Optimal Linear Estimation Fusion—Part IV: Optimality and Efficiency of Distributed Fusion*

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Abstract – *This paper is concerned with the performance of distributed and centralized fusion with best linear unbiased estimation (BLUE), also known as linear minimum mean-square error (LMMSE) estimation, and optimal weighted least squares (WLS) estimation. Necessary and sufficient conditions for optimal distributed fusion rules to have identical performance as their centralized counterparts are presented. The conditions are general—e.g., no assumption is made that measurements are linear in the estimatee. Further, measures of relative efficiency of distributed fusion compared with centralized fusion are proposed. General and explicit formulas in terms of MSE matrix for performance degradation of the optimal distributed fusion relative to the optimal centralized fusion are given. It is shown both theoretically and by simulation results that the optimal distributed and centralized fusion rules using linear measurements have identical performance in general when measurement errors are uncorrelated across sensors and the measurement matrix has full column rank; the former is inferior to the latter in general when the measurement errors are correlated across sensors or are correlated with the estimatee. Numerical examples that demonstrate the relative efficiency of the distributed fusion are given. It is also illustrated that the optimal distributed fusion could be quite poor compared with the optimal centralized fusion.*

Keywords: Distributed fusion, centralized fusion, track fusion, BLUE, least squares

1 Introduction

Distributed fusion has certain advantages over centralized fusion in terms of survivability, autonomy, communication requirements, etc. In target tracking, for example, distributed fusion and centralized fusion are known as track fusion and measurement fusion, respectively.

An important issue in distributed fusion is its performance relative to that of centralized fusion. It is well known that under linear-Gaussian assumption (i.e., linear measurements with jointly Gaussian noise), optimal distributed fusion is algebraically equivalent to centralized fusion if measurement noises are uncorrelated across sensors. It was, however, not known what conditions are necessary and sufficient for such an equivalence in performance to hold in any (general or special) case. Probably more important, in the case such an equivalence does not hold, how much is the performance degradation of the distributed fusion relative to the centralized fusion? This knowledge is very helpful for such tasks as system design in data fusion.

This issue is the topic of numerous publications, including [5, 1, 8, 2, 3, 9, 4]. To the authors' knowledge, however, neither general necessary and sufficient conditions for the distributed and centralized fusion to have identical performance, nor formulas for performance degradation of the former relative to the latter are available. Such formulas are quite useful in, e.g., design of a distributed system.

In this paper, we present such general necessary and sufficient conditions for BLUE fusion and optimal WLS fusion and propose measures of relative efficiency of distributed fusion as compared with the corresponding optimal centralized fusion performance. Although the results will be presented for fusing estimates of x, they are perfectly valid for fusing estimates of x rather than x_k.

The rest of the paper is organized as follows. Sec. 2 presents optimal distributed and centralized BLUE fusion rules for any two-sensor case in a form particularly handy for comparing their performance. A necessary and sufficient condition for the optimal distributed and centralized BLUE fusion to have identical performance is presented in Sec. 3 in a two-sensor system, along with several sufficient conditions. Sec. 4 is dedicated to the important special case of linear observations. A necessary and sufficient optimality condition for a general distributed system of an arbitrary number of sensors is presented in Sec. 5. Sec. 6 presents optimal WLS fusion rules and a necessary and sufficient optimality condition for the optimal distributed fusion. Measures of efficiency of distributed fusion relative to the optimal centralized fusion are proposed in Sec. 7. Numerical examples are provided in Sec. 8. Sec. 9 concludes the paper with a summary.

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2 Optimal Centralized and Distributed BLUE Fusers

Centralized, standard distributed, and distributed fusion architectures have been defined in [7]. The corresponding fusion rules have been presented there, with illustrative examples given in [6]. This paper does not address nonstandard distributed fusion directly and the standard distributed fusion is simply referred to as distributed fusion.

Notation and terminology. The best linear unbiased estimator (BLUE) and the optimal weighted least squares (WLS) estimator using data y are defined by

$$\hat{x}^{\text{BLUE}} = \arg \min_{\hat{x} = a + By} E[(x-\hat{x})(x-\hat{x})'|y] \quad (1)$$

$$\hat{x}^{\text{WLS}} = \arg \min_{\hat{x}} (y - H\hat{x})' R^{-1} (y - H\hat{x}) \quad (2)$$

where a and B do not depend on data y and R is the measurement noise covariance. They minimize mean-square error and fitting error, respectively.

It is well known that the BLUE estimator of x using data z_i is given by

$$\begin{aligned} \hat{x}_i &= \hat{x}(z_i) = E^*[x|z_i] = \bar{x} + K_i \tilde{z}_i \\ P_i &= C_x - K_i S_i K_i' \end{aligned}$$

where[1]

$$\begin{aligned} \hat{x}(z_i) &= \text{estimator of } x \text{ using data } z_i \\ E^*[x|z_i] &= \text{BLUE of } x \text{ using data } z_i \\ \bar{y} &= E[y] = \text{prior mean of } y \\ C_y &= \text{cov}(y) = E[(y-\bar{y})(y-\bar{y})'] \\ C_{uv} &= \text{cov}(u,v) = E[(u-\bar{u})(v-\bar{v})'] \\ S_i &= C_{z_i} \\ K_i &= C_{xz_i} S_i^+ \\ \tilde{z}_i &= z_i - \bar{z}_i = z_i - E[z_i] \\ \tilde{x}_i &= x - \hat{x}_i = \text{estimation error} \\ P_i &= \text{MSE}(\hat{x}_i) = E[\tilde{x}_i \tilde{x}_i'] = \text{MSE matrix of } \hat{x}_i \end{aligned}$$

where MSE stands for mean-square error matrix, and A^+ stands for the unique Moore-Penrose pseudoinverse, MP inverse for short, of A ($A^+ = A^{-1}$ if A^{-1} exists).

Remarks. Use of the MP inverse makes the formulas presented completely general in the sense they are valid for all matrices. An MP inverse A^+ used below can be interpreted as matrix inverse A^{-1} for an easier understanding (at a price of a slight loss of generality). Also note that Matlab function `pinv(A)` computes the MP inverse of any matrix `A` that could be singular or rectangular.

The following theorem is instrumental for our results on BLUE fusion.

Theorem 2.1 (**BLUE Fusion Rule**). The unique BLUE fusion rule using $z = [z_1', z_2']'$ with two arbitrary observations z_1 and z_2 can always be written in the following form:

$$\begin{aligned} \hat{x}^{cf} &= E^*[x|z_1, z_2] = E^*[x|\hat{x}_1, \tilde{z}_{2|1}] \\ &= \hat{x}_1 + C_{\tilde{x}_1 \tilde{z}_{2|1}} C_{\tilde{z}_{2|1}}^+ \tilde{z}_{2|1} \\ P_{cf} &= \text{MSE}(\hat{x}^{cf}) = P_1 - C_{\tilde{x}_1 \tilde{z}_{2|1}} C_{\tilde{z}_{2|1}}^+ C_{\tilde{x}_1 \tilde{z}_{2|1}}' \end{aligned}$$

where

$$\begin{aligned} \hat{x}_1 &= E^*[x|z_1] = \bar{x} + K_1 \tilde{z}_1 \\ P_1 &= \text{MSE}(\hat{x}_1) = C_x - K_1 S_1 K_1' \\ \tilde{z}_{2|1} &= z_2 - E^*[z_2|z_1] = \tilde{z}_2 - C_{z_2 z_1} C_{z_1}^+ \tilde{z}_1 \end{aligned}$$

By the orthogonality principle, we have $\tilde{x}_1 \perp \tilde{z}_1$ and $\tilde{z}_{2|1} \perp \tilde{z}_1$, and then it can be easily shown

$$\begin{aligned} C_{\tilde{x}_1 \tilde{z}_{2|1}} &= C_{x \tilde{z}_{2|1}} = C_{\tilde{x}_1 z_2} = C_{x z_2} - K_1 C_{z_1 z_2} \\ C_{\tilde{z}_{2|1}} &= S_2 - C_{z_2 z_1} S_1^+ C_{z_2 z_1}' \end{aligned}$$

This theorem indicates that use of $z = [z_1', z_2']'$ and $\tilde{z} = [\hat{x}_1', \tilde{z}_{2|1}']'$ is equivalent for BLUE fusion. The use of the notation $E^*[x|\hat{x}_1, \tilde{z}_{2|1}]$ above is justified by the fact that as shown in Appendix, $E^*[x|\hat{x}_1] = \hat{x}_1$.

Note that Theorem 2.1 is valid for any observations z_1 and z_2. In particular, it is valid for $z_1 = \hat{x}_1$ and $z_2 = \hat{x}_2$. In other words, distributed fusion is a special case of centralized fusion with $z_1 = \hat{x}_1$ and $z_2 = \hat{x}_2$. Also, Theorem 1 of [7] for centralized fusion in the two-sensor case is a special case of this theorem in the case of linear observations.

Thus, as a corollary of Theorem 2.1 by treating $\hat{x}_2$ as the observation z_2 of x, we have the following theorem.

Theorem 2.2 (**Distributed BLUE Fusion Rule**). The distributed BLUE fusion rules can always be written in the following form:

$$\begin{aligned} \hat{x}^{df} &= E^*[x|\hat{x}_1, \hat{x}_2] = \hat{x}_1 + C_{\tilde{x}_1 \tilde{x}_{2|1}} C_{\tilde{x}_{2|1}}^+ \tilde{x}_{2|1} \\ P_{df} &= \text{MSE}(\hat{x}^{df}) = P_1 - C_{\tilde{x}_1 \tilde{x}_{2|1}} C_{\tilde{x}_{2|1}}^+ C_{\tilde{x}_1 \tilde{x}_{2|1}}' \end{aligned}$$

where

$$\begin{aligned} \tilde{x}_1 &= x - \hat{x}_1 = \tilde{x} - K_1 \tilde{z}_1 \\ \tilde{x}_{2|1} &= \hat{x}_2 - E^*[\hat{x}_2|\hat{x}_1] = K_2 \tilde{z}_2 - C_{\hat{x}_2 \hat{x}_1} C_{\hat{x}_1}^+ K_1 \tilde{z}_1 \\ C_{\tilde{x}_1 \tilde{x}_{2|1}} &= C_{x \tilde{x}_{2|1}} = C_{\tilde{x}_1 \hat{x}_2} \end{aligned}$$

This theorem and Theorem 1 of [7] for standard distributed fusion are equivalent for the two-sensor case.

By the orthogonality principle, we have $\tilde{x}_1 \perp (\hat{x}_1 - \bar{x}_1)$ and $\tilde{x}_{2|1} \perp (\hat{x}_1 - \bar{x}_1)$, and then it can be easily shown that

$$\begin{aligned} C_{\hat{x}_i} &= K_i S_i K_i' \\ C_{\hat{x}_2 \hat{x}_1} &= K_2 C_{z_2 z_1} K_1' \\ C_{\tilde{x}_1 \tilde{x}_{2|1}} &= C_{x \tilde{x}_{2|1}} = C_{\tilde{x}_1 \hat{x}_2} = C_{\tilde{x}_1 z_2} K_2' = C_{\hat{x}_2} - C_{\hat{x}_2 \hat{x}_1}' \\ C_{\tilde{x}_{2|1}} &= K_2 S_2 K_2' \\ &\quad - (K_2 C_{z_2 z_1} K_1')(K_1 S_1 K_1')^+ (K_2 C_{z_2 z_1} K_1')' \end{aligned}$$

Although optimal distributed and centralized fusion rules have other equivalent forms, the two forms presented above are particularly convenient for the establishment of a necessary and sufficient condition for them to have identical performance. This is the topic of the next section.

[1] This is because BLUE estimator has many properties akin to those of conditional mean, although $E^*[x|z]$ is not necessarily conditional mean.

3 Optimality Condition for Two-Sensor Distributed BLUE Fusion

We first present a necessary and sufficient condition for the optimal distributed BLUE fusion rule to have the same performance as the optimal centralized BLUE fusion rule. It is clear from a comparison of Theorems 2.1 and 2.2 that this is the case if and only if the two fusion rules have the same MSE matrix:

$$P_{cf} = P_{df} \tag{3}$$

Since

$$\begin{aligned} P_{cf} &= P_1 - C_{\tilde{x}_1 \tilde{z}_{2|1}} C^+_{\tilde{z}_{2|1}} C'_{\tilde{x}_1 \tilde{z}_{2|1}} \\ &= P_1 - C_{\tilde{x}_1 z_2} C^+_{\tilde{z}_{2|1}} C'_{\tilde{x}_1 z_2} \\ P_{df} &= P_1 - C_{\tilde{x}_1 \tilde{x}_{2|1}} C^+_{\tilde{x}_{2|1}} C'_{\tilde{x}_1 \tilde{x}_{2|1}} \\ &= P_1 - C_{\tilde{x}_1 z_2} K'_2 C^+_{\tilde{x}_{2|1}} K_2 C'_{\tilde{x}_1 z_2} \end{aligned}$$

(3) is equivalent to

$$C_{\tilde{x}_1 z_2} (C^+_{\tilde{z}_{2|1}} - K'_2 C^+_{\tilde{x}_{2|1}} K_2) C'_{\tilde{x}_1 z_2} = O$$

Note that the BLUE estimator that minimizes MSE matrix is unique almost surely (i.e., unique except possibly for a set of measurement space with zero probability). If this condition is satisfied, by the uniqueness of the BLUE, the distributed and centralized fusion rules are identical (almost surely).

We formally state the above necessary and sufficient condition as a theorem.

Theorem 3.1 (**Optimality Condition for Distributed BLUE Fusion**). Assume that local estimators are BLUE. The optimal distributed and centralized BLUE fusion results are identical (almost surely) if and only if the following condition is satisfied

$$C_{\tilde{x}_1 z_2} (C^+_{\tilde{z}_{2|1}} - K'_2 C^+_{\tilde{x}_{2|1}} K_2) C'_{\tilde{x}_1 z_2} = O \tag{4}$$

We present below as corollaries of Theorem 3.1 several sufficient conditions for the optimal distributed and centralized BLUE fusion rules to be equivalent.

Corollary 3.1. The optimal distributed and centralized BLUE fusion rules are identical (almost surely) if $K_1 C_{z_1 z_2} = C_{x z_2}$ or $K_2 C_{z_2 z_1} = C_{x z_1}$.

Proof. This corollary follows immediately from (4) since in this case $C_{\tilde{x}_1 z_2} = C_{x z_2} - K_1 C_{z_1 z_2} = O$. Condition $K_2 C_{z_2 z_1} = C_{x z_1}$ follows from the symmetry of z_1 and z_2 since the results cannot be dependent on the arbitrary labelling of sensor identity.

Corollary 3.2. The optimal distributed and centralized BLUE fusion rules are identical (almost surely) if measurements are uncorrelated across sensors, that is, $C_{z_1 z_2} = O$.

Proof. When $C_{z_1 z_2} = O$, we have $C_{\tilde{z}_{2|1}} = S_2$, $C_{\tilde{x}_1 z_2} = C_{x z_2}$, and $C_{\tilde{x}_{2|1}} = K_2 S_2 K'_2$, and thus (4) follows from below

$$\begin{aligned} & C_{\tilde{x}_1 z_2} K'_2 C^+_{\tilde{x}_{2|1}} K_2 C'_{\tilde{x}_1 z_2} \\ &= C_{x z_2} C^+_{z_2} C'_{x z_2} (C_{x z_2} C^+_{z_2} C'_{x z_2})^+ C_{x z_2} C^+_{z_2} C'_{x z_2} \\ &= C_{x z_2} C^+_{z_2} C'_{x z_2} = C_{\tilde{x}_1 z_2} C^+_{\tilde{z}_{2|1}} C'_{\tilde{x}_1 z_2} \end{aligned}$$

4 BLUE Fusion with Linear Observations

Consider the special case that observations are linear in the estimatee (i.e., the quantity to be estimated) x:

$$z_i = H_i x + v_i$$

with known $E[v_i] = \bar{v}_i$, $E[x] = \bar{x}$, $\text{cov}(v_i) = R_i$, $\text{cov}(v_i, v_j) = R_{ij}$, $\text{cov}(x, v_i) = V_i$, and $\text{cov}(x) = C_x$.

In this case, the measurement residual covariance and the gain are

$$\begin{aligned} S_i &= C_{z_i} = H_i C_x H'_i + H_i V_i + (H_i V_i)' + R_i \\ K_i &= C_{x z_i} C^+_{z_i} = (C_x H'_i + V_i) S^+_i \end{aligned}$$

and the centralized and distributed BLUE fusion rules are

$$\begin{aligned} \hat{x}^{cf} &= \bar{x} + K_1 \tilde{z}_1 + C_{\tilde{x}_1 z_2} C^+_{\tilde{z}_{2|1}} (\tilde{z}_2 - K_1 \tilde{z}_1) \\ P_{cf} &= C_x - K_1 S_1 K'_1 - C_{\tilde{x}_1 z_2} C^+_{\tilde{z}_{2|1}} C'_{\tilde{x}_1 z_2} \\ \hat{x}^{df} &= \hat{x}_1 + C_{\tilde{x}_1 z_2} K'_2 C^+_{\tilde{x}_{2|1}} [K_2 \tilde{z}_2 \\ & \quad -(K_2 C'_{z_1 z_2} K'_1)(K_1 S_1 K'_1)^+ K_1 \tilde{z}_1] \\ P_{df} &= C_x - K_1 S_1 K'_1 - C_{\tilde{x}_1 z_2} K'_2 C^+_{\tilde{x}_{2|1}} K_2 C'_{\tilde{x}_1 z_2} \end{aligned}$$

where

$$\begin{aligned} \tilde{z}_i &= z_i - (H_i \bar{x} + \bar{v}_i) \\ C_{z_1 z_2} &= H_1 C_x H'_2 + H_1 V_2 + V'_1 H'_2 + R_{12} \\ C_{\tilde{x}_1 z_2} &= C_x H'_2 + V_2 - K_1 C_{z_1 z_2} \\ C_{\tilde{z}_{2|1}} &= S_2 - C'_{z_1 z_2} S^+_1 C_{z_1 z_2} \\ C_{\tilde{x}_{2|1}} &= K_2 S_2 K'_2 \\ & \quad - (K_2 C'_{z_1 z_2} K'_1)(K_1 S_1 K'_1)^+ (K_2 C'_{z_1 z_2} K'_1)' \end{aligned}$$

The two BLUE fusion rules are equivalent if and only if

$$C_{\tilde{x}_1 z_2} (C^+_{\tilde{z}_{2|1}} - K'_2 C^+_{\tilde{x}_{2|1}} K_2) C'_{\tilde{x}_1 z_2} = O \tag{5}$$

Theorem 4.1. Assume that R_i, and C_x are positive definite rather than just positive semidefinite, H_i have full column rank. For BLUE fusion with linear observations, the distributed fusion and centralized fusion are equivalent if $C_{z_1 z_2} = (H_2 C_{x z_1})'$; that is, if

$$H_1 V_2 + R_{12} = O \tag{6}$$

or equivalently (by symmetry of two sensors), $C_{z_2 z_1} = (H_1 C_{x z_2})'$ or

$$H_2 V_1 + R'_{12} = O \tag{7}$$

Corollary. Under the stated assumption for BLUE fusion with linear observations, the distributed fusion and centralized fusion are equivalent if the observation errors are uncorrelated across sensors (i.e., $R_{12} = O$) and uncorrelated with the estimatee (i.e., $V_i = O$).

5 Optimality Condition for General Distributed BLUE Fusion

A necessary and sufficient condition for the distributed fusion to have identical performance as the centralized fusion for a two-sensor system is presented in Sec. 3. It can be generalized to a general distributed system with an arbitrary n sensors by treating $z_1 := [z_1', z_2', \ldots, z_{n-1}']'$, $z_2 := z_n$, $\hat{x}_1 := [\hat{x}_1', \hat{x}_2', \ldots, \hat{x}_{n-1}']'$, and $\hat{x}_2 := \hat{x}_n$. This, however, will lead to a condition that is quite messy. Instead, we present in this section an alternative but equivalent condition for such a general system, which is much more elegant.

Consider a parallel distributed system with n sensors. Denote the stacked vectors of local observations and estimates as

$$z = [z_1', z_2', \ldots, z_n']', \quad y = [\hat{x}_1', \hat{x}_2', \ldots, \hat{x}_n']'$$

Note that for any local estimates $\hat{x}_i = \bar{x} + K_i \tilde{z}_i$, we have

$$y = \bar{x} + K\tilde{z}, \qquad K = \text{diag}(K_1, \ldots, K_n)$$

where $\tilde{z} = z - E[z]$. As established in [7], centralized and distributed BLUE fusion rules for this system are nothing but BLUE estimators using z and y, respectively. Then, from the BLUE theory, we have the following theorem.

Theorem 5.1 (**Optimal BLUE Fusion**). The centralized and distributed BLUE fusion rules for the above n-sensor system are given respectively by

$$\begin{aligned}
\hat{x}^{cf} &= E^*[x|z] = \bar{x} + C_{xz} C_z^+ \tilde{z} \\
P_{cf} &= \text{MSE}(\hat{x}^{cf}) = C_x - C_{xz} C_z^+ C_{xz}' \\
\hat{x}^{df} &= E^*[x|y] = \bar{x} + C_{xy} C_y^+ \tilde{y} \\
P_{df} &= \text{MSE}(\hat{x}^{df}) = C_x - C_{xy} C_y^+ C_{xy}'
\end{aligned}$$

where $C_z = \text{cov}(z)$ and

$$\begin{aligned}
C_{xz} &= \text{cov}(x, z) = [C_{xz_1}, \ldots, C_{xz_n}] \\
C_{xy} &= \text{cov}(x, y) = C_{xz} K' \\
C_y &= \text{cov}(y) = K C_z K'
\end{aligned}$$

Based on this theorem and following the same steps as in Sec. 3, we immediately have the following theorem.

Theorem 5.2 (**Optimality Condition for General Distributed BLUE Fusion**). The optimal distributed and centralized BLUE fusion rules are identical (almost surely) if and only if the following condition is satisfied

$$C_{xz}[C_z^+ - K'(KC_zK')^+ K] C_{xz}' = O \tag{8}$$

Note that the local estimators need not be BLUE for Theorems 5.1 and 5.2 to hold. Theorem 5.2 is not only more elegant but also more general than Theorem 3.1, which is valid only for a two-sensor system with BLUE local estimators, although it can be extended to the general case. However, this condition involves matrices that have much higher dimensions than those involved in Theorem 3.1. For example, let $\dim(z_i) = m, \forall i$. Then, C_z of this theorem is $(nm) \times (nm)$ while $C_{\tilde{z}_{2|1}}$ of Theorem 3.1 is only $m \times m$.

6 Optimal Weighted LS Fusion

Consider the measurement model of Sec. 4. Let

$$\begin{aligned}
z &= [z_1', \ldots, z_n']', \quad H = [H_1', \ldots, H_n']' \\
R &= \text{cov}(v) = [R_{ij}] = [\text{cov}(v_i, v_j)], \quad v = [v_1, \ldots, v_2]'
\end{aligned}$$

Theorem 6.1 (**Optimal Centralized WLS Fusion**). The optimal centralized weighted least-squares fusion using $z = Hx + v$ with $R = cov(v)$ is given by

$$\begin{aligned}
\hat{x}^{cf} &= \hat{x}(z) = (H'R^{-1}H)^{-1} H'R^{-1} z \\
P_{cf} &= \text{MSE}(\hat{x}(z)) = (H'R^{-1}H)^{-1}
\end{aligned}$$

This centralized fusion rule is well known.

Consider now distributed fusion. Let $y = [\hat{x}_1', \ldots, \hat{x}_n']'$. Note that

$$\hat{x}_i = x - (x - \hat{x}_i) = x + \tilde{v}_i$$

or

$$\begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{bmatrix} = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} x + \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix}$$

Then as established in [7], by treating y as a vector-valued measurement of x, we have the following theorem.

Theorem 6.2 (**Optimal Distributed WLS Fusion**). The optimal distributed weighted least-squares fusion using $y = [\hat{x}_1', \ldots, \hat{x}_n']'$ is given by

$$\begin{aligned}
\hat{x}^{df} &= \hat{x}(y) = (\tilde{H}'\tilde{R}^{-1}\tilde{H})^{-1} \tilde{H}'\tilde{R}^{-1} y \\
P_{df} &= \text{MSE}(\hat{x}(y)) = (\tilde{H}'\tilde{R}^{-1}\tilde{H})^{-1}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{H} &= [I, \ldots, I]' \\
\tilde{R} &= \left[\tilde{R}_{ij}\right] = [\text{cov}(\tilde{x}_i, \tilde{x}_j)], \quad \tilde{x}_i = x - \hat{x}_i
\end{aligned}$$

Therefore, the optimal centralized and distributed weighted least-squares fusion rules have the same performance if and only if $\text{MSE}(\hat{x}(z)) = \text{MSE}(\hat{x}(y))$, that is,

$$(H'R^{-1}H)^{-1} = (\tilde{H}'\tilde{R}^{-1}\tilde{H})^{-1}$$

or equivalently, $H'R^{-1}H = \tilde{H}'\tilde{R}^{-1}\tilde{H}$. We state this fact formally as a theorem.

Theorem 6.3 (**Optimality Condition for Distributed WLS Fusion**). The optimal centralized and distributed weighted least-squares fusion rules have identical performance if and only if

$$H'R^{-1}H = \tilde{H}'\tilde{R}^{-1}\tilde{H} \tag{9}$$

Note that the local estimators need not be optimal WLS estimators Theorems 6.2 and 6.3 to hold. If local estimates are unbiased and given by $\hat{x}_i = K_i z_i$, then $K_i H_i = I$ due to the unbiasedness requirement and

$$\tilde{x}_i = x - \hat{x}_i = K_i H_i x - K_i z_i = K_i(H_i x - z_i) = -K_i v_i$$

and thus

$$\tilde{R} = \left[K_i R_{ij} K_j'\right] = KRK', \quad K = \text{diag}(K_1, \ldots, K_n)$$

With this, we present the following theorem.

Theorem 6.4 (**Sufficient Condition for Optimality of Distributed WLS Fusion**). Assume that local estimates are from optimal WLS estimators. Then the optimal centralized and distributed weighted least-squares fusion rules have identical performance if the measurement noises are uncorrelated across sensors:

$$R_{ij} = O, \quad i \neq j$$

Proof. When local estimates are from optimal WLS estimators, $\hat{x}_i = K_i z_i = (H_i' R_i^{-1} H_i)^{-1} H_i' R_i^{-1} z_i$ and

$$\begin{aligned} \tilde{R} &= \left[K_i R_{ij} K_j'\right] = \text{diag}(K_1 R_1 K_1', \ldots, K_n R_n K_n') \\ \tilde{R}^{-1} &= \text{diag}[(K_1 R_1 K_1')^{-1}, \ldots, (K_n R_n K_n')^{-1}] \\ &= \text{diag}(H_1' R_1^{-1} H_1, \ldots, H_n' R_n^{-1} H_n) \\ \tilde{H}' \tilde{R}^{-1} \tilde{H} &= H_1' R_1^{-1} H_1, + \cdots + H_n' R_n^{-1} H_n = H' R^{-1} H \end{aligned}$$

The theorem thus follows from Theorem 6.3.

7 Relative Efficiency of Distributed Fusion

Since distributed fusion does not yield in general the performance of the centralized fusion, a question arises naturally: How much performance degradation is the (optimal) distributed fusion relative to the optimal centralized fusion?

This can be answered by comparing the MSE matrices of the optimal centralized and distributed fusion rules. Clearly, the *performance degradation* in terms of MSE matrix is given by MSE($\hat{x}^{df}$)$-$ MSE($\hat{x}^{cf}$). For BLUE fusion, it is given by

$$\begin{aligned} &\text{MSE}(\hat{x}^{df}) - \text{MSE}(\hat{x}^{cf}) \\ &= \begin{cases} C_{xz}[C_z^+ - K'(KC_zK')^+K]C_{xz}' & \text{any } n \\ C_{\tilde{x}_1 z_2}(C_{\tilde{z}_{2|1}}^+ - K_2' C_{\tilde{x}_{2|1}}^+ K_2) C_{\tilde{x}_1 z_2}' & n = 2 \end{cases} \end{aligned}$$

For optimal WLS fusion, it is given by

$$\text{MSE}(\hat{x}^{df}) - \text{MSE}(\hat{x}^{cf}) = (\tilde{H}' \tilde{R}^{-1} \tilde{H})^{-1} - (H' R^{-1} H)^{-1}$$

This degradation is matrix-valued and not convenient to use. It may be replaced by the following scalar measure of *performance degradation* in terms of scalar MSE: mse($\hat{x}^{df}$)$-$ mse($\hat{x}^{cf}$), or

$$\begin{aligned} &\text{mse}(\hat{x}^{df}) - \text{mse}(\hat{x}^{cf}) \\ &= \begin{cases} \text{tr}\left(C_{xz}[C_z^+ - K'(KC_zK')^+K]C_{xz}'\right), & \text{BLUE} \\ \text{tr}[(\tilde{H}' \tilde{R}^{-1} \tilde{H})^{-1} - (H' R^{-1} H)^{-1}], & \text{WLS} \end{cases} \end{aligned}$$

where tr(A) stands for trace of matrix A and

$$\begin{aligned} \text{MSE}(\hat{x}) &= E[(x - \hat{x})(x - \hat{x})'] \\ \text{mse}(\hat{x}) &= E[(x - \hat{x})'(x - \hat{x})] \end{aligned}$$

This is an absolute difference. Its magnitude does not reflect the performance efficiency of the optimal distributed fusion relative to the optimal centralized fusion. To judge how well a distributed fusion rule is relative to the optimal centralized fusion, it is intuitively more appealing to look into relative difference, rather than the above absolute difference. Clearly, the smaller the relative difference the more efficient. As a measure or index for the relative efficiency of the distributed fusion, it would be more desirable that a greater value implies better efficiency.

In view of the above, we introduce two measures of relative efficiency of the distributed fusion: mse ratio (MSER) and generalized error variance ratio (GEVR), defined by

$$\text{MSER} = \frac{\text{mse}(\hat{x}^{cf})}{\text{mse}(\hat{x}^{df})} = \frac{\text{tr}[\text{MSE}(\hat{x}^{cf})]}{\text{tr}[\text{MSE}(\hat{x}^{df})]} \quad (10)$$

$$\text{GEVR} = \frac{\det[\text{MSE}(\hat{x}^{cf})]}{\det[\text{MSE}(\hat{x}^{df})]} \quad (11)$$

where $\det(A)$ stands for determinant of matrix A. Note that the determinant of a covariance matrix is known as generalized variance in statistics. Both measures have the following nice property:

$$0 \leq \text{MSER} \leq 1, \quad 0 \leq \text{GEVR} \leq 1$$

Clearly, the greater these measures, the more efficient the distributed fusion rule is. If any of the above necessary and sufficient condition is satisfied, then MSER $=$ GEVR $= 1$.

The above concepts and measures are general. They are valid for any distributed fusion rule that is not necessarily optimal or linear.

8 Numerical Examples

Several simple numerical examples are given in this section to verify the formulas presented and to demonstrate the relative efficiency of the optimal distributed fusion compared with the optimal centralized fusion. In all these examples, a distributed system of two or three sensors is considered and the following linear measurements are used

$$z_l^i(k) = H_l^i x(k) + v_l^i, \ \forall i; \ l = 1, 2, 3, 4, \ k = 1, \ldots, 100$$

where superscript i and subscript l stand for quantities pertaining to sensor i and lth measurement, and

$$H_l^i = \begin{cases} \frac{1}{20}[1, 5] & l = 1 \\ \frac{l}{20}[0, 1] & l \neq 1 \end{cases}, \ R_l^i(k) = \text{cov}(v_l^i(k)) = 1$$

For BLUE fusion, $x(k) \sim \mathcal{N}[\bar{x}(k), C_x]$ was generated, where

$$\bar{x}(k) = \begin{bmatrix} 10\cos(2k\pi/100) \\ 10\sin(2k\pi/100) \end{bmatrix}, \ C_x = \text{cov}[x(k)] = (1/3)I$$

are known to the BLUE fuser. For the WLS fusion, the true $x(k)$ was not generated as random.

The observation noises are both white noise sequences and thus the short notation v_l^i is used.

All simulation results are averages over 100 Monte-Carlo runs.

8.1 Example 1: Uncorrelated Error

In this example, the system consists of three sensors and the observation errors are uncorrelated across sensors (i.e., $R_{ij} = O$) and uncorrelated with the state (i.e., $V_i = O$) for the BLUE (for WLS, x is not random). The following parameters were used:

$$\text{cov}(v_l^i, v_m^j) = \delta_{l-m}\delta_{i-j}R_l^i, \quad \text{cov}(x, v_l^i) = O$$

where stands δ_{i-j} for the Kronecker delta function.

It can be easily checked that the necessary and sufficient conditions (8) and (9) are satisfied in this case.

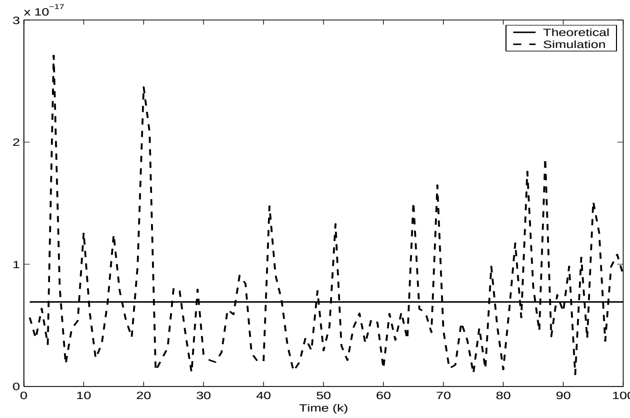
Let the difference in the (theoretical and sample) MSE matrices of distributed and centralized fusion be

$$E = \begin{cases} C_{xz}[C_z^+ - K'(KC_zK')^+K]C'_{xz} & \text{BLUE} \\ (\tilde{H}'\tilde{R}^{-1}\tilde{H})^{-1} - (H'R^{-1}H)^{-1} & \text{optimal WLS} \end{cases}$$

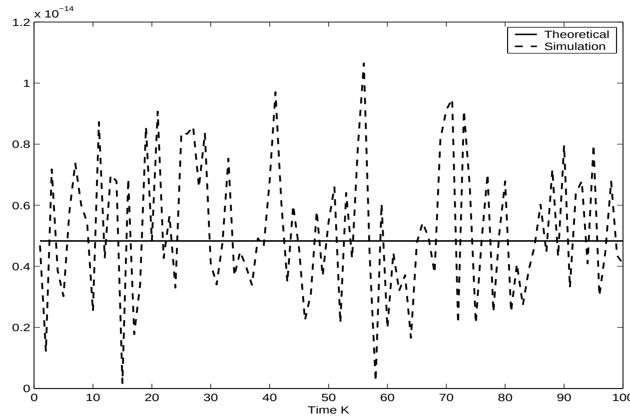
As such, the optimality condition (8) or (9) is satisfied if and only if $E = O$, or its Frobenius norm is zero:

$$\|E\|_F = \left(\sum_{i,j} a_{ij}^2\right)^{1/2} = 0$$

The arithmetic averages of the norms for the theoretical (calculated by the computer) and sample MSE matrices over 100 Monte-Carlo runs are plotted in Fig. 1. It is seen to be of the order of computer rounding errors (10^{-16}) and thus it verifies that optimal distributed and centralized fusion estimates are equal in every run.



(a) BLUE



(b) Optimal WLS

Fig. 1: Average normalized Frebenius norm of MSE matrix difference for Example 1.

8.2 Example 2: Correlation of Error and State for BLUE Fusion

In this example, the observation errors are uncorrelated across sensors (i.e., $R_{ij} = O$) but correlated with the state (i.e., $V_i \neq O$). The following parameters were used:

$$\text{cov}(v_l^i, v_m^j) = \delta_{l-m}\delta_{i-j}R_l^i,$$
$$\text{cov}(x_k, v_l^i) = \frac{1}{(k+m)}\begin{bmatrix}1\\3\end{bmatrix}, \; m = \begin{cases}16, & 2 \text{ sensors}\\ 18, & 3 \text{ sensors}\end{cases}$$

It can be easily checked that the optimality condition (8) is not satisfied in this case. Fig. 2(a) shows that the difference in the scalar MSE between optimal distributed and centralized fusion is not zero. This verifies that they are not equivalent and thus the distributed fusion is inferior, although the performance degradation is small. This demonstrates that the optimality conditions presented are exact. Fig. 2(b) plots the relative efficiency of the optimal distributed fusion in terms of the theoretical mean-square error ratio (MSER) and generalized error variance ratio (GEVR), defined by (10)–(11). It also verifies how small the performance degradation is. The corresponding sample MSER and GEVR are not plotted because the small changes in theoretical MSER and GEVR would become unnoticeable due to the relatively large fluctuations in the sample MSER and GEVR. Note that the optimal distributed fusion becomes more efficient because the correlation between the observation noise and the state becomes weaker as time goes.

8.3 Example 3: Correlated Error

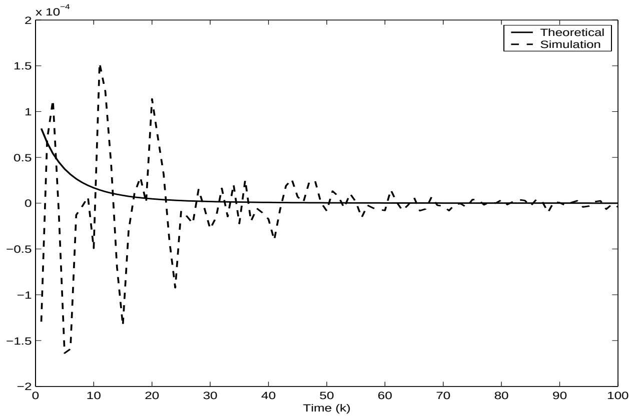
As shown in a forthcoming paper, the observation errors of a sample system of a continuous-time multiple-sensor system are correlated. The performance degradation reported in [4] can also be attributed to this correlation across sensors in the presence of a common process.

This example is a minor modification of the example considered in [6]. In this example, the observation errors are correlated across sensors (i.e., $R_{ij} \neq O$) but uncorrelated with the state (i.e., $V_i = O$). Both 2-sensor and 3-sensor cases were considered. The following parameters were used:

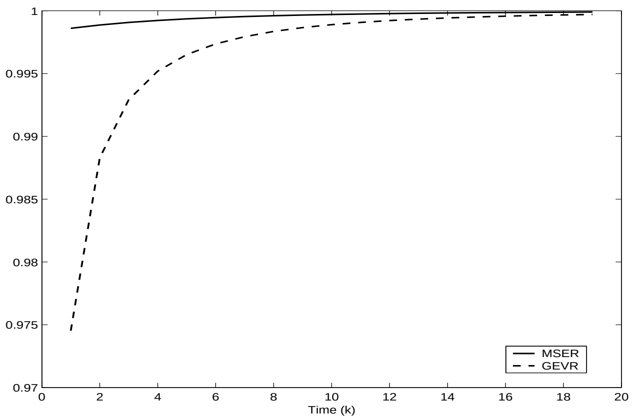
$$\text{cov}(v_{l_1}^i, v_{l_2}^j) = 3^{\min\{l_1,l_2\}-1}(0.8)^{(l_1+l_2)}\begin{bmatrix}\alpha_k & 0\\0 & 0\end{bmatrix}, \; i \neq j$$
$$\alpha_k = \begin{cases}-0.129 + \frac{0.258n}{99} & \text{2-sensor case}\\ -0.065 + \frac{0.13n}{99} & \text{3-sensor case}\end{cases}$$
$$\text{cov}(v_l^i, v_m^i) = \delta_{l-m}R_l^i, \quad \text{cov}(x, v_l^i) = 0$$

It can be checked that none of the corresponding optimality conditions (4), (8), and (9) are satisfied in this case. Simulation results (not shown) verify this.

Fig. 3 plots the relative efficiency of distributed fusion in terms of the theoretical MSER and GEVR for BLUE and

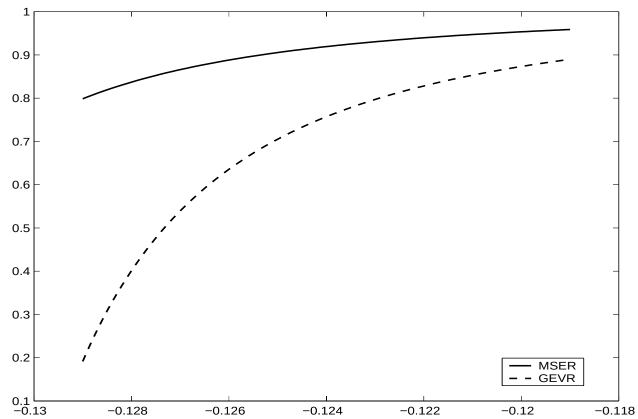


(a) Difference in scalar MSE

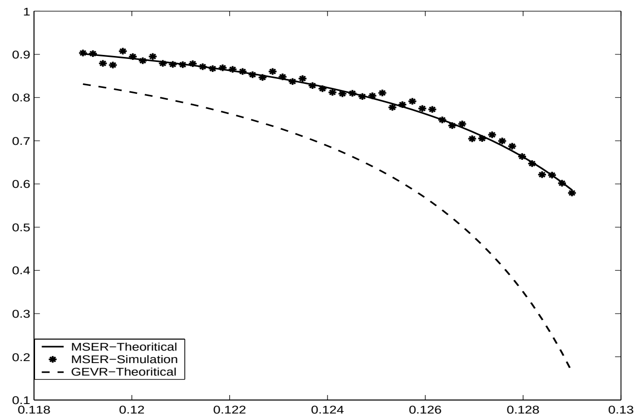


(b) MSER and GEVR

Fig. 2: Relative efficiency of distributed fusion for Example 2.



(a) BLUE



(b) Optimal WLS

Fig. 3: Relative efficiency of distributed fusion for Example 3 (2-sensor case).

optimal WLS fusion rules for the 2-sensor case. The horizontal axis is the value of α_k. Also plotted is the simulation result of MSER, which verifies the theoretical MSER. Fig. 4 plots the corresponding results for BLUE for the 3-sensor case and a comparison between BLUE and optimal WLS fusion for the 2-sensor case. These plots illustrate that the relative efficiency of the optimal (BLUE or WLS) distributed fusion could be quite low—the optimal distributed fusion could be quite inferior to the optimal centralized fusion. It is interesting to note that the relative efficiency of the optimal distributed fusion deteriorates as the covariance is getting closer to be singular. Note that although the distributed BLUE fusion is more efficient than the distributed optimal WLS fusion over the interval shown, it can be less efficient for other values of α_k.

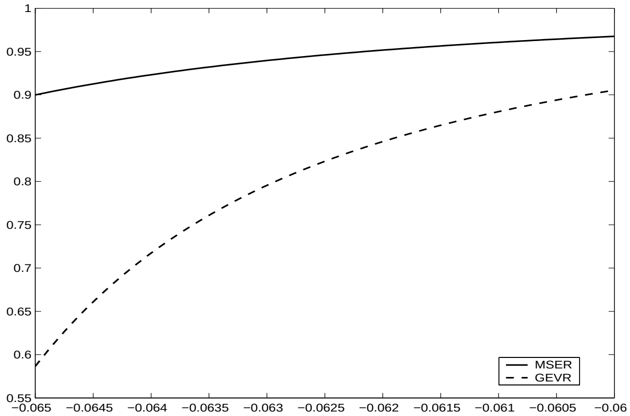
Fig. 5 shows a comparison between 2- and 3-sensor cases, where Fig. 5(a) plots $\log(1-\text{MSER})$, which is over a larger interval of the α_k values than those shown in Figs. 3 and 4. The smaller $\log(1-\text{MSER})$ is, the better. Note that α_k cannot have an arbitrarily large magnitude due to positive semidefiniteness of the noise covariance matrix.

As demonstrated in Fig. 5, the performance degradation is more significant when more sensors are involved, as expected.

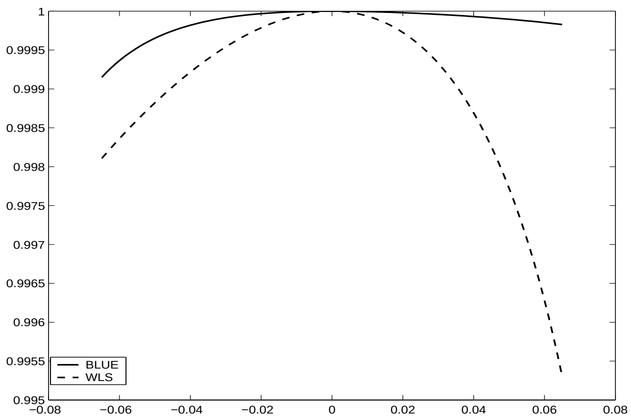
We found from all cases tested that GEVR is more sensitive than MSER to the relative efficiency.

9 Summary

Simple necessary and sufficient conditions for the optimal distributed and centralized BLUE fusion and optimal WLS fusion to have identical performance have been presented, which can be easily checked. Several measures of efficiency of distributed fusion relative to centralized fusion have been proposed, which include the ratio of the mean-square errors of centralized fusion and distributed fusion. These measures quantify the performance degradation of distributed fusion to the optimal centralized fusion. It has been shown both theoretically and by simulation results that the optimal distributed and centralized fusion rules using linear measurements have identical performance in general only when measurement errors are uncorrelated across sensors; the former is inferior to the latter in general when the measurement errors are correlated across sensors or are correlated with the estimatee (i.e., state). Numerical examples provided have verified the theoretical results presented and demonstrated that the optimal distributed fusion could be

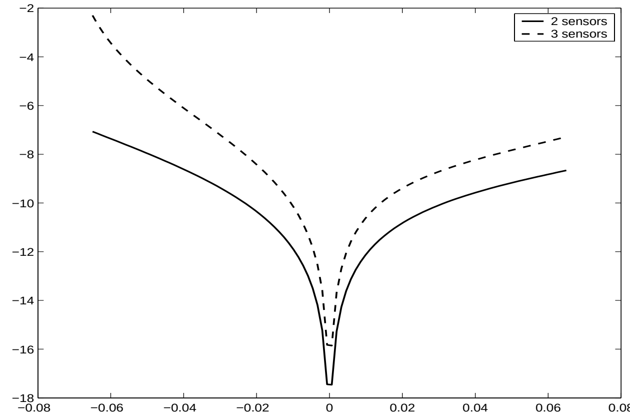


(a) BLUE for 3-sensor case

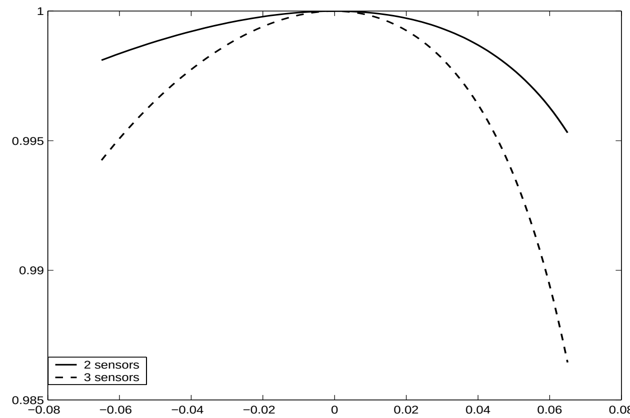


(b) BLUE vs. WLS for 2-sensor case (MSER)

Fig. 4: Relative efficiency of distributed fusion for Example 3.



(a) BLUE $[\log(1 - \text{MSER})]$



(b) Optimal WLS (MSER)

Fig. 5: Relative efficiency of distributed fusion for Example 3 (2-sensor and 3-sensor cases).

quite inferior in performance to the optimal centralized fusion. We emphasize that the results presented are directly valid for fusion of state estimates by treating the state x_k as our x above (i.e., viewing the above as a snapshot at the time of fusion).

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