Lexicographic Max-Min Fairness in a Wireless Ad Hoc Network with Random Access

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Abstract

We consider the lexicographic max-min fair rate control problem at the link layer in a random access wireless network. In lexicographic max-min fair rate allocation, the minimum link rates are maximized in a lexicographic order. For the Aloha multiple access model, we propose iterative approaches that attain the optimal rates under very general assumptions on the network topology and communication pattern; the approaches are also amenable to distributed implementation. The algorithms and results in this paper generalize those in our previous work [7] on maximizing the minimum rates in a random access network, and nicely connects to the "bottleneck-based" lexicographic rate optimization algorithm popularly used in wired networks [1].

1 Introduction

In a wireless network, the Medium Access Control (MAC) protocol defines rules by which nodes regulate their transmission onto the shared broadcast channel. An efficient MAC protocol should ensure high system throughput, and distribute the available bandwidth fairly among the competing nodes. In this paper, we consider the problem of optimizing a random access MAC protocol with the goal of attaining *lexicographic max-min* fair rate allocations at the link layer. Fairness is a key consideration in designing MAC protocols, and the lexicographic max-min fairness metric [1] is one of the most widely used notions of fairness. The objective, stated simply, is to maximize the minimum rates in a lexicographic manner. More specifically, a lexicographic max-min fair rate allocation algorithm should maximize the minimum rate, then maximize the second minimum rate, then maximize the third minimum rate, and so on.

In our previous work [7], we have proposed algorithms that maximizes the minimum rate in a wireless ad hoc network in a distributed manner, and showed that the proposed algorithms can achieve lexicographic max-min fairness under very restrictive "symmetric" communication patterns. However, the question of achieving lexicographic max-min fair rate allocation in a more general wireless ad hoc network, preferably in a distributed manner, remained an open question. In this paper, we propose algorithms that solve this question.

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Specifically, we propose two multi-step approaches that can achieve lexicographic max-min fair allocation. Intuitively, both algorithms attain lexicographic max-min fairness in the network by solving a sequence of max-min rate optimization problem and identifying *bottleneck links* at each step. Loosely speaking, a bottleneck link is a link that has the minimum rate in the network, in all possible optimal allocations. We will prove rigorously the convergence of the two algorithms, and discuss the possibility of implementing the algorithms in a distributed manner.

The paper is organized as follows. Section 2 describes the system model and problem formulation. Section 3 provides a few important definitions which are used later in describing the solution approach. In Section 4, we propose an approach for providing lexicographic max-min fair rate, which is based on identifying a subset of the bottleneck links. In Section 5, we discuss how we can identify all bottleneck links so as to improve the efficiency of the algorithm. Section 5 concludes the work, and all proofs are presented in the appendix.

2 Problem Formulation

2.1 System Model

A wireless network can be modelled as an undirected graph $G = (N, E)$, where N and E respectively denote the set of *nodes* and the set of undirected *edges*. An edge exists between two nodes if and only if they can receive each other's signals (we assume a symmetric hearing matrix). Note that there are $2|E|$ possible communication pairs, but only a subset of these may be actively communicating. The set of active communication pairs is represented by the set of *links*, L. Each link $(i, j) \in L$ is always backlogged. Without loss of generality we assume that all the nodes share a single wireless channel of unit capacity.

For any node i, the set of i's *neighbors*, $K_i = \{j : (i, j) \in E\}$, represents the set of nodes that can receive i's signals. For any node i, the set of *out-neighbors* of i, $O_i = \{j : (i, j) \in L\} \subseteq K_i$, represents the set of neighbors to which i is sending traffic. Also, for any node i, the set of *in-neighbors* of i, $I_i = \{j : (j, i) \in L\} \subseteq K_i$, represents the set of neighbors from which i is receiving traffic. A transmission from node i reaches all of i's neighbors. Each node has a single transceiver. Thus, a node can not transmit and receive simultaneously. We do not assume any capture, i.e., node j can not receive any packet successfully if more than one of its neighbors are transmitting simultaneously. Therefore, a transmission in link $(i, j) \in L$ is successful if and only if no node in $K_j \cup \{j\} \setminus \{i\}$, transmits during the transmission on (i, j).

We focus on random access wireless networks, and use the slotted Aloha model [1] for modeling interference and throughput. In this model, i transmits a packet with probability P_i in a slot. If i does not have an outgoing edge, i.e., $O_i = \phi$, then $P_i = 0$. Once i decides to transmit in a slot, it selects a destination $j \in O_i$ with probability p_{ij}/P_i, where $\sum_{j \in O_i} p_{ij} = P_i$. Therefore, in each slot, a packet is transmitted on link (i, j) with probability p_{ij}. Let $\mathbf{p} = (p_{ij}, (i, j) \in L)$ be the vector of transmission probabilities on all edges, and let $\mathbf{P}_f$ denote the feasible region for $\mathbf{p}$, i.e. $\mathbf{P}_f = \{\mathbf{p} : 0 \leq p_{ij} \leq 1, \forall (i, j) \in L, P_i = \sum_{j \in O_i} p_{ij}, 0 \leq P_i \leq 1, \forall i \in$

$N\}$. Then, the rate or throughput on link (i,j), x_{ij}, is given by

$$x_{ij}(\mathbf{p}) = p_{ij}(1-P_j)\prod_{k\in K_j\setminus\{i\}}(1-P_k), \qquad \mathbf{p}\in\mathbf{P}_f \ . \tag{1}$$

Note that $(1-P_j)\prod_{k\in K_j\setminus\{i\}}(1-P_k)$ is the probability that a packet transmitted on link (i,j) is successfully received at j.

2.2 Lexicographic Max-Min Fair Rate

Let $\mathbf{x} = (x_{ij},(i,j)\in L)$ denote the vector of rates for all links in the active communication set E (also referred to as the *allocation vector*), and $\vec{\mathbf{x}}$ be the allocation vector $\mathbf{x}$ sorted in nondecreasing order. An allocation vector $\mathbf{x}_1$ is said to be *lexicographically greater* than another allocation vector $\mathbf{x}_2$, denoted by $\mathbf{x}_1 \succ \mathbf{x}_2$, if the first non-zero component of $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ is positive. Consequently, an allocation vector $\mathbf{x}_1$ is said to be *lexicographically no less than* than another allocation vector $\mathbf{x}_2$, denoted by $\mathbf{x}_1 \succeq \mathbf{x}_2$, if $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2 = 0$, or the first non-zero component of $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ is positive.

A rate allocation is said to be *lexicographic max-min fair* if the corresponding rate allocation vector is lexicographically no less than any other feasible rate allocation vector. In the lexicographic max-min fair rate allocation vector, therefore, a rate component can be increased only at the cost of decreasing a rate component of equal or lesser value, or by making the vector infeasible.

3 Preliminaries

In this section we introduce a few definitions which are used later in the paper in describing our solution approach, and stating and proving our results. We also define the max-min fair rate allocation problem, and the concept of bottleneck link, which are two main components of our solution approach in solving the lexicographic max-min fair rate allocation problem.

3.1 Directed Link Graph and Its Component Graph

3.1.1 Directed Link Graph

Recall that the transmission on link (i,j) is successful if and only if node j, as well as all neighbors of node j (except node i), are silent. From this, it is straightforward to see that the interference relationship between two links (i,j) and (s,t), may not be symmetric. As an example, consider two links (i,j) and (j,k), $i\neq k$. Obviously, transmission on (i,j) is successful only if (j,k) is silent. However, if i is not in the neighborhood of k, then the successful transmission on link (j,k) does not require that link (i,j) be silent.

We define a directed graph, called *directed link graph* $G_L = (V_L, E_L)$, where each vertex stands for a link in the original network. There is an edge from link (i,j) to link (s,t) in the directed link graph if and only if a successful transmission on link (s,t) requires that link (i,j) be silent.

We use the notation $(i, j) \rightsquigarrow (s, t)$ to denote the case when there is a path from link (i, j) to link (s, t) in the directed link graph. We have the following lemma regarding to the property of the directed link graph.

Lemma 1 (Proof in the Appendix) *Let x^*_{ij} and x^*_{st} respectively denote the lexicographic max-min fair rates for the two links (i, j) and (s, t). If there is a path from (i, j) to (s, t) in a directed link graph, i.e. $(i, j) \rightsquigarrow (s, t)$, then we have $x^*_{ij} \leq x^*_{st}$.*

For a directed graph $G = (V, E)$, the set of predecessors of $u \in V$ is defined as $\mathcal{P}_u = \{v \in V : v \rightsquigarrow u\} \bigcup \{u\}$. Also, for any vertex set $U \subseteq V$, we define $\mathbb{G}_U = (U, \mathbb{E}_U)$ as a subgraph of G for U, where $\mathbb{E}_U = ((u, v) : (u, v) \in E, u \in U, v \in U)$.

3.1.2 Component Graph

In the directed graph $G_L = (V_L, E_L)$, a strongly connected component is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u and v in C, we have both $v \rightsquigarrow u$ and $u \rightsquigarrow v$, that is, vertices u and v are reachable from each other. The following corollaries easily follow from Lemma 1.

Corollary 1 *The lexicographic max-min fair rates of all links belonging to the same strongly connected component of $G_L = (V_L, E_L)$ is the same.*

Corollary 2 *Let C_1 and C_2 be two strongly connected components in the directed link graph $G_L = (V_L, E_L)$, and x^*_1 and x^*_2 be the lexicographic max-min fair rates for C_1 and C_2, respectively. If $u \in C_1$ and $v \in C_2$ such that $u \rightsquigarrow v$, then we have $x^*_1 \leq x^*_2$.*

For a directed link graph $G_L = (V_L, E_L)$, we can decompose it into its strongly connected components, and construct the *component graph* $\mathcal{G}_L = (\mathcal{V}_L, \mathcal{E}_L)$, which we define as follows. Suppose G_L has strongly connected components C_1, C_2, ..., C_k. The vertex set $\mathcal{V}_L$ is $\{v_1, v_2, ..., v_k\}$, and it contains a vertex v_i for each strongly connected component C_i of G_L. There is a directed edge $(v_i, v_j) \in \mathcal{E}_L$ if G_L contains a directed edge (x, y) for some $x \in C_i$ and some $y \in C_j$. Viewed another way, we obtain $\mathcal{G}_L$ from G_L by contracting all edges whose incident vertices belong to the same strongly connected component of G_L. From Lemma 22.13 of [6], it follows that the component graph is a directed acyclic graph.

For any $v \in \mathcal{V}_L$, we denote $\mathfrak{C}(v)$ as the set of links in C, where C is the corresponding strongly connected component in the directed link graph. For a set of vertices $\mathcal{U} \subseteq \mathcal{V}_L$, we define $\mathfrak{C}(\mathcal{U}) = \bigcup_{v_i \in \mathcal{U}} \mathfrak{C}(v_i)$.

3.1.3 Illustrative Example

We use a small wireless ad hoc network to illustrate the concept of a directed link graph. The network we consider is composed of 8 nodes and 9 links, and is shown in Fig. 1. In this graph, the set of undirected edges (computed based on the symmetric hearing matrix), E, is given by $E = \{(A, B), (B, C), (C, D), (D, E), (D, F), (F, G), (F, H), (G, H)\}$.

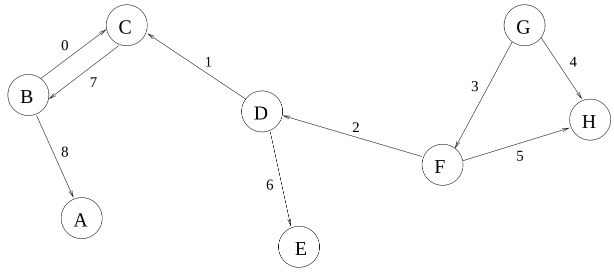

Figure 1: An example wireless ad hoc network.

In the directed link graph, there are 9 vertices, representing the 9 links. It can be seen from Fig. 1 that a successful transmission on link 0 requires that node C and its neighboring nodes (node D) keep silent. Therefore there are edges $(7,0)$, $(1,0)$ and $(6,0)$ in the directed link graph. Also, when link 0 is scheduled from node B, all other links from node B should not be scheduled, i.e. there is edge $(8,0)$ in the directed link graph. Similarly we find all other edges in the directed link graph, and the result is shown in Fig. 2.

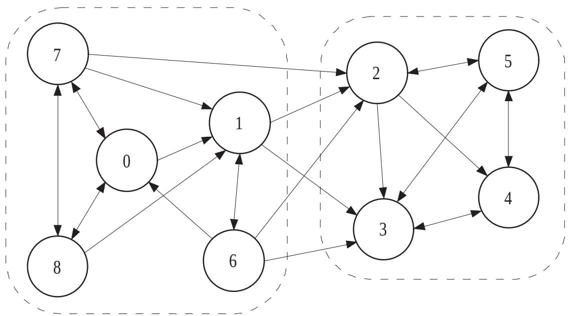

Figure 2: The directed link graph for the wireless ad hoc network considered.

It is obvious in Fig. 2 that there are two strongly connected components in this directed link graph, both of which are highlighted by dashed square box. The first strongly connected component, denoted as C_1, contains link 0, link 1, link 6, link 7, and link 8, and the second strongly connected component, denoted as C_2, contains link 2, link 3, link 4 and link 5. Also it can be seen that there are edges from vertices in C_1 to vertices in C_2, and therefore at the lexicographic max-min fairness, $x_1^* \leq x_2^*$, where x_1^* and x_2^* are the lexicographic max-min fair rates for links in C_1 and C_2, respectively.

3.2 Max-Min Fair Rate Allocation Problem

The objective of max-min rate allocation is to maximize the minimum rate over all links. Note that whereas the lexicographic max-min fairness optimizes the entire sorted vector of link rates in a lexicographic manner, max-min fairness only maximizes the minimum component in the rate vector. Therefore, max-min fairness is a weaker notion of fairness compared to lexicographic max-min fairness. We will show, however, that we can solve the lexicographic max-min fair rate allocation problem by solving a sequence of max-min fair rate allocation problems.

In our context, the max-min fair rate allocation problem can be formulated as follows [7]:

$$\begin{array}{lll} \max & x, & \\ \text{s.t.} & x \leq x_{ij}(\mathbf{p}), & \forall (i,j) \in L, \\ & \mathbf{p} \in \mathbf{P}_f, & \end{array} \tag{2}$$

where x is the max-min rate, and $x_{ij}(\mathbf{p})$ is given in (1). It is worth noting that (2) is equivalent to the following convex program:

$$\begin{array}{ll} \max & f(\mathbf{z}), \\ \text{s.t.} & h_{ij}(\mathbf{z}) \leq 0, \quad \forall (i,j) \in L, \end{array} \tag{3}$$

where $\mathbf{z} = (y, \mathbf{p})$ and $y = \log(x)$, i.e. the logarithmic value of the max-min rate. $f(\mathbf{z}) = y$, and $h_{ij}(\mathbf{z}) = y - \log(x_{ij}(\mathbf{p}))$. Note that $h_{ij}(\mathbf{z})$ is the transformed function of capacity constraint on link $(i, j) \in L$ and is convex. Also note that $\mathbf{p} \in \mathbf{P}_f$ is removed as the logarithmic function automatically ensures the feasibility of $\mathbf{p}$.

3.3 Bottleneck Link

Next we define the notion of a *bottleneck link*. Loosely speaking, a bottleneck link is a link that has the minimum rate in (2) and hence decides the max-min rate.

Define $g_{ij}(x, \mathbf{p}) = x - x_{ij}(\mathbf{p})$, and denote an optimal solution to (2) as $(x^*, \mathbf{p}^*)$. It is easy to argue that x^* is unique while $\mathbf{p}^*$ could be non-unique. If $g_{ij}(x^*, \mathbf{p}^*) = 0$ for any optimal solution $(x^*, \mathbf{p}^*)$, i.e. the constraint for link (i, j) is active at all optimal solutions, then link (i, j) is called a *bottleneck link*.

An alternative definition of a bottleneck link is as follows. Consider perturbing (3) with a perturbation on link (i, j),

$$\begin{array}{lll} \max & f(\mathbf{z}), & \\ \text{s.t.} & h_{ij}(\mathbf{z}) \leq -\epsilon, & \\ & h_{uv}(\mathbf{z}) \leq 0, & \forall (u, v) \in L \setminus \{(i, j)\}. \end{array} \tag{4}$$

The optimal value of (4) is a function on ϵ, and we denote it as $U^*_{ij}(\epsilon)$. We define link (i, j) as a bottleneck link if $U^*_{ij}(0) > U^*_{ij}(\epsilon)$ for any positive ϵ. It can be easily argued that this definition of a bottleneck link is consistent with the previous one.

The result below follows directly from Lemma 1.

Corollary 3 *If link (i, j) is a bottleneck link, and if links (i, j) and (s, t) belong to the same strongly connected component, then link (s, t) is also a bottleneck link.*

Furthermore, the following property also holds:

Lemma 2 (See the proof in the appendix) *If link l is a bottleneck link, and $l \in \mathfrak{C}(v)$ where v is a vertex in the component graph, then all links that belong to $\mathfrak{C}(\mathcal{P}_v)$ are bottleneck links, where $\mathcal{P}_v$ is the set of predecessors for v in the component graph.*

From Corollary 3 and Lemma 2, we can identify one or more strongly connected components (in the directed link graph) that consist only of bottleneck links.

4 Algorithm Based on Identifying a Subset of the Bottleneck Links

4.1 Identifying Bottleneck Links Using Lagrange Multipliers

Direct identification of bottleneck links, using the definition or the alternative definition, has extremely high computational cost and maybe practically infeasible. In this section we discuss how we can identify at least one bottleneck link, using Lagrange multipliers, in an efficient manner.

We consider (3), the transformed convex program for the max-min fair rate problem. It is clear that the Slater Constraint Qualification holds for equation (3). Thus the global optimality of a feasible solution of (3) is characterized by the Karush-Kuhn-Tucker (KKT) conditions:

$$0 = \nabla f(\mathbf{z}) + \sum_{(i,j)\in L} \lambda_{ij}\, \nabla h_{ij}(\mathbf{z}), \qquad 0 \le \boldsymbol{\lambda} \;\perp\; \mathbf{h}(\mathbf{z}) \;\le\; 0, \tag{5}$$

where $\mathbf{h}(\mathbf{z})$ is the $|L|$-vector with components $h_{ij}(\mathbf{z})$ and $\boldsymbol{\lambda}$ is the $|L|$-vector with components λ_{ij}, the Lagrange multipliers for h_{ij}. The $\perp$ notation means orthogonality or the complementary slackness condition.

The relationship between Lagrange multipliers and bottleneck links is stated in the following lemma. Its proof uses the *cross complementarity property* of the solutions of the KKT system, which results from the convexity of such solutions. Namely, if $(\mathbf{z}^i, \boldsymbol{\lambda}^i)$ for $i = 1, 2$ are two KKT pairs, then we must have $\lambda^1_{ij}\; h_{ij}(\mathbf{z}^2) = 0$ for all $(i,j) \in L$.

Lemma 3 *For any KKT pair $(\mathbf{z}^*, \boldsymbol{\lambda}^*)$, if $\lambda^*_{ij} > 0$, then link (i,j) is a bottleneck link.*

The lemma above is a special case of Lemma 4. Note that, at least one Lagrange multiplier is non-zero for a KKT pair, as ∇f must be non-zero at optimality. Therefore we can always identify at least one bottleneck link using Lagrange multipliers. Following Corollary 3 and Lemma 2, we can further identify a set of bottleneck links.

However, it is worth noting that we cannot identify all bottleneck links using an arbitrary Lagrange multiplier, since it is possible that a bottleneck link has zero Lagrange multipliers. In fact, identifying all bottleneck links is in general a difficult problem. We will discuss this in more detail in Section 5.

4.2 Solution Approach

To attain lexicographic max-min fairness, we adopt the following procedure:

1. For a given wireless Aloha network $G = (N, E)$, compute the directed link graph $G_L = (V_L, E_L)$, and construct the component graph $\mathcal{G}_L = (\mathcal{V}_L, \mathcal{E}_L)$ for the directed link graph;

2. Set $k = 0$, $G_k = \mathcal{G}_L$, $V_k = \mathcal{V}_L$, and $E_k = \mathcal{E}_L$;

3. Solve the transformed convex program of (2) for all links that belong to $\mathfrak{C}(V_k)$,

$$\begin{array}{ll} \max & y \\ \text{s.t.} & y \leq \log(x_{ij}(\mathbf{p})), \quad \forall (i,j) \in \mathfrak{C}(V_k), \\ & \mathbf{p} \in \mathbf{P}_f. \end{array} \tag{6}$$

Denote the max-min fair rate as $x_k^* = e^{y^*}$, where y^* is the optimal solution for (6);

4. Find at least one bottleneck link using the Lagrange multipliers, and find the corresponding vertex v in the component graph;

5. Set $U_k = \mathcal{P}_v$. The lexicographic max-min rate for the links in $\mathfrak{C}(U_k)$ is x_k^*;

6. Fix the link attempt probabilities for all the links that belong to $\mathfrak{C}(U_k)$, and set $V_{k+1} = V_k \setminus U_k$;

7. If V_{k+1} is nonempty, we construct the subgraph of G_k for V_{k+1} and denote it as G_{k+1}, i.e. $G_{k+1} = \mathbb{G}_{V_{k+1}} = (V_{k+1}, \mathbb{E}_{V_{k+1}})$, increment k by 1, and go to step 3;

8. Terminate if V_{k+1} is empty.

Intuitively, the above procedure repeatedly solves the problem (6), which maximizes the next minimum rate in the network. Note that, at the optimum of (6), we can identify at least one bottleneck using Lagrange multipliers. By following Corollary 3 and Lemma 2, we can furthermore identify one or more strongly connected components (in the directed link graph) that consist only of bottleneck links. We fix the attempt probabilities for those bottleneck links identified, and go to the next step, i.e. solving (6) again for the rest of the links in the network. Note that the rate on a link, once identified as a bottleneck link, remain unchanged in the later steps, and the procedure is repeated until all links have been identified as bottleneck links (at different step k) and their attempt probabilities have been fixed.

The following theorem states that the above procedure converges to a lexicographic max-min fair rate allocation.

Theorem 1 (Proof in the appendix) *Denote $\mathbf{x}^*$ and $\mathbf{p}^*$ as the vector of link rates and link attempt probabilities attained by the procedure 1) to 8) given above. Then $\mathbf{x}^*$ is the lexicographic max-min fair rate allocation, and $\mathbf{p}^*$ is the link attempt probabilities that makes $\mathbf{x}^*$ feasible, i.e. $x_{ij}^* \leq x_{ij}(\mathbf{p}^*)$ for any link (i,j).*

In addition we can show that the optimal solution of $\mathbf{x}^*$ and $\mathbf{p}^*$ are unique, and $x_{ij}^* = x_{ij}(\mathbf{p}^*)$ for any link (i,j).

It is worth noting that the procedure above can be implemented in a distributed manner. The construction of a strongly connected component is realized if each vertex in the directed link graph finds the set of vertices that it has a path to, and the set of vertices who have a path to it, and hence can be achieved through a distributed path-search algorithm. Also, (6) can be solved in a distributed manner to obtain both primal variables and Lagrange multipliers (refer [7] for details). Also note that the procedure only identify a subset of bottleneck links at each step. In the worst case it might identify only one strongly connected component at a step, and solve (6) repeatedly more than necessary. This motivates a solution approach based on identifying all bottleneck links discussed in Section 5.

5 Algorithm Based on Identifying All Bottleneck Links

5.1 Bottleneck Links, Maximally Complementary Solutions, and the Interior Point Methods

The concept of a bottleneck link is closely tied to that of a *maximally complementary solution* [8, 9] of a monotone nonlinear complementarity problem (NCP) [5] derived from the KKT conditions of a convex program satisfying a suitable CQ. In order to define the corresponding concept of maximal complementarity for the KKT system (5), we introduce three basic sets, $\alpha(\mathbf{z}, \boldsymbol{\lambda}) = \{(i, j) : \lambda_{ij} > 0 = h_{ij}(\mathbf{z})\}$, $\beta(\mathbf{z}, \boldsymbol{\lambda}) = \{(i, j) : \lambda_{ij} = 0 = h_{ij}(\mathbf{z})\}$, and $\gamma(\mathbf{z}, \boldsymbol{\lambda}) = \{(i, j) : \lambda_{ij} = 0 > h_{ij}(\mathbf{z})\}$, associated with every KKT pair $(\mathbf{z}, \boldsymbol{\lambda})$ satisfying (5).

A KKT pair $(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ is said to be *maximally complementary* if the index set $\alpha(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}}) \cup \gamma(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ is maximal among all KKT pairs; i.e., if $(\mathbf{z}, \boldsymbol{\lambda})$ is another KKT pair such that $\alpha(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}}) \cup \gamma(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}}) \subseteq \alpha(\mathbf{z}, \boldsymbol{\lambda}) \cup \gamma(\mathbf{z}, \boldsymbol{\lambda})$, then equality holds in the above inclusion. It is not difficult to show (see [9, page 627]) that if $(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ is a maximally complementary solution, then $\alpha(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ and $\gamma(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ are respectively maximal as two separate sets among all the KKT pairs.

An important fact is that the respective index sets $\alpha(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$, $\beta(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$, and $\gamma(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ are the same among all maximally complementary KKT pairs. This fact is easy to prove using the convexity of the solutions to the KKT system. Therefore we can label the common sets as $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$, respectively.

By definition, link $(i, j) \in L$ is a bottleneck link if its capacity constraint h_{ij} is satisfied as an equality by *all* optimal solutions of (3). Let $L_B \subset L$ denote the set of all bottleneck links, and $L_{NB} \subset L$ denote the set of all non-bottleneck links. Obviously $L = L_B \bigcup L_{NB}$. The main connection between a bottleneck link and the maximally complementary solution is described in the result below.

Lemma 4 (Proof in the appendix) Link (i, j) is a bottleneck link, i.e. $h_{ij}(\mathbf{z})$ is satisfied as an equality by *all* optimal solutions, if and only if $(i, j) \in \widehat{\alpha} \bigcup \widehat{\beta}$, i.e. $L_B = \widehat{\alpha} \bigcup \widehat{\beta}$ and $L_{NB} = \widehat{\gamma}$.

This relation of maximally complementarity and bottleneck links can be illustrated by considering the case below. Suppose we have three links in an Aloha network, namely link 1, link 2, and link 3. The network is setup in a way such that $x_1(\mathbf{p}) = p_1(1 - p_2)$, $x_2(\mathbf{p}) = p_2(1 - p_1)$, and $x_3(\mathbf{p}) = p_3(1 - p_2)$. The max-min fair rate problem is therefore formulated as follows:

$$\begin{array}{ll} \max & x, \\ \text{s.t.} & x \leq p_1(1 - p_2), \\ & x \leq p_2(1 - p_1), \\ & x \leq p_3(1 - p_2). \end{array} \tag{7}$$

Obviously the optimization solution is $x^* = 0.25$ when $p_1 = 0.5$, $p_2 = 0.5$, and $0.5 \leq p_3 \leq 1$. Note that only link 1 and link 2 are bottleneck links, i.e. $B = \{1, 2\}$. Also note that $\widehat{\alpha} = \{1, 2\}$, $\widehat{\beta} = \phi$, and $\widehat{\gamma} = \{3\}$ in this case. Therefore we have $L_B = \widehat{\alpha} \bigcup \widehat{\beta}$ and $L_{NB} = \widehat{\gamma}$.

The significance of Lemma 4 is that the index sets $\widehat{\alpha}$ and $\widehat{\beta}$ can be obtained by an interior-point algorithm (see [9], [5, Chapter 11]) applied to the KKT formulation (5). Nevertheless, enjoying both polynomial complexity and local quadratic convergence, such an interior-point algorithm cannot easily be implemented in a distributed manner.

5.2 The Barrier Method

5.2.1 Identifying All the Bottleneck Links Using the Barrier Method

In this section, we discuss how to identify all the bottleneck links using the barrier method.

For the convex program of the max-min fair rate optimization, (3), the barrier problem is formulated below.

$$\min \quad \theta(\mu), \qquad \text{s.t.} \quad \mu \geq 0, \tag{8}$$

where $\theta(\mu) = \inf\{-f(\mathbf{z}) + \mu B(\mathbf{z}) : h_{ij}(\mathbf{z}) < 0, \forall(i,j) \in L, \mathbf{p} \in \mathbf{P}_f\}$. Here B is the *barrier function* that is nonnegative and continuous over the region $\{\mathbf{z} : h_{ij}(\mathbf{p}) < 0, \mathbf{p} \in \mathbf{P}_f\}$, and approaches ∞ as the boundary of the region $\{\mathbf{z} : h_{ij}(\mathbf{p}) < 0, \mathbf{p} \in \mathbf{P}_f\}$ is approached from the interior.

More specifically, the barrier function B is defined by

$$B(\mathbf{z}) = \sum\nolimits_{(i,j)\in L} \phi\left[h_{ij}(\mathbf{z})\right],$$

where ϕ is a function of one variable that is continuous over $\{s : s < 0\}$ and satisfies:

$$\phi(s) \geq 0 \text{ if } s < 0 \qquad \text{and} \qquad \lim_{s\to 0^-} \phi(s) = \infty$$

One typical $\phi(s)$ is $\phi(s) = -1/s$, and another is $\phi(s) = \log(-s)$. We refer to the function $-f(\mathbf{z}) + \mu B(\mathbf{z})$ as the *auxiliary function*.

Intuitively, when solving the max-min fair rate optimization problem (3) using the barrier method, a very large penalty (the barrier function) is added in the objective to convert the originally constrained optimization problem into an unconstrained optimization problem (8). Lemma **??** (see the statement of the lemma in the appendix) ensure that for any positive μ, there exists $\mathbf{z}_\mu$ so that $\theta(\mu) = -f(\mathbf{z}_\mu) + \mu B(\mathbf{z}_\mu)$, and that the limit of any convergent subsequence of $\{\mathbf{z}_\mu\}$ is an optimal solution to the primal problem (3) when μ approaches to zero, and therefore ensure the validity of the barrier method.

It is worth noting that, when searching for the optimal solution of $\theta(\mu)$ at any given positive μ, the solution tries to stay away from the boundaries as a solution close to the boundary always incurs a very large penalty on the objective. In this manner, the constraints are active only for those bottleneck links. For non-bottleneck links that can be inactive at some optimal solutions, the constraint will not be active. Therefore, the optimal solution given by the barrier method naturally divides all the links into the set of bottleneck links and the set of non-bottleneck links. We make this argument rigorous in the following theorem.

Theorem 2 (Proof in the appendix) *Suppose $\phi(s)$ satisfies that the auxiliary function $f(\mathbf{z})+\mu B(\mathbf{z})$ is strictly convex on $\mathbf{z}$. If the limit point of a convergent subsequence of $\{\mathbf{z}_\mu\}$ is denoted by $\mathbf{z}^*$, then $h_{ij}(\mathbf{z}^*) = 0$ for all $(i,j) \in L_B$ and $h_{ij}(\mathbf{z}^*) < 0$ for all $(i,j) \in L_{NB}$, where L_B and L_{NB} denote the set of bottleneck links and the set of non-bottleneck links respectively.*

5.2.2 Distributed Implementation

To provide lexicographic max-min fairness in a distributed manner, we need to solve the max-min fair rate problem (8) in a distributed manner.

In [7] it has been shown that (3) is equivalent to the convex problem below,

$$\begin{array}{ll} \min & \sum_{(i,j)\in L} y_{ij}^2, \\ \text{s.t.} & y_{ij} - \log\left(x_{ij}(\mathbf{p})\right) \leq 0, \qquad \forall (i,j) \in L, \\ & y_{ij} \leq y_{st}, \quad \forall (i,j) \in L, (s,t) \in \mathfrak{L}(i,j), \\ & \mathbf{p} \in \mathbf{P}_f. \end{array} \tag{9}$$

where y_{ij} is the logarithmic value of the max-min fair rate on link (i,j). $\mathfrak{L}(i,j)$ is defined as the neighboring links of link (i,j) in the directed link graph, i.e. $\mathfrak{L}(e) = \{\hat{e} : (e,\hat{e}) \in E_L \text{ or } (\hat{e},e) \in E_L, \hat{e} \in V_L)\}$ for $e = (i,j) \in V_L$.

Intuitively, (9) introduces y_{ij} for each link (i,j) and forces them to be equal (in the second constraint) so that the originally centralized problem can be solved in a distributed manner. It applies logarithmic transformation to each capacity constraint to make it a convex set. The objective function is rewritten as $\min \sum y_{ij}^2$ since $y_{ij} \leq 0$ for any link (i,j) and we want to make y_{ij} maximized (hence its square should be minimized).

Based on the convex program (9), we construct a barrier problem for $\mu > 0$ as below,

$$\begin{array}{ll} \min & \sum_{(i,j)\in L} y_{ij}^2 + \mu \sum_{(i,j)\in L} \frac{1}{\log\left(x_{ij}(\mathbf{p})\right) - y_{ij}}, \\ \text{s.t.} & y_{ij} \leq y_{st}, \quad \forall (i,j) \in L, (s,t) \in \mathfrak{L}(i,j), \\ & \mathbf{p} \in \mathbf{P}_f. \end{array} \tag{10}$$

Note that (10) actually transfers the capacity constraints to the objective by using the barrier function $\phi(s) = -1/s$. From Lemma 6 (refer to the appendix), (10) converges to the optimal solution of (9) when $\mu \to 0$.

According to the scalar composition [2], for $\phi : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$, $\phi \circ g$ is convex if ϕ is convex and nondecreasing, and g is convex. For the barrier problem considered in (10), $\phi(s) = -1/s$ and $g_{ij}(\mathbf{y},\mathbf{p}) = y_{ij} - \log\left(x_{ij}(\mathbf{p})\right)$ for link (i,j), where $\mathbf{y} = (y_{ij} : (i,j) \in L)$. Therefore the barrier function $B(\mathbf{y},\mathbf{p})$ defined as

$$B(\mathbf{y},\mathbf{p}) = \sum_{(i,j)\in L} \frac{1}{\log\left(x_{ij}(\mathbf{p})\right) - y_{ij}}$$

is a convex function on $(\mathbf{y},\mathbf{p})$. Therefore (10) is a convex program, and can be solved in a distributed manner. We now present a distributed algorithm to solve (10) iteratively.

Let $p_{ij}^{(n)}$ and $y_{ij}^{(n)}$ denote the attempt probability on link (i,j) and the logarithmic value of the link rate at the nth iteration respectively. Define the "link relation indicator" for link (i,j) and its neighboring link (s,t), $\nu_{ij,st}^{(n)}$, as

$$\nu_{ij,st}^{(n)} = \begin{cases} 0 & \text{when } u_{ij} \leq u_{st}, \\ 1 & \text{when } u_{ij} > u_{st}. \end{cases} \tag{11}$$

Let κ be a positive constant, and γ_n be the step size at the nth iteration. The logarithmic value of rate on link (i, j), y_{ij}, is updated as

$$y_{ij}^{(n+1)} = y_{ij}^{(n)} - \gamma \left(2y_{ij}^{(n)} + \mu \frac{\partial B}{\partial y_{ij}} + \kappa \sum_{(s,t) \in \mathfrak{L}(i,j)} \left(\nu_{ij,st}^{(n)} - \nu_{st,ij}^{(n)} \right) \right), \tag{12}$$

and the attempt probability on link (i, j), p_{ij}, is updated as

$$p_{ij}^{(n+1)} = p_{ij}^{(n)} - \gamma \mu \frac{\partial B}{\partial p_{ij}}. \tag{13}$$

Following the procedure based on the subgradient method [3], we can show that the max-min rates (the exponential of y_{ij} for link (i, j)) and link attempt probabilities converge to a neighborhood around the optimum value, and the size of the neighborhood becomes arbitrarily small with decreasing stepsize.

5.3 Solution Approach

If we let $\mathfrak{C}(U_k)$ be all the bottleneck links identified when we repeatedly solve (3) for the kth time, not only the procedure given in Section 4.2 applies here, but also Theorem 1 on the convergence holds true. However, there is some considerable difference between these two procedures that is worth noting here. Since the procedure given above can identify all bottleneck links every time when (3) is solved, (3) will be solved for the least number of times and hence greatly lower the computation cost. Also, in the barrier method, identification of the set of bottleneck links does not require any information on the structure of the component graph, and this can greatly reduce the complexity in the distributed implementation. This simplicity in computation/implementation comes at a cost: note that in practice, the barrier method would only yield approximate solutions, since its solution converges to the optimum only when μ approaches zero. However, the solution provided by the barrier method will become closer to the optimum when μ is decreased, and can be arbitrarily close to the optimum by making μ sufficiently small.

6 Conclusion

In this paper, we address the problem of providing lexicographic max-min fair rate allocations at the link layer in a wireless ad hoc network with random access. We propose two efficient approaches which attain the globally optimal solutions and are amenable to distributed implementation.

Our algorithms are based on solving the problem of maximizing the minimum rate repeatedly, and identifying bottleneck links at each iteration. In this respect, our approaches share an intuitive similarity with the well-known bottleneck-based algorithm for computing the lexicographic max-min fair rates in a wired network [1, Chapter 6]. However, the lexicographic max-min fair rate allocation problem in our context is significantly more complex than that for wired networks. In particular, whereas the problem constraints in our case are non-linear, non-convex and non-separable, the corresponding link capacity constraints in a wired network are linear. Naturally, the notion of a bottleneck link, and the proof of optimality of our approaches are considerably more involved than their counterparts for wired networks. Finally, note that the bottleneck-based algorithm in [1, Chapter 6] considers multi-hop end-to-end sessions in a wired network, as is therefore designed to attain fair rates at the level of the transport layer (the corresponding problem at the link layer, where we we need to consider single-hop connections, is trivial to solve). In contrast, the approaches in our paper are applicable at the level of the link layer, where only single-hop connections need to be considered. The question of attaining lexicographic max-min fair rates for end-to-end multi-hop wireless sessions remains open for future investigation.

Appendix

A Proof of Lemma 1

Proof: First we show that, if there is an edge from (i, j) to (s, t) in a directed link graph, we have $x^*_{ij} \leq x^*_{st}$.

By the definition of an edge in the directed link graph, it must be one of the following two cases to have an edge from (i, j) to (s, t),

1. link (i, j) and (s, t) have the same source nodes;
2. i is either the receiver t or a neighboring node of receiver t for link (s, t).

We then show that in either of the above cases, $x^*_{ij} \leq x^*_{st}$.

Assume that $x^*_{ij} > x^*_{st}$, and at the lexicographic max-min fairness the corresponding attempt probabilities are p^*_{uv} for $(u, v) \in E$.

In case 1), i and s are the same node. Obviously we can find $\delta > 0$, and define $p'_{ij} = p^*_{ij} - \delta$, $p'_{st} = p^*_{st} + \delta$, and $p'_{uv} = p^*_{uv}$ such that

$$x'_{ij} = p'_{ij}(1 - P'_j) \prod_{k \in K_j \setminus \{i\}} (1 - P'_k) = p'_{ij}(1 - P^*_j) \prod_{k \in K_j \setminus \{i\}} (1 - P^*_k)$$

$$x'_{st} = p'_{st}(1 - P'_t) \prod_{k \in K_t \setminus \{s\}} (1 - P'_k) = p'_{st}(1 - P^*_t) \prod_{k \in K_t \setminus \{s\}} (1 - P^*_k)$$

It is worth noting that rates of all other links except (i, j) and (s, t) remain unchanged. Since rate of (s, t) is increased while rate of all the links whose rate is smaller than x^*_{st} remains unchanged, this contradicts the fact that x^*_{st} is the lexicographic max-min fair rate.

In case 2), i is either the receiver of link (s, t) or a neighboring node of node t. We can find $\delta > 0$, and define $p'_{ij} = p^*_{ij} - \delta$ and $p'_{uv} = p^*_{uv}$ otherwise, such that $x^*_{st} < x'_{st} < x'_{ij} < x^*_{ij}$. Note that when changing from $\mathbf{p}^*$ to $\mathbf{p}'$, rates of all links except (i, j) are non-decreased. Since rate of (s, t) is increased while rate of all the links whose rate is smaller than x^*_{st} is not decreased, this contradicts the fact that x^*_{st} is the lexicographic max-min fair rate.

We can then conclude that in either case 1) or 2), there is contradiction. Therefore if $\mathbf{x}^*$ is the vector of lexicographic max-min fair rates, then $x^*_{ij} \leq x^*_{st}$.

If $(i,j) \rightsquigarrow (s,t)$ in the directed link graph, we can denote the links along the path as (u_1, v_1), (u_2, v_2), ..., (u_{n-1}, v_{n-1}), (u_n, v_n). Since there is an edge from (i,j) to (u_1, v_1), $x^*_{ij} \leq x^*_{u_1 v_1}$. Similarly, we have $x^*_{u_1 v_1} \leq x^*_{u_2 v_2}$, ..., $x^*_{u_{n-1} v_{n-1}} \leq x^*_{u_n v_n}$, and $x^*_{u_n v_n} \leq x^*_{st}$. Therefore $x^*_{ij} \leq x^*_{st}$. This completes the proof. ∎

B Proof of Lemma 2

Proof: Denote the max-min fair rate for (2) as x^*. For any link r in the directed link graph, denote its lexicographic max-min rate as x^*_r. Obviously $x^*_r \geq x^*$, as x^* won't be the max-min rate for (2) otherwise. Since $r \in \mathfrak{C}(\mathcal{P}_v)$, $r \rightsquigarrow l$ in the directed link graph. According to Lemma 1, $x^*_r \leq x^*_l$, where x^*_r and x^*_l are the lexicographic max-min rates for link r and link l respectively. Therefore the lexicographic max-min fair rates for link r and link l must be equal. Since l is a bottleneck link, link r is also a bottleneck link. ∎

C Proof Outline of Theorem 1

Proof: First, we show that for the bottleneck links in $\mathfrak{C}(U_k)$, their rates will remain fixed in any steps $l > k$. By the definition of a directed link graph, if the rate of link (i,j) depends on the attempt probability of link (s,t), then $(s,t) \rightsquigarrow (i,j)$ in the directed link graph, i.e., the rate of a link (i,j) depends on the attempt probability of link (s,t) only if (s,t) is a predecessor of link (i,j). From Lemma 2, $\mathcal{P}_v \subset \bigcup_{l=0,1,\dots,k} U_l$ for any $v \in U_k$. Since all links that belong to $\mathfrak{C}(\bigcup_{l=0,1,\dots,k} U_l)$ have fixed link attempt probabilities for iteration $l > k$, the rates for the links that belong to $\mathfrak{C}(U_k)$ will remain fixed.

We now show that attempt probabilities of the links in $\mathfrak{C}(U_k)$, solved from (6), are unique.

Lemma 5 *Consider the max-min fair rate problem*

$$\begin{array}{ll} \max & y, \\ s.t. & y \leq \log(x_{ij}(\mathbf{p})), \quad \forall (i,j) \in \mathfrak{C}(V_k), \\ & \mathbf{p} \in \mathbf{P}_f. \end{array}$$

At the optimal solution, attempt probabilities of the links in $\mathfrak{C}(U_k)$ are unique.

Proof: If $l_1 \in \mathfrak{C}(V_k \setminus U_k)$ and $l_2 \in \mathfrak{C}(U_k)$, we have $x^*_{l_1} \geq x^*_{l_2}$ since $l_2 \rightsquigarrow l_1$ in the directed link graph, where $x^*_{l_1}$ and $x^*_{l_2}$ denote the lexicographic max-min fair rate for link l_1 and l_2 respectively. Therefore we can remove the constraints that correspond to the links in $\mathfrak{C}(V_k \setminus U_k)$ and obtain the following equivalent problem.

$$\begin{array}{ll} \max & y, \\ \text{s.t.} & y \leq \log(x_{ij}(\mathbf{p})), \quad \forall (i,j) \in \mathfrak{C}(U_k), \\ & \mathbf{p} \in \mathbf{P}_f. \end{array} \tag{14}$$

As all the links in $\mathfrak{C}(U_k)$ are bottleneck links, all the constraints in (14) are active at the optimum. The max-min rate x and the attempt probabilities for the links in $\mathfrak{C}(U_k)$ will be decided in (14).

Denote $\mathbf{p}^{U_k} = (p_{ij} : (i,j) \in \mathfrak{C}(U_k))$. Note that if $(i,j) \in \mathfrak{C}(U_k)$, then $x_{ij}(\mathbf{p})$ only depends on $\mathbf{p}^{U_l}$, where $l = 1, ..., l-1$. Note that $\mathbf{p}^{U_l}$ is fixed for $l < k$, we can write $x_{ij}(\mathbf{p}^{U_k})$.

Denote the optimal value of (14), as y^*. Assume that both $\mathbf{p}_1^{U_k}$ and $\mathbf{p}_2^{U_k}$ are both optimal solutions. Since all constraints of (14) are active at the optimum, we have $y^* = \log(x_{ij}(\mathbf{p}_1^{U_k}))$ and $y^* = \log(x_{ij}(\mathbf{p}_2^{U_k}))$.

Denote $\mathbf{p}_\lambda^{U_k} = \lambda\mathbf{p}_1^{U_k} + (1-\lambda)\mathbf{p}_2^{U_k}$, for $0 < \lambda < 1$. Since $\log(x_{ij}(\mathbf{p}^{U_k})$ is strictly concave on $\mathbf{p}^{U_k}$, for any $(i,j) \in \mathfrak{C}(U_k)$ we have $\log(x_{ij}(\mathbf{p}_\lambda^{U_k}) = \log(x_{ij}(\lambda\mathbf{p}_1^{U_k} + (1-\lambda)\mathbf{p}_2^{U_k}) > \lambda y^* + (1-\lambda)y^* = y^*$. This contradicts with the fact that y^* is the optimal value for (14). Therefore (14) has a unique solution. ■

From Lemma 2, we conclude that all links in $\mathfrak{C}(U_k)$ are bottleneck links, and their max-min fair rate is x_k^*. We have also shown that the max-min rate for those bottleneck links will remain fixed in the later steps.

From Corollary 2 and Lemma 2 we can easily see that $x_0^* \leq x_1^* \leq x_2^* \leq ... \leq x_n^*$ for step 1 to step n.

We now show that, if the algorithm terminates at the nth step, $x_0^* \leq x_1^* \leq x_2^* \leq ... \leq x_n^*$ is the lexicographic max-min fair rate. From the above discussion, it can be seen that our procedure first maximizes the minimum rate in the network, and then maximizes the rate that's second to the minimum, and so on. Therefore the procedure guarantees that the rate of any link cannot be increased without decreasing the rate of a link that has smaller rate. Also, we have shown that for each step, the attempt probabilities solved from (14) is unique, and hence there is no possibilities that the rate of a link is increased while the rates of the links, who have smaller rates, remain unchanged. Therefore our procedure gives the lexicographic max-min fair rate. ■

D Proof of Lemma 4

Proof: If (i,j) is a bottleneck link, then it is clear that $(i,j) \notin \widehat{\gamma}$ by the cross complementarity property. Conversely, suppose that (i,j) is not a bottleneck link. Then there exists an optimal solution $\widetilde{\mathbf{z}}$ of (3) such that $h_{ij}(\widetilde{\mathbf{z}}) < 0$. Let $\widetilde{\boldsymbol{\lambda}}$ be any KKT multiplier corresponding to $\widetilde{\mathbf{z}}$ and let $(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}})$ be a maximally complementary KKT pair. The pair $(\mathbf{z}^{1/2}, \boldsymbol{\lambda}^{1/2}) \equiv \frac{1}{2}(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}}) + \frac{1}{2}(\widetilde{\mathbf{z}}, \widetilde{\boldsymbol{\lambda}})$ is also a KKT pair, by the convexity of the solution set of the KKT system. Clearly, we have $\widehat{\gamma} = \gamma(\widehat{\mathbf{z}}, \widehat{\boldsymbol{\lambda}}) \subseteq \gamma(\mathbf{z}^{1/2}, \boldsymbol{\lambda}^{1/2})$. The maximality of $\widehat{\gamma}$ implies that $(i,j) \in \gamma(\mathbf{z}^{1/2}, \boldsymbol{\lambda}^{1/2}) = \widehat{\gamma}$. ■

E Proof of Theorem 2

Proof: The following lemma ensures the validity of using barrier functions for solving a constrained problem by converting them into a single unconstrained problem or into a sequence of unconstrained problems.

Lemma 6 *The following statements hold for the barrier method applied to our problem:*

1. *For each $\mu > 0$ there exists an $\mathbf{z}_\mu \in Z$ with $\mathbf{g}(\mathbf{z}_\mu) < \mathbf{0}$ such that*

$$\begin{aligned} \theta(\mu) &= f(\mathbf{z}_\mu) + \mu B(\mathbf{z}_\mu) \\ &= \inf\{f(\mathbf{z}) + \mu B(\mathbf{z}) : \mathbf{g}(\mathbf{z}) < 0, \mathbf{z} \in Z\}, \end{aligned}$$

2. $\inf\{f(\mathbf{z}) : \mathbf{g}(\mathbf{z}) \leq 0, \mathbf{z} \in Z\} \leq \inf\{\theta(\mu) : \mu > 0\}$

3. *For $\mu > 0$, $f(\mathbf{z})$ and $\theta(\mu)$ are nondecreasing functions of μ, and $B(\mathbf{z}_\mu)$ is a non-increasing function of μ,*

4. $\min\{f(\mathbf{z}) : \mathbf{g}(\mathbf{z}) \leq \mathbf{0}, \mathbf{z} \in Z\} = \lim_{\mu\to 0^-} \theta(\mu) = \inf_{\mu>0} \theta(\mu)$,

5. *The limit of any convergent subsequence of $\{\mathbf{z}_\mu\}$, at least one of which must exist, is an optimal solution to the primal problem, and furthermore $\mu B(\mathbf{z}_\mu) \to 0$ as $\mu \to 0^+$.*

Proofs of Lemma 6 follows from standard results for the barrier method [4].

Let $A_\mu(y, \mathbf{p})$ denote the auxiliary function for the barrier problem (8) at the given $\mu > 0$, i.e.

$$A_\mu(y, \mathbf{p}) = -y + \mu \sum_{(i,j)\in L} \phi(h_{ij}(y, \mathbf{p})), \tag{15}$$

and let $(y^*, \mathbf{p}^*)$ denote the limit point of a convergent of subsequence of $\{\mathbf{z}_\mu\} = \{(y_\mu, \mathbf{p}_\mu)\}$. Since ϕ is assumed to make $A_\mu(y, \mathbf{p})$ a strictly convex function, $(y^*, \mathbf{p}^*)$ is an optimal solution to (3) from Lemma 6.

We write $\mathbf{p} = (\mathbf{p}^B, \mathbf{p}^{NB})$, where $\mathbf{p}^B$ is the vector of attempt probabilities for bottleneck links, and $\mathbf{p}^{NB}$ is the vector of attempt probabilities for non-bottleneck links.

Since the limit point $(y^*, \mathbf{p}^*)$ is an optimal solution to (3), by definition of a bottleneck link we have

$$h_{ij}(y^*, \mathbf{p}^*) = 0, \qquad \forall (i,j) \in L_B. \tag{16}$$

According to the definition of a non-bottleneck link, there exists an $\tilde{\mathbf{p}}$ such that for any $(i,j) \in L_{NB}$, we have

$$h_{ij}(y^*, \tilde{\mathbf{p}}) < 0.$$

Therefore there exists an $\zeta > 0$ such that

$$h_{ij}(y^*, \tilde{\mathbf{p}}) < -\zeta, \qquad \forall (i,j) \in L_{NB}. \tag{17}$$

We now show that for any convergent subsequence $\{(y_\mu, \mathbf{p}_\mu)\}$ and for any $\epsilon > 0$, there exists δ^1_ϵ such that for any $0 < \mu < \delta^1_\epsilon$, there exists $(y_\mu, \tilde{\mathbf{p}}_\mu)$ that satisfies for any $(i,j) \in L_{NB}$ we have

$$h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu) < -\zeta + \epsilon, \tag{18}$$

where $|L_{NB}|$ denotes the size of the set L_{NB}, i.e. the number of non-bottleneck links.

Noting that $\mathbf{p}^{B*}$ is unique from Lemma 5, we conclude that the convergent subsequence $\{(y_\mu, \mathbf{p}_\mu)\}$ satisfies that $\{\mathbf{p}^B_\mu\}$ converges to $\tilde{\mathbf{p}}^B = \mathbf{p}^{B*}$.

We construct $\tilde{\mathbf{p}}_\mu = (\mathbf{p}^B_\mu, \tilde{\mathbf{p}}^{NB})$, and therefore for any $\epsilon > 0$ there exists $\delta^1_\epsilon > 0$, such that for any $0 < \mu < \delta^1_\epsilon$ and any $(i,j) \in L_{NB}$ we have

$$|y_\mu - y^*| < 0.5\epsilon, \qquad |\log(x_{ij}(\tilde{\mathbf{p}})) - \log(x_{ij}(\tilde{\mathbf{p}}_\mu))| < 0.5\epsilon.$$

Therefore we have

$$\begin{aligned}
& |h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu)| = |y_\mu - \log(x_{ij}(\tilde{\mathbf{p}}_\mu))| \\
= \; & |y_\mu - y^* + y^* - \log(x_{ij}(\tilde{\mathbf{p}})) + \log(x_{ij}(\tilde{\mathbf{p}})) - \log(x_{ij}(\tilde{\mathbf{p}}_\mu))| \\
\geq \; & |y^* - \log(x_{ij}(\tilde{\mathbf{p}}))| - |y_\mu - y^*| - |\log(x_{ij}(\tilde{\mathbf{p}})) - \log(x_{ij}(\tilde{\mathbf{p}}_\mu))| \\
> \; & \zeta - \epsilon.
\end{aligned}$$

Since $h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu) < 0$, it follows that

$$h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu) < -\zeta + \epsilon, \qquad \forall (i,j) \in L_{NB}.$$

We then show that for a convergent subsequence $\{(y_\mu, \mathbf{p}_\mu)\}$, the limit point $(y_0, \mathbf{p}_0)$ must satisfy that $h_{ij}(y_0, \mathbf{p}_0) < 0$. We prove this result by contradiction.

Suppose that for link $l \in L_{NB}$, $h_l(y_0, \mathbf{p}_0) = 0$. Therefore for any $\epsilon > 0$, we can find δ_ϵ^2 such that for any $0 < \mu < \delta_\epsilon^2$, it holds $-\epsilon < h_l(y_\mu, \mathbf{p}_\mu) < 0$. From the previous discussion, we see that when $0 < \mu < \delta_\epsilon^1$, we can find $(y_\mu, \tilde{\mathbf{p}}_\mu)$ such that $h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu) < -\zeta + \epsilon$ for all $(i,j) \in L_{NB}$.

Since $\phi(s)$ approaches infinity when s approaches 0 where $s < 0$, there exists $\epsilon_1 > 0$ such that for any $0 < \epsilon < \epsilon_1$, we have $\phi(-\epsilon) > |L_{NB}|\phi(-0.5\zeta)$. We then define $\epsilon_2 = 0.5\zeta$. Denote $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, and denote $\delta_{\epsilon_0} = \min\{\delta_{\epsilon_0}^1, \delta_{\epsilon_0}^2\}$.

Noting that for a bottleneck link (i,j), h_{ij} only depends on $\mathbf{p}^B$, and that $\tilde{\mathbf{p}}_\mu = (\mathbf{p}_\mu^B, \tilde{\mathbf{p}}^{NB})$, we have $h_{ij}(\tilde{\mathbf{p}}_\mu) = h_{ij}(\mathbf{p}_\mu)$ for any link $(i,j) \in L_B$. Therefore, for any $0 < \mu < \delta_{\epsilon_0}$ we then have

$$\begin{aligned}
& A_\mu(y_\mu, \mathbf{p}_\mu) - A_\mu(y_\mu, \tilde{\mathbf{p}}_\mu) \\
= \; & \left[-y_\mu + \mu \sum_{(i,j)\in L} \phi(h_{ij}(y_\mu, \mathbf{p}_\mu)) \right] - \left[-y_\mu + \mu \sum_{(i,j)\in L} \phi(h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu)) \right] \\
= \; & \mu \sum_{(i,j)\in L_B} [\phi(h_{ij}(y_\mu, \mathbf{p}_\mu)) - \phi(h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu))] \\
& + \mu \sum_{(i,j)\in L_{NB}} [\phi(h_{ij}(y_\mu, \mathbf{p}_\mu)) - \phi(h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu))] \\
= \; & \mu \sum_{(i,j)\in L_{NB}} [\phi(h_{ij}(y_\mu, \mathbf{p}_\mu)) - \phi(h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu))] \\
\geq \; & \mu \left[\phi(h_l(y_\mu, \mathbf{p}_\mu)) - \sum_{(i,j)\in L_{NB}} \phi(h_{ij}(y_\mu, \tilde{\mathbf{p}}_\mu)) \right] \\
\geq \; & \mu(\phi(-\epsilon_0) - |L_{NB}\phi(-0.5\zeta)|) > 0.
\end{aligned}$$

This contradicts with the fact that $(y_\mu, \mathbf{p}_\mu)$ is the optimal solution to $A_\mu(y, \mathbf{p})$. Therefore the assumption that $h_l(y_0, \mathbf{p}_0) = 0$ for link $l \in L_{NB}$ is incorrect, and we conclude that $h_l(y_0, \mathbf{p}_0) < 0$ for any link $l \in L_{NB}$. ■

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