots \mathcal{V}_n$. A linear space is a special case of space arrangements.*

Definition 2 *Let $V = \bigcup_{i=1}^{n} \mathcal{V}_i$ be a space arrangement, and let $x_0 \in V$. We denote the tangent space arrangement of V at x_0 as $TV(x_0)$. It is defined as*

$$TV(x_0) := \bigcup_{i \in I(x_0)} \mathcal{V}_i,$$

where $I(x_0) = \{i \mid i \leq n, x_0 \in \mathcal{V}_i\}$

In [1, 2], the tangent space arrangement is called the *contingent cone*. We will use the concepts about space arrangements and tangent space arrangements to compute $V^*(\Sigma,\mathcal{K})$ in general.

Proposition 3 *A space arrangement $V \subset \mathcal{K}$ is a controlled invariant space arrangement of $\Sigma(A,B,I)$ contained in $\mathcal{K}$ if for all $x \in V$ there exists an $i \in I$ such that $A_i x \in TV(x) + \operatorname{Im} B_i$.*

Proof: Assume that for all $x \in V$ there exists an $i \in I$ such that $A_i x \in TV(x) + \operatorname{Im} B_i$. It implies the existence of a u such that $(A_i x + B_i u) \in TV(x)$. Hence we can always choose a mode and an input such that the trajectory remains in V. ■

Consider the sequence of space arrangements formed by the following iteration.

$$V_0 = \mathcal{K}, \tag{8a}$$

$$V_{i+1} = V_i \cap \left(\bigcup_{j=1}^{n_i} \bigcup_{k=1}^{m} \mathcal{V}_{ij} \cap A_k^{-1} (\mathcal{V}_{ij} + \operatorname{Im} B_k) \right). \tag{8b}$$

Here $\mathcal{V}_{ij},\ j = 1 \dots n_i$, are the linear spaces that build up V_i, i.e.

$$V_i = \bigcup_{j=1}^{n_i} \mathcal{V}_{ij}. \tag{9}$$

The fixed points of the iteration have a special property, which is explained in the following lemma.

Lemma 4 *A space arrangement $\hat{V}$ is called a* fixed point *of the iteration (8b), if $\hat{V}$ satisfies the following relation*

$$\hat{V} = \hat{V} \cap \left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \hat{\mathcal{V}}_j \cap A_k^{-1} \left(\hat{\mathcal{V}}_j + \operatorname{Im} B_k \right) \right), \tag{10}$$

where $\hat{\mathcal{V}}_j,\ j = 1 \dots n$ are the linear spaces that build up $\hat{V}$. $\hat{V}$ is a fixed point if and only if it is a controlled invariant space arrangement.

Proof: ($\Longrightarrow$) Assume that $\hat{V}$ is a fixed point of (8b), i.e. (10) is true. It follows that

$$\hat{V} \subset \left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \hat{\mathcal{V}}_j \cap A_k^{-1} \left(\hat{\mathcal{V}}_j + \operatorname{Im} B_k \right) \right). \tag{11}$$

Hence for every $x \in \hat{V}$, there exist j and k such that the following relations are all valid.

$$1 \leq j \leq n, \tag{12a}$$

$$1 \leq k \leq m, \tag{12b}$$

$$x \in \hat{\mathcal{V}}_j, \tag{12c}$$

and

$$\exists u \in \mathcal{U} \ s.t.\ (A_k x + B_k u) \in \hat{\mathcal{V}}_j \subset T\hat{V}(x). \tag{13}$$

Therefore $\hat{V}$ is a controlled invariant space arrangement.

($\Longleftarrow$) Assume that $\hat{V}$ is a controlled invariant space arrangement. Then for every $x \in \hat{V}$ there exists a $k \in \{1 \dots m\}$ such that

$$A_k x \in T\hat{V}(x) + \operatorname{Im} B_k. \tag{14}$$

It implies the existence of $j,k \in \{1 \dots n\}$ such that $x \in \hat{\mathcal{V}}_j$ and $A_k x \in \hat{\mathcal{V}}_j + \operatorname{Im} B_k$. Hence (11) is satisfied and therefore $\hat{V}$ is a fixed point of (8b). ■

Lemma 5 *Consider the iteration (8b). Let $\hat{V}$ be a controlled invariant space arrangement. If $\hat{V}$ is contained in some V_i then $\hat{V}$ is also contained in all $V_j,\ j \geq i$.*

Proof: Assume that $\hat{V}$ is contained in V_i and that $\hat{V}$ and V_i are built up by linear spaces as follows

$$V_i = \bigcup_{j=1}^{n_i} \mathcal{V}_{ij},$$

$$\hat{V} = \bigcup_{j=1}^{n} \hat{\mathcal{V}}_j.$$

Since $\hat{V}$ is contained in V_i, for every $j \in \{1 \dots n\}$ there exists a $k \in \{1 \dots n_i\}$ such that

$$\hat{\mathcal{V}}_j \subset \mathcal{V}_{ik}. \tag{15}$$

Because of this relation, it is not difficult to show that

$$\left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \hat{\mathcal{V}}_j \cap A_k^{-1} \left(\hat{\mathcal{V}}_j + \operatorname{Im} B_k \right) \right) \subset \left(\bigcup_{j=1}^{n_i} \bigcup_{k=1}^{m} \mathcal{V}_{ij} \cap A_k^{-1} (\mathcal{V}_{ij} + \operatorname{Im} B_k) \right). \tag{16}$$

Hence we have that

$$V_i \cap \left(\bigcup_{j=1}^{n_i} \bigcup_{k=1}^{m} \mathcal{V}_{ij} \cap A_k^{-1} (\mathcal{V}_{ij} + \mathrm{Im} B_k) \right) = V_{i+1},$$

$$\hat{V} \cap \left(\bigcup_{j=1}^{n_i} \bigcup_{k=1}^{m} \mathcal{V}_{ij} \cap A_k^{-1} (\mathcal{V}_{ij} + \mathrm{Im} B_k) \right) \subset V_{i+1},$$

$$\hat{V} \cap \left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \hat{\mathcal{V}}_j \cap A_k^{-1} \left(\hat{\mathcal{V}}_j + \mathrm{Im} B_k\right) \right) \subset V_{i+1},$$

$$\hat{V} \subset V_{i+1}. \qquad (17)$$

Here we use the fact that $\hat{V}$ is a controlled invariant space arrangement and Lemma 4 to obtain

$$\hat{V} = \hat{V} \cap \left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \hat{\mathcal{V}}_j \cap A_k^{-1} \left(\hat{\mathcal{V}}_j + \mathrm{Im} B_k\right) \right). \qquad (18)$$

Using (17) and the principle of mathematical induction, the lemma is proven. ■

With the help of Lemma 4 and 5 the following theorem can be proven for the maximal controlled invariant space arrangement of $\Sigma(A,B,I)$ contained in $\mathcal{K}$.

Theorem 6 *Let the iteration (8) converge to V, i.e. there is a $p > 0$ such that*

$$i \geq p \Longleftrightarrow V_{i+1} = V_i = V.$$

V is the maximal controlled invariant space arrangement of $\Sigma(A,B,I)$ contained in $\mathcal{K}$.

Proof: V is a fixed point of (8b). By Lemma 4 we know that V is a controlled invariant space arrangement. Moreover, any controlled invariant space arrangement in $\mathcal{K}$ is included in V_0. By Lemma 5 it follows that any controlled invariant space arrangement in $\mathcal{K}$ is included in V. Hence V is the maximal controlled invariant space arrangement. ■

The next logical question to ask is whether the iteration will terminate at all. Indeed, it can be proven that the iteration is guaranteed to converge after at most $\dim(\mathcal{K})$ steps. However, due to space limitation, the proof will not be presented here.

In addition, given that the iteration terminates, we can infer the following result.

Corollary 7 *Let V be the maximal controlled invariant space arrangement of $\Sigma(A,B,I)$ contained in $\mathcal{K}$. The following relation is true.*

$$V = \bigcup_{i \in I} V^*(A_i, B_i, \mathcal{K}). \qquad (19)$$

This means that the maximal controlled invariant space arrangement is the union of the maximal controlled invariant space of the individual modes.

Proof: Let the iteration terminates after N step. For every $\mathcal{V}_{Nj}$ there exists a $k(j) \in I$ such that

$$\mathcal{V}_{Nj} = \mathcal{V}_{Nj} \cap A_{k(j)}^{-1} \left(\mathcal{V}_{Nj} + \mathrm{Im} B_{k(j)}\right).$$

The maximal controlled invariant space arrangement V is then the union of all $\mathcal{V}_{Nj}$ for all $j \in I$, i.e.

$$V = \bigcup_{j \in I} \mathcal{V}_{Nj} \qquad (20)$$

Using (2) and (20) we can infer

$$\mathcal{V}_{Nj} \subset V^*(A_{k(j)}, B_{k(j)}, \mathcal{K}). \qquad (21)$$

Consequently we have that

$$V \subset \bigcup_{i \in I} V^*(A_i, B_i, \mathcal{K}). \qquad (22)$$

Combining (22) with the trivial inclusion

$$V \supset \bigcup_{i \in I} V^*(A_i, B_i, \mathcal{K}), \qquad (23)$$

we prove the corollary.

■

3.2 Systems with fast switches

For some technical reasons, the notion of control invariance for systems with fast switches is a bit different from that of the systems with regular switches. A set V is called controlled invariance if for any $x_0 \in V$ we can stay steer the trajectory to remain arbitrarily close to V for an arbitrary long finite time interval. Before we proceed with the construction of the MCIS of the systems with fast switches, consider the following definitions.

Definition 8 *A point $x \in \mathcal{X}$ is included in the open convex hull of $\{x_1, \cdots, x_m\} \in \mathcal{X}$, if and only if there is $\alpha_1, \cdots, \alpha_m > 0$ such that*

$$x = \sum_{i=1}^{m} \alpha_i x_i, \qquad (24)$$

$$\sum_{i=1}^{m} \alpha_i = 1. \qquad (25)$$

Notation 9 *For $J \subset \{1,2,\cdots,m\}$, we introduce the notations*

$$\mathcal{B}_J := \sum_{i \in J} \mathrm{Im} B_i.$$

$$D_J(x) := \text{the open convex hull of } \{A_i x\}_{i \in J}.$$

Definition 10 *The set $vel(x)$, $x \in \mathcal{X}$ is defined as*

$$vel(x) := \bigcup_{J \in \mathcal{P}(I)} D_J(x) + \mathcal{B}_J, \qquad (26)$$

where $\mathcal{P}(I)$ is the power set of I.

Theorem 11 *For any $t \in \mathbb{R}^+$, let $x(t) = x_0$. We are able to steer the trajectory (possibly with fast switches) such that $\dot{x}(t) = w$ if and only if $w \in vel(x_0)$*

Proof: ($\Longleftarrow$) Assume that $w \in vel(x_0)$. Hence there is a $J \in \mathcal{P}(I)$, $(\alpha_i)_{i\in J} > 0$ and $(u_i)_{i\in J}$ such that

$$\begin{aligned} w &= \sum_{i\in J} \alpha_i A_i x_0 + \sum_{i\in J} B_i u_i, \\ &= \sum_{i\in J} \alpha_i (A_i x_0 + B_i \frac{u_i}{\alpha_i}). \end{aligned} \tag{27}$$

and

$$\sum_{i\in J} \alpha_i = 1$$

Compare (27) with (7). We see that we can steer $\dot{x} = w$.

($\Longrightarrow$) This implies the existence of $(\alpha_i)_{i\in I} \geq 0$ and $(u_i)_{i\in I}$ such that

$$w = \sum_{i=1}^{m} \alpha_i (A_i x_0 + B_i u_i), \ \sum_{i=1}^{m} \alpha_i = 1.$$

Hence if $J := \{i | \alpha_i \neq 0\}$, then $w \in D_J(x_0) + \mathcal{B}_J \subset vel(x_0)$. ∎

The following definition is taken from [1].

Definition 12 *A* contingent cone *[1] of a set V at point $x \in V$, denoted as $T_V(x)$, is the collection of all vectors v such that*

$$\liminf_{h\to 0^+} \frac{d_V(x+hv)}{h} = 0,$$

where $d_V(y)$ denotes the distance of y to V, given by

$$d_V(y) := \inf_{z\in V} \|y - z\|.$$

Intuitively, the contingent cone gives the directions to which we can move without leaving the set V. Following this definition, we hypothesize that the following iteration can be used to compute the maximal controlled invariant set of $\Sigma(A,B,I)$ contained in $\mathcal{K}$. Similar iterations for impulse differential inclusions are given in [2].

$$V_0 = \mathcal{K} \tag{28a}$$

$$V_{i+1} = \{x \in V_i \,|\, vel(x) \cap T_{V_i}(x) \neq \emptyset\} \tag{28b}$$

The following proposition can be proved about this iteration.

Proposition 13 *If a set V is a fixed point of the iteration (28b), then V is a controlled invariant set.*

Proof: V is a fixed point of the iteration (28b) means

$$V = \{x \in V \,|\, vel(x) \cap T_V(x) \neq \emptyset\}.$$

Hence for every $x \in V$, there is a $v \in vel(x)$ such that v is also an element of $T_V(x)$. From Theorem 11 we know that we can steer $\dot{x}$ to v and thus keeping the trajectory from leaving V. ∎

We now know that if iteration (28b) converges to a fixed point V, V is a controlled invariant set. In addition, since it can be seen easily that $V_{i+1} \subset V_i$, we also know that $V \subset \mathcal{K}$.

In order to establish the maximality of the fixed point of iteration (28), let us consider the following lemma, which is analogous to Lemma 5.

Lemma 14 *Let V be a controlled invariant set. If V is contained in some V_i then V is also contained in all V_j, $j \geq i$.*

Proof: Take any element $x \in V \subset V_i$. Since V is a controlled invariant set, there is a $v \in vel(x)$ such that v is also an element of $T_V(x)$. But, this also means that $v \in T_{V_i}(x)$. Hence $vel(x) \cap T_{V_i}(x) \neq 0$, and from (28b) this implies that $x \in V \subset V_{i+1}$. The lemma is proved by incorporating this argument in an induction scheme. ∎

Now we are ready to state the theorem about the maximality of the fixed point. Consider the following theorem, which is analogous to Theorem 6.

Theorem 15 *Let the iteration (28) converge to V, i.e. there is a $p > 0$ such that*

$$i \geq p \Longleftrightarrow V_{i+1} = V_i = V.$$

V is the maximal controlled invariant set of $\Sigma(A,B,I)$ contained in $\mathcal{V}$.

Proof: V is a fixed point of (28b). By Proposition 13, we know that V is a controlled invariant set. Moreover, using Lemma 14, we also know that every controlled invariant set in $\mathcal{V}$ are also in V. Hence V is maximal. ∎

The algorithm for systems with fast switches given in (28) is a generalization of that given in (8) for systems with only regular switches. Notice that in this case

$$vel(x) = \bigcup_{k=1}^{m} (A_k x + \mathrm{Im}\, B_k). \tag{29}$$

Iteration (28) starts with a linear space $\mathcal{K}$ which is also a space arrangement. We will show that (28b) equals (8b), given (29) and the fact that V_i is a space arrangement. Let

$V_i = \bigcup_{j=1}^{n_i} \mathcal{V}_{ij}$ describe the decomposition of the space arrangement V_i into its spaces. We then have the following relations.

$$\begin{aligned}
&\{x \in V \,|\, vel(x) \cap T_V(x) \neq \emptyset\} = \\
&= \bigcup_{j=1}^{n_i} \{x \in \mathcal{V}_{ij} \,|\, vel(x) \cap T_V(x) \neq \emptyset\}, \\
&= \bigcup_{j=1}^{n_i} \{x \in \mathcal{V}_{ij} \,|\, vel(x) \cap \mathcal{V}_{ij} \neq \emptyset\}, \\
&= \bigcup_{j=1}^{n_i} \left\{ x \in \mathcal{V}_{ij} \,\middle|\, \bigcup_{k=1}^{m} (A_k x + \mathrm{Im}\, B_k) \cap \mathcal{V}_{ij} \neq \emptyset \right\}, \\
&= \bigcup_{j=1}^{n_i} \left(\mathcal{V}_{ij} \cap \bigcup_{k=1}^{m} A_k^{-1} (\mathrm{Im}\, B_k + \mathcal{V}_{ij}) \right), \\
&= \bigcup_{j=1}^{n_i} \bigcup_{k=1}^{m} \left(\mathcal{V}_{ij} \cap A_k^{-1} (\mathrm{Im}\, B_k + \mathcal{V}_{ij}) \right). \qquad (30)
\end{aligned}$$

Observe that (30) is equivalent to (8b).

4 Example

The following example is adopted from [7]. It is a model of a lossless electric circuit. The states of the system, $(x_1, x_2, x_3)^T$ represent the currents flowing through the inductors and the voltage across the capacitor, as shown in Figure 1. The capacitance and inductance values of the

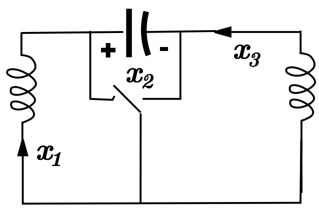

Figure 1: The circuit in the example.

components are taken to be 1, for simplicity. The system can operate in two modes of dynamics, depending on the position of the switch. We can derive that the system is a switched linear system $\Sigma(A, B, I)$ where $I = \{1, 2\}$, and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Mode 1 corresponds to the switch being closed to the left and mode 2 to the switch being closed to the right. We want to find the MCIS of $\Sigma(A, B, I)$ contained in $\mathcal{K} = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, i.e. we want to keep the voltage across the capacitor zero. If we assume that only regular switches are admissible, using Corollary 7 we can obtain the MCIS

$$V_{reg} = \bigcup_{i=1}^{2} V^*(A_i, B_i, \mathcal{K}). \qquad (31)$$

Using iteration (2), we can obtain

$$\begin{aligned}
V^*(A_1, B_1, \mathcal{K}) &= \mathrm{Im}\; \mathrm{col}(1, 0, 0), \\
V^*(A_2, B_2, \mathcal{K}) &= \mathrm{Im}\; \mathrm{col}(0, 0, 1).
\end{aligned}$$

If we assume that fast switches are allowed, we shall use iteration (28) to construct the MCIS. First, we set $V_0 = \mathcal{K}$. Hence $T_{V_0}(x) = \mathcal{K}$, $x \in \mathbb{R}^3$. We can immediately observe that $\mathcal{B}_I = \emptyset$.

The set $vel(x)$ is then the closed line segment between $(0, -x_3, x_2)$ and $(-x_2, x_1, 0)$. Hence,

$$\begin{aligned}
V_1 &= \{x \in V_0 | vel(x) \cap T_{V_0}(x) \neq \emptyset\}, \\
&= \{x \in \mathcal{K} | \exists \alpha, 0 \leq \alpha \leq 1, (1-\alpha)x_1 - \alpha x_3 = 0\}, \\
&= \{x \in \mathcal{K} | x_1 x_3 \geq 0\}. \qquad (32)
\end{aligned}$$

Continuing the iteration will reveal that V_1 is maximum. This result agrees with the result presented in [7], that in $\mathcal{K} \backslash V_1$ it is not possible to find a suitable duty ratio such that $\mathcal{K}$ is controlled invariant.

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