

# Price-Driven Charging of Plug-in Electric Vehicles: Nash Equilibrium and Best-Response Convergence Analysis

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**Abstract**—As the number of charging Plug-in Electric Vehicles (PEVs) increase, it is crucial to control the charging of PEVs in order to minimize energy generation and transmission costs, and ensure grid stability. In this work, we analyze the equilibrium properties of a natural price-driven charging control game in the distribution grid, between the utility (that sets the time-dependent energy usage price) and selfish PEVs (that choose their own charging schedules to minimize individual cost). We demonstrate through analysis and simulations that individual best-response strategies converge to socially optimal charging profiles (also equilibrium solutions) under fairly weak assumptions on the (asynchronous) charging profile update processes. We also discuss how the framework can be extended to consider the topology of the distribution tree and associated transmission line capacity constraints.

## I. INTRODUCTION

The use of Plug-in Electric Vehicles (PEVs) can reduce the cost of the fuel consumption, greenhouse gas emissions, and at the same time increase vehicle engine performance efficiency. Naturally, PEVs have been getting popular in recent years [1], and almost all major automobile manufacturers are currently either developing or contemplating development of PEVs [2].

As PEVs derive some or all of their energy from the electric grid, increase in PEV usage would imply a significant increase in the overall load on the electric grid. Based on survey data [3],[4], if every household owns just one PEV in the near future, the peak demand from charging the PEVs can increase the peak load of the grid to 2.5 times their current values. Therefore efficient management of the this excess demand is important for the overall efficiency and stability of the grid [5], [6], [7]. This has led researchers to study the question of coordinating/optimizing the charging of PEVs [8], who have demonstrated that the undesirable effects of PEV charging on the distribution grid can be largely avoided through this coordination. The question of coordinated PEV charging has been approached from different perspectives, including game theory [9], [10], gradient optimization [11], sequential quadratic optimization [12], [13], dynamic programming [14], and heuristic methods [1], [15].

In this paper, we study the problem of price-driven charging of PEVs in a distribution network (grid), posing it as a non-cooperative game between the PEVs (users). In this game formulation, the utility sets time-dependent prices that PEVs must pay for energy usage, based on which PEVs choose their charging schedule so as to (selfishly) optimize

their individual charging costs. We study the equilibrium properties of this game under the scenario that the per-unit prices set by the utility are load (congestion) dependent (Section III). We show that under some very general assumptions of the pricing function, Nash equilibrium of this game exists, although it is not necessarily unique. Moreover, every Nash equilibrium is also socially optimal (and vice versa) in the sense that it minimizes the variance of the overall load in the distribution grid. This implies that the price-of-anarchy of the game is unity. Furthermore, we show that the best-response updates by individual PEVs results in the system converging to socially optimal solutions, even when the charging profile updates by PEVs are asynchronous and may occur at different frequencies (Section IV). We also discuss how the model and results can be extended to consider the topology of the distribution tree and associated transmission line capacity constraints (Section V). Finally, we investigate through simulations the convergence of best-response charging profile updates while varying different system parameters (Section VI).

Our work is related to some of the recent work in this domain on decentralized PEV charging, [9], [11] and [10] in particular. Among these, [9] is perhaps closest to our work as [9] also models decentralized charging of PEVs in a non-cooperative game framework, and studies similar issues. The study in [11] derives an optimization based algorithm for decentralized charging of PEVs towards attaining social optimality. While a game theoretic model is not explicitly used in [11], the proposed algorithm can be viewed as a best-response update policy for a specific time-dependent pricing function. Despite these apparent similarities, there are some important differences between our model and results and those in [9], [11]. The approaches in [9], [11] can be viewed in a price-driven game theoretic setting in the following way. The utility uses a linear pricing policy (i.e., cost of charging for a PEV is the product of this price and the charging rate) that depends on the load in the network. However, an additional quadratic term must be built into either the user utility or the price itself, to guarantee convergence or optimality of the converged solutions. This specific quadratic term depends on the deviation of the PEV's charging rate from the mean charging rate in the distribution grid [9] at that time, or the difference from the PEV's charging rate last chosen by the PEV [11]. If this term is built into the price itself then the pricing policy would be non-linear. The pricing policy that we consider is non-linear as well, but has a nicer (arguably simpler and more intuitive) interpretation, particularly in comparison to that of [11] (based on its price-

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driven game-theoretic interpretation). Our results are shown for a much more general setting than that of [9], which derives similar properties of the Nash equilibrium but in the “large population” regime, and for a homogeneous system where the charging start and finish times are the same. In our model and results, none of these assumptions are needed. Note that the use of the large population regime, a PEV can ignore the effect of its own choice of charging rate in the overall load on the grid. However, in the finite population regime that we consider in our work, a PEV does take into account (while determining its charging cost) the effect of its charging rate on the overall load on the grid, and therefore the price changes as a result of the additional load it places in the grid. Finally, whereas [9] prove convergence of the best-response strategy under synchronous updates (using contraction mapping arguments), we show that the best-response strategy in our framework converges under asynchronous updates, under fairly weak assumptions on the timing and frequency of the update processes by individual PEVs. This is achieved by extending the convergence result of the classical block coordinate descent (Gauss-Seidel iteration) method [16] to a partial asynchronous model.

The model considered in [10], is different from ours in that it analyzes a Stackelberg game between the utility (leader) and PEVs (followers), where the utility sets a price to maximize its own revenue, while the PEVs are interested in maximizing their own utility function discounted by the charging cost. Compared to the [10], our game model differs in the modeling of both the price setting (by the utility) and charge rate control (by the PEVs) behavior. In our model, energy pricing is based on the actual load on the grid at a particular time, and not something that maximizes the revenue of the utility. We also do not associate utility functions with PEV, which are needed in [10] to make the Stackelberg game definition meaningful. The pricing policy in our model (which reflects the cost of energy production and transmission) is better motivated when considering a regulated energy market. The pricing and user model in [10] is likely more meaningful in an unregulated energy market where PEVs have access to alternate sources of energy, say from other electricity suppliers, or fuel (hybrid vehicles).

## II. SYSTEM MODEL

We model a system where an electric utility negotiates with the finite set of  $\mathcal{K} = \{1, \dots, K\}$  plug-in electric vehicles (indexed  $1, \dots, K$ ) determining the charging schedule in advance (like day-ahead schedule), over a distribution network. We discretize the time of day into  $T$  units, indexed as  $1, 2, \dots, T$ . Let  $D(t)$  denote the given base demand (aggregated non-PEV demand) over the entire distribution network, and  $p_k(t)$  denote the charging power of PEV  $k$  at time  $t$ , for all  $t \in \{1, \dots, T\}$ . The PEV charging rate  $p_k(t)$  are variables that get determined based on the time-dependent energy prices set by the utility, as we will see shortly. Let the vector  $\mathbf{p}_k = (p_k(1), \dots, p_k(T))$  denote the *charging profile* of PEV  $k$  and the vector  $\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_K) = (p_1(1), p_1(2), \dots, p_K(T))$  denote the charging profile of all

the  $K$  PEVs in the system. The vector  $\mathbf{p}$  is also denoted as  $\mathbf{p} = (\mathbf{p}_{-k}, \mathbf{p}_k)$ , where  $\mathbf{p}_{-k} = (\mathbf{p}_1, \dots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \dots, \mathbf{p}_K)$  is the vector  $\mathbf{p}$  excluding the components corresponding to vector  $\mathbf{p}_k$ . Assume that PEV  $k$  must charge over the time interval  $[t_k^s, t_k^f]$ , where  $t_k^s$  is the charging start time and  $t_k^f$  is the charging finish time for PEV  $k$ ,  $1 \leq t_k^s < t_k^f \leq T$ .

For PEV  $k$ , as per its battery specification, the charging rate should be within a range:

$$0 \leq p_k(t) \leq p_k^{max}, \quad \forall t. \quad (1)$$

Let us define  $W_k$  as the total energy PEV  $k$  requests to get charged in the interval  $[t_k^s, t_k^f]$  and  $0 \leq W_k \leq \sum_{t=t_k^s}^{t_k^f} p_k^{max}$ . Hence the charging rate variables  $p_k(t)$  must satisfy

$$\sum_{t=t_k^s}^{t_k^f} p_k(t) = W_k, \quad 1 \leq k \leq K. \quad (2)$$

If PEV  $k$  customer sets  $W_k$  as the total energy needed to fully charge the battery, then  $W_k$  is calculated based on the PEV battery capacity,  $B_k$ , charging efficiency,  $\eta_k$ , and charging percentage of the vehicle  $k$  at time zero,  $s_k(0)$ , as  $W_k = B_k(1 - s_k(0))/\eta_k$ .

We define  $\mathcal{D}_k = \{p_k(t) : t = 1, \dots, T | 0 \leq p_k(t) \leq p_k^{max}, p_k(t) = 0 \text{ if } t \notin [t_k^s, t_k^f], \sum_{t=t_k^s}^{t_k^f} p_k(t) = W_k\}$  as the set of charging profiles for PEV  $k$  satisfying constraints (1) and (2), as well as start and finish time constraints. The set  $\mathcal{D}_k$  defines a bounded subset of  $\mathbb{R}^T$ . Furthermore, let  $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_K$  denote the set of charging profiles of all PEVs. As  $\mathcal{D}_k$  is compact and convex for all  $k$ , the set  $\mathcal{D}$  as a product of  $K$  independent compact sets is also compact and convex.

We assume that the utility charges according to per-unit price function (kernel)  $\pi(\cdot)$  that depends on the aggregate load (base plus PEV) at that time. We assume that the individual PEVs are selfish, and just interested in minimizing their own charging cost, taking into consideration the aggregate load at different times, that in turn depend on the collection of charging strategies adopted by other PEVs in the distribution grid. In particular, PEV  $k$  is interested in its own cost function (subject to the constraint  $\mathbf{p}_k \in \mathcal{D}_k$ ), defined as follows:

$$\mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}) = \sum_{t=1}^T \left[ \int_{x_k^t=0}^{p_k(t)} \pi \left( D(t) + \sum_{j=1, j \neq k}^K p_j(t) + x_k^t \right) dx_k^t \right], \quad (3)$$

where  $\pi(x)$  is defined as the energy pricing function and it is assumed to be a strictly increasing, continuous function of  $x$ . To motivate the above pricing (cost) function, note that  $D(t) + \sum_{j=1, j \neq k}^K p_j(t)$  represents the total load due to the base demand and the other PEVs (all PEVs other than  $k$ ) at time  $t$ . Assuming this “other load”, when PEV  $k$  addition of a “small” load of  $\Delta x_k^t$  when the PEV has already added a load of  $x_k^t$  to the grid at time  $t$ , costs the PEV an amount of  $\pi \left( D(t) + \sum_{j=1, j \neq k}^K p_j(t) + x_k^t \right) \Delta x_k^t$  for charging at time

$t$ . Letting  $\Delta x_k^t$  tend to zero, and varying  $x_k^t$  from 0 to  $p_k(t)$ , the charging rate demanded by PEV  $k$  at time  $t$ , results in the formula in (3).

Let  $\tilde{\mathbf{p}}$  denote the Nash equilibrium charging profile of the PEVs, based on the individual cost functions defined by (3). In other words,  $\tilde{\mathbf{p}}$  is a fixed point solution (assuming one exists) of the following set of equations:

$$\tilde{\mathbf{p}}_k = \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \mathcal{J}_k(\mathbf{p}_k, \tilde{\mathbf{p}}_{-k}). \quad (4)$$

From the perspective of the utility, the cost of energy production and distribution is minimized when its peak-to-average ratio of the total load, or equivalently the variance of the total load, is minimized [9], [11]. This is represented by the social cost function,  $\mathcal{L}(\mathbf{p})$ , defined as:

$$\mathcal{L}(\mathbf{p}) = \sum_{t=1}^T \mathcal{V} \left( D(t) + \sum_{k=1}^K p_k(t) \right), \quad (5)$$

where the variance function,  $\mathcal{V}(x)$ , is a differentiable, increasing, strictly convex function of  $x$ .

In the case that the pricing function,  $\pi(x)$ , is defined as the derivative of the variance function,  $\mathcal{V}(x)$ , i.e.  $\pi(x) = \mathcal{V}'(x)$ , taking the integral in equation (3), results in:

$$\begin{aligned} \mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}) &= \sum_{t=1}^T \mathcal{V} \left( D(t) + \sum_{k=1}^K p_k(t) \right) \\ &- \sum_{t=1}^T \mathcal{V} \left( D(t) + \sum_{j=1, j \neq k}^K p_j(t) \right) = \mathcal{L}(\mathbf{p}) - \mathcal{L}(\mathbf{0}, \mathbf{p}_{-k}). \end{aligned} \quad (6)$$

In the following two sections, we assume that  $\pi(x) = \mathcal{V}'(x)$ , and we show that under this assumption, (i) the set of Nash equilibria coincides with the set of socially optimal solutions; (ii) best-response charging profile updates by individual PEVs converge to the social optimum solution, under fairly general assumptions on the (possibly asynchronous) update processes. Note that this pricing function varies non-linearly with the charging load added by individual PEVs (the loads by other PEVs remain the same): thus the price *per unit load added* at any time  $t$  increases as the PEV puts more load on the grid at that time. However, as  $\mathcal{V}$  can be any arbitrary differentiable, increasing, strictly convex function, a very large class of pricing functions  $\pi(\cdot)$  satisfy the above two desirable properties, (i) and (ii).

### III. PROPERTIES OF NASH EQUILIBRIUM

Minimizing the cost function  $\mathcal{J}_k$  as in (3) is equivalent to maximizing the payoff function  $\mathcal{U}_k(\mathbf{p}_k, \mathbf{p}_{-k}) = -\mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k})$ . The price-driven charging profile choice by selfish PEVs (minimizing individual costs), as defined in the previous section, can be represented as a strategic form game  $\langle \mathcal{K}, (\mathcal{D}_k), (\mathcal{U}_k) \rangle$ .

#### A. Existence of Nash Equilibrium

The following holds for each  $k \in \mathcal{K}$ :

1.  $\mathcal{D}_k$  is a non-empty, convex and compact subset of a finite-dimensional Euclidean space.

2.  $\mathcal{U}_k(\mathbf{p}) = -\mathcal{J}_k(\mathbf{p})$  is continuous in  $\mathbf{p}$  as  $\pi(x)$  is continuous in  $x$ .

3.  $\mathcal{U}(\mathbf{p}_k, \mathbf{p}_{-k}) = -\mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}) = -\mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}) + \mathcal{L}(\mathbf{0}, \mathbf{p}_{-k})$  is concave in  $\mathbf{p}_k$  as  $\mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k})$  is convex in  $\mathbf{p}_k$ . Then based on the *Debreu, Glicksberg, Fan theorem* [17], the game  $\langle \mathcal{K}, (\mathcal{D}_k), (\mathcal{U}_k) \rangle$  has a pure strategy Nash equilibrium.

#### B. Non-uniqueness of Nash Equilibrium

We provide a simple example to show that the Nash equilibrium is not necessarily unique. Let consider the charging profile of two PEVs with  $T = 2$  hours,  $t_1^s = t_2^s = 1$ ,  $t_1^f = t_2^f = 2$ ,  $U_1 = 1$ ,  $U_2 = 2$ . Also,  $p_k^{max}(t) = 1.96KW$  for  $k = 1, 2, t = 1, 2$ , and  $D(1) = 420KW$ ,  $D(2) = 421KW$ . Let the pricing function,  $\pi(x) = x$ , and the variance function,  $\mathcal{V}(x) = \frac{1}{2}x^2$ . The two charging profiles  $\tilde{\mathbf{p}}^1 = ((1, 0), (1, 1))KW$  and  $\tilde{\mathbf{p}}^2 = ((\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{1}{2}))KW$  are both Nash equilibria. This implies that the Nash equilibria in our problem may not be unique in general. In fact, it is possible that an infinite number of Nash equilibria exists for this problem, which is also implied indirectly by the result we present next.

#### C. Social Optimality of Nash Equilibrium

*Theorem 1:* Any socially optimal charging profile is a Nash equilibrium, and vice versa.

*Proof:* Let  $\mathbf{p}^*$  be a social optimum charging profile, therefore it minimizes  $\mathcal{L}(\mathbf{p})$  over  $\mathcal{D}$ . We use contradiction to show that  $\mathbf{p}^*$  is also a NE charging profile. Let assume  $\mathbf{p}^*$  is not an NE. Thus there exists a PEV  $k$  such that it would deviate from  $\mathbf{p}_k^*$  to  $\tilde{\mathbf{p}}_k \neq \mathbf{p}_k^*$  in order to reduce its individual cost from  $\mathcal{J}_k^*$  to  $\tilde{\mathcal{J}}_k$ . As we show in Lemma 1 in the next section, the best-response charging profile  $\tilde{\mathbf{p}}_k$  that minimizes  $\mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}^*)$  is unique, therefore  $\tilde{\mathcal{J}}_k < \mathcal{J}_k^*$ . Evaluating the integrals in equation (3) under the assumption  $\pi(x) = \mathcal{V}'(x)$ , and using the definition of  $\mathcal{L}(\mathbf{p})$  in (5), we have:

$$\begin{aligned} \tilde{\mathcal{J}}_k &= \mathcal{L}(\tilde{\mathbf{p}}_k, \mathbf{p}_{-k}^*) - \mathcal{L}(\mathbf{0}, \mathbf{p}_{-k}^*) \\ \mathcal{J}_k^* &= \mathcal{L}(\mathbf{p}^*) - \mathcal{L}(\mathbf{0}, \mathbf{p}_{-k}^*). \end{aligned} \quad (7)$$

Therefore,  $\tilde{\mathcal{J}}_k < \mathcal{J}_k^*$  implies  $\mathcal{L}(\tilde{\mathbf{p}}_k, \mathbf{p}_{-k}^*) < \mathcal{L}(\mathbf{p}^*)$ . This contradicts the social optimality of  $\mathbf{p}^*$  that implies  $\mathbf{p}^*$  minimizes  $\mathcal{L}(\mathbf{p})$  over  $\mathcal{D}$ .

To prove the converse, we need to show that the Nash equilibrium charging profile  $\tilde{\mathbf{p}}$  is a social optimum. As  $\tilde{\mathbf{p}}_k$  minimizes  $\mathcal{J}_k(\mathbf{p}_k, \tilde{\mathbf{p}}_{-k})$  over  $\mathcal{D}_k$ , evaluating the integral in (3) with  $\pi(x) = \mathcal{V}'(x)$ , it follows that  $\tilde{\mathbf{p}}_k$  also minimizes  $\mathcal{L}(\mathbf{p}_k, \tilde{\mathbf{p}}_{-k})$  over  $\mathcal{D}_k$ . Using the first order optimality constraints for any  $k$ , have:

$$\nabla_k \mathcal{L}(\tilde{\mathbf{p}})^T (\mathbf{p}_k - \tilde{\mathbf{p}}_k) \geq 0, \forall \mathbf{p}_k \in \mathcal{D}_k. \quad (8)$$

Summing up all the inequalities in (8) and using the Cartesian product structure of the set  $\mathcal{D}$ , we conclude that  $\nabla \mathcal{L}(\tilde{\mathbf{p}})^T (\mathbf{p} - \tilde{\mathbf{p}}) \geq 0, \forall \mathbf{p} \in \mathcal{D}$ . As  $\mathcal{L}$  is convex over  $\mathcal{D}$ , then

from Proposition 3.1 (b) in [16],  $\hat{\mathbf{p}}$  minimizes  $\mathcal{L}$  over  $\mathcal{D}$  and therefore is a social optimum charging profile. ■

#### IV. CONVERGENCE OF THE BEST RESPONSE STRATEGY

In this section, we explore the convergence of an iterative best-response strategy where each PEV iteratively updates its charging profile *asynchronously*, based on the latest prices, so as to minimize its individual cost computed according to (3). In the asynchronous update model that we analyze, updates by the PEVs occur in real time, with two assumptions: (i) only a single update (by a single PEV) occurs at any time (step), and is immediately reflected in the price; (ii) the time interval between any two successive updates by the same PEV is bounded. Note that (i) can be achieved by the PEV communicating its updated charging profile to the utility immediately after the update, and the utility communicating the new prices (based on the updated load) to all PEVs immediately as well. In the following analysis as the best-response charging profile is unique (Lemma 1), we only consider *non-redundant* updates, ignoring any two successive updates by the same PEV (which does not change the system state). The time difference between any two consecutive update steps can be arbitrary (but bounded), but these updates must be due to two different PEVs.

Suppose the update at step  $m + 1$  (i.e., the  $(m + 1)^{th}$  non-redundant update) be due to PEV  $k$ . In this step, PEV  $k$  updates its charging profile  $\hat{\mathbf{p}}_k$  (given  $\mathbf{p}_{-k}^m$ ) to minimize the cost function  $\mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}^m)$ :

$$\begin{aligned} \mathbf{p}_k^{m+1} &= \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \mathcal{J}_k(\mathbf{p}_k, \mathbf{p}_{-k}^m) \\ &= \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \sum_{t=1}^T \int_{x=0}^{p_k(t)} \pi \left( D(t) + \sum_{j=1, j \neq k}^K p_j^m(t) + x \right) dx \\ &= \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \sum_{t=1}^T \mathcal{V} \left( D(t) + \sum_{j=1, j \neq k}^K p_j^m(t) + p_k(t) \right) \\ &= \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}^m), \end{aligned} \quad (9)$$

where  $\pi(x) = \mathcal{V}'(x)$ . As  $\mathbf{p}_k^{m+1}$  minimizes  $\mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}^m)$ , therefore the best-response charging profile also minimizes the variance of the total load at each update.

The best-response strategy is amenable to distributed implementation. The pricing function (kernel)  $\pi(\cdot)$  is assumed to be known to all PEVs (pre-established by the utility), and the utility iteratively broadcasts the total load ( $D(t) + \sum_{j=1}^K p_j^m(t)$ ) at different times  $t = 1, \dots, T$ . From this, since PEV  $k$  knows its own load  $p_k^m(t)$ , it can easily calculate  $(D(t) + \sum_{j=1, j \neq k}^K p_j^m(t))$  that is required for the update at step  $m + 1$ , according to (9).

*Lemma 1:* The best-response charging profile computed at any step, according to (9), is unique.

*Proof:* Let  $\hat{\mathbf{p}}_k$  be the best-response charging profile of PEV  $k$  (computed in any arbitrary step  $m + 1$ ),  $b_k(t) = D(t) + \sum_{j=1, j \neq k}^K p_j^m(t)$  and  $z(t) = b_k(t) + \hat{p}_k(t)$

is the total load at time  $t$ . Let assume  $\exists t_1, t_2 \in [t_k^s, t_k^f]$  such that  $z(t_1) < z(t_2)$  with  $\hat{p}_k(t_1) < \hat{p}_k^{max}(t_1)$  and  $0 < \hat{p}_k(t_2)$ . Let  $0 < \delta p < \min\{p_k^{max}(t_1) - \hat{p}_k(t_1), \frac{z(t_2) - z(t_1)}{2}, \hat{p}_k(t_2)\}$ . As  $\pi(x) = \mathcal{V}'(x)$ , therefore based on the mean value theorem, there exists  $c_1, c_2 : z(t_1) \leq c_1 \leq z(t_1) + \delta p < z(t_2) - \delta p \leq c_2 \leq z(t_2)$  such that  $\frac{\mathcal{V}(z(t_1) + \delta p) - \mathcal{V}(z(t_1))}{\delta p} = \pi(c_1) < \pi(c_2) = \frac{\mathcal{V}(z(t_2)) - \mathcal{V}(z(t_2) - \delta p)}{\delta p}$ . This results:  $\mathcal{V}(z(t_1) + \delta p) + \mathcal{V}(z(t_2) - \delta p) < \mathcal{V}(z(t_1)) + \mathcal{V}(z(t_2))$ , that contradicts with  $\mathcal{L}(\hat{\mathbf{p}}_k)$  is minimized over  $\mathcal{D}_k$ .

Therefore if there exists  $t_k^s \leq t_1, t_2 \leq t_k^f$  such that  $z(t_1) < z(t_2)$ , then  $\hat{p}_k(t_1) = \hat{p}_k^{max}(t_1)$  or  $\hat{p}_k(t_2) = 0$ . Thus, for the time slots  $t : 0 < \hat{p}_k(t) < p_k^{max}(t)$ , there exists a constant level  $\lambda > 0$  such that  $z(t) = \lambda$ . The best response solution for  $t \in [t_k^s, t_k^f]$  is as in the following:

$$\hat{p}_k(t) = \begin{cases} 0, & b_k(t) \geq \lambda, \\ \lambda - b_k(t), & \lambda - p_k^{max}(t) < b_k(t) < \lambda, \\ p_k^{max}(t), & b_k(t) < \lambda - p_k^{max}(t). \end{cases} \quad (10)$$

It shows that  $\hat{p}_k(t)$  can be written as  $\hat{p}_k(t) = \min\{\lambda - b_k(t), p_k^{max}(t)\}^+$  where  $x^+ = \max\{x, 0\}$ . Let define  $Y_k(\lambda) = \sum_{t=t_k^s}^{t_k^f} \min\{\lambda - b_k(t), p_k^{max}(t)\}^+$ . Let consider the case that  $0 < Y_k(\lambda), W_k < \sum_{t=1}^T p_k^{max}(t)$ , therefore  $\exists t_0 \in [t_k^s, t_k^f]$  such that  $0 < \lambda - b_k(t_0) < p_k^{max}(t_0)$  and  $W_k(\lambda)$  is a continuous and strictly increasing function of  $\lambda$ . Thus there exists a unique  $\lambda > 0$  such that  $Y_k(\lambda) = W_k$  and the best response solution is uniquely derived from (10). Furthermore, for the extreme cases  $Y_k(\lambda) = W_k = 0$  and  $Y_k(\lambda) = W_k = \sum_{t=1}^T p_k^{max}(t)$ , the unique solutions are  $\hat{p}_k(t) = 0$  and  $\hat{p}_k(t) = p_k^{max}(t)$ , for all  $k$  and  $t$ , respectively. ■

The uniqueness of the solution of (9) implies that successive updates by the same PEV can be ignored.

To proceed with the convergence proof, we define a *cycle* as a sequence of steps in which each vehicle has updated its charging profile at least once. Note that assumption (ii) (stated towards the beginning of this section) implies that the cycle-length is bounded.

*Theorem 2:* Every limit point of the best-response update process is socially optimal, in other words,  $\mathcal{L}^* = \lim_{m \rightarrow \infty} \mathcal{L}(\mathbf{p}^m)$  minimizes  $\mathcal{L}(\mathbf{p})$  over  $\mathcal{D}$ .

*Proof:* We base our proof on the convergence analysis in Proposition 3.9 of [16] for the synchronous non-linear Gauss-Seidel method; this analysis is adapted to the case of asynchronous updates under the assumptions (i) and (ii) mentioned earlier in this section. Let denote  $\mathcal{L}^m = \mathcal{L}(\mathbf{p}^m)$ . If there exists an  $M > 0$  such that  $\mathcal{L}^m = \mathcal{L}^*$  for all  $m \geq M$ , then none of the PEVs can decrease its cost function by deviating from  $\mathbf{p}^* = \mathbf{p}_k^M$  and satisfies  $\mathbf{p}_k^* = \arg \min_{\mathbf{p}_k \in \mathcal{D}_k} \mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}^*)$  for all  $k$ . From the first order optimality constraint, for all  $k$ , we have:

$$\nabla_k \mathcal{L}(\mathbf{p}_k^*)(\mathbf{p}_k - \mathbf{p}_k^*) \geq 0, \forall \mathbf{p}_k \in \mathcal{D}_k. \quad (11)$$

Adding up the inequalities for all  $k$  in (11) and using the Cartesian product structure of the set  $\mathcal{D}$ , we conclude that  $\nabla \mathcal{L}(\mathbf{p}^*)(\mathbf{p} - \mathbf{p}^*) \geq 0$  for all  $\mathbf{p} \in \mathcal{D}$ . As  $\mathcal{L}$  is convex over

$\mathcal{D}$  then  $\mathbf{p}^*$  is a global minimum of  $\mathcal{L}(\mathbf{p})$  over  $\mathbf{p} \in \mathcal{D}$  and therefore  $\mathbf{p}^*$  is a social optimum.

In the only other possible case,  $\mathcal{L}^m$  is an infinite length decreasing sequence of real numbers, i.e.  $\mathcal{L}(\mathbf{p}^{m+1}) \leq \mathcal{L}(\mathbf{p}^m)$  for  $m \geq 1$ , and for any  $M > 0$  there exists  $m \geq M$  such that  $\mathcal{L}(\mathbf{p}^{m+1}) < \mathcal{L}(\mathbf{p}^m)$ . As  $\mathcal{L}^m$  is bounded below by  $\sum_{t=1}^T D(t)^2$ , based on the monotone convergence theorem, there exists some  $\mathcal{L}^* = \inf_m \mathcal{L}^m$  such that  $\lim_{m \rightarrow \infty} \mathcal{L}^m = \mathcal{L}^*$ . As  $\mathcal{D}$  is compact and  $\mathcal{L}$  is continuous on  $\mathcal{D}$  and  $\lim_{m \rightarrow \infty} \mathcal{L}(\mathbf{p}^m) = \mathcal{L}^*$ , based on Weierstrass theorem, there exists  $\mathbf{p}^* \in \mathcal{D}$  and a subsequence  $\{\mathbf{p}^{m_i}\}_{i=1}^{\infty}$  of  $\{\mathbf{p}^m\}_{m=1}^{\infty}$  such that  $\lim_{i \rightarrow \infty} \mathbf{p}^{m_i} = \mathbf{p}^*$ . It now remains to show that  $\mathbf{p}^*$  minimizes  $\mathcal{L}$  over the set  $\mathcal{D}$ .

Towards that end, we first show that  $\{\mathbf{p}^{m_i+1}\} - \{\mathbf{p}^{m_i}\}$  converges to zero. Assume the contrary, that is,  $\{\mathbf{p}^{m_i+1}\} - \{\mathbf{p}^{m_i}\}$  does not converge to zero. Let  $\gamma^{m_i} = \|\mathbf{p}^{m_i+1} - \mathbf{p}^{m_i}\|_2$ . By possibly restricting to a subsequence  $\{a_i\} \subseteq \{m_i\}$ , we may assume that there exists some  $\gamma_0 > 0$  such that  $\gamma^{a_i} > \gamma_0$  for all  $i$ . As the number of the PEVs,  $K$ , is finite and the sequence  $\{a_i\}$  is infinite in length, there exists a vehicle  $k$ ,  $1 \leq k \leq K$ , that updates its profile  $\mathbf{p}_k$  in an infinite length subsequence of  $\{b_i + 1\} \subseteq \{a_i + 1\}$ . Let  $s_k^{b_i} = (\mathbf{p}^{b_i+1} - \mathbf{p}^{b_i})/\gamma^{b_i}$ . Thus,  $\mathbf{p}^{b_i+1} = \mathbf{p}^{b_i} + \gamma^{b_i} s_k^{b_i}$ ,  $\|s_k^{b_i}\|_2 = 1$ , and  $s_k^{b_i}$  differs from zero only along the  $k^{th}$  block-component. Notice that  $s_k^{b_i}$  belongs to a compact set and therefore has a limit point  $\bar{s}_k$ . By restricting to a further subsequence of  $\{n_i\} \subseteq \{b_i\}$ , we assume that  $s_k^{n_i}$  converges to  $\bar{s}_k$ .

Let us fix some  $\epsilon \in [0, 1]$ . Notice that  $0 \leq \epsilon\gamma_0 \leq \gamma^{n_i}$ . Therefore,  $\mathbf{p}^{n_i} + \epsilon\gamma_0 s_k^{n_i}$  lies on the segment joining  $\mathbf{p}^{n_i}$  and  $\mathbf{p}^{n_i} + \gamma^{n_i} s_k^{n_i} = \mathbf{p}^{n_i+1}$  and belongs to  $\mathcal{D}$  because  $\mathcal{D}$  is convex. Using the convexity of  $\mathcal{L}$ , and the fact that  $\mathbf{p}^{n_i+1}$  minimizes  $\mathcal{L}$  over all  $\mathbf{p}$  that differ from  $\mathbf{p}$  along the  $k^{th}$  block-component, we obtain:

$$\begin{aligned} \mathcal{L}(\mathbf{p}^{n_i+1}) &= \mathcal{L}(\mathbf{p}^{n_i} + \gamma^{n_i} s_k^{n_i}) \\ &\leq \mathcal{L}(\mathbf{p}^{n_i} + \epsilon\gamma_0 s_k^{n_i}) \leq \mathcal{L}(\mathbf{p}^{n_i}). \end{aligned} \quad (12)$$

Since  $\mathcal{L}(\mathbf{p}^{n_i})$  converges to  $\mathcal{L}(\mathbf{p}^*)$ ,  $\mathcal{L}(\mathbf{p}^{n_i+1})$  also converges to  $\mathcal{L}(\mathbf{p}^*)$ . We now take the limit as  $i$  tends to infinity, to obtain  $\mathcal{L}(\mathbf{p}^*) \leq \mathcal{L}(\mathbf{p}^* + \epsilon\gamma_0 \bar{s}_k) \leq \mathcal{L}(\mathbf{p}^*)$ . We conclude that  $\mathcal{L}(\mathbf{p}^*) = \mathcal{L}(\mathbf{p}^* + \epsilon\gamma_0 \bar{s}_k)$ , for every  $\epsilon \in [0, 1]$ . Since  $\gamma_0 \bar{s}_k \neq 0$ , this contradicts the strict convexity of  $\mathcal{L}$  as a function of the  $k^{th}$  block-component. This contradiction establishes that  $\mathbf{p}^{n_i+1} - \mathbf{p}^{n_i}$  converges to zero. In particular,  $\mathbf{p}^{n_i+1}$  converges to  $\mathbf{p}^*$ .

Using this approach successively results that for every sequence  $\{\mathbf{p}^{m_i}\}$  converging to  $\mathbf{p}^*$ , the sequence  $\{\mathbf{p}^{m_i+M}\}$  also converges to  $\mathbf{p}^*$  for all  $M \in \mathbb{Z}^+$ . Let  $N$  denotes the maximum cycle length that is defined based on the update time obligations for each PEV. Let  $\mathbf{p}^{m_i}$  be the subsequence converging to  $\mathbf{p}^*$ . Let  $\{n_i\}$  is a subsequence of  $\{m_i\}$  such that  $n_{i+1} - n_i > N$ . Let  $Z_i = \{\mathbf{p}^{n_i}, \dots, \mathbf{p}^{n_i+N}\}$ . As  $N$  is the maximum cycle length, therefore each vehicle  $k$  updates its profile at least once in  $Z_i$ . Let  $k_i$  is the first time that PEV  $k$  updates its profile at  $Z_i$ . All the elements in  $Z_i$  converges to

$\mathbf{p}^*$ , therefore any sequence  $\{\mathbf{p}^{n_i+k_i}\}$  also converges to  $\mathbf{p}^*$ :

$$\begin{aligned} \forall \delta > 0 : M &= \max_{M_j} \{i > M_j : \|\mathbf{p}^{n_i+j} - \mathbf{p}^*\| \leq \delta\} \\ \Rightarrow i > M : \|\mathbf{p}^{n_i+j} - \mathbf{p}^*\| &\leq \delta, \quad \forall j : 1 \leq j \leq N. \end{aligned} \quad (13)$$

From the definition (9) of the algorithm, for every PEV  $k$ , we have:

$$\mathcal{L}(\mathbf{p}^{n_i+k_i}) \leq \mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}^{n_i+k_i-1}), \forall \mathbf{p}_k \in \mathcal{D}_k. \quad (14)$$

Taking the limit as  $i$  tends to infinity, we obtain:

$$\mathcal{L}(\mathbf{p}^*) \leq \mathcal{L}(\mathbf{p}_k, \mathbf{p}_{-k}^*), \forall \mathbf{p}_k \in \mathcal{D}_k. \quad (15)$$

Using the first order optimality conditions for constrained optimization, we conclude that:

$$\nabla_k \mathcal{L}(\mathbf{p}^*)^T (\mathbf{p}_k - \mathbf{p}_k^*) \geq 0, \forall \mathbf{p}_k \in \mathcal{D}_k.$$

Adding these inequalities for all  $k$ , and using the Cartesian product structure of the set  $\mathcal{D}$ , we conclude that  $\nabla \mathcal{L}(\mathbf{p}^*)^T (\mathbf{p} - \mathbf{p}^*) \geq 0$  for every  $\mathbf{p} \in \mathcal{D}$ . In view of the convexity of  $\mathcal{L}$ , this shows that  $\mathbf{p}^*$  minimizes  $\mathcal{L}$  over the set  $\mathcal{D}$ . The result in Theorem 2 follows. ■

As an illustration of the best response strategy, let us consider the simple example with two PEVs,  $\mathcal{K} = \{1, 2\}$ , and the parameters:  $t_1^s = 1, t_1^f = 2, t_2^s = 2, t_2^f = 3, U_1 = 1KWh, U_2 = 1KWh$  and  $D(t) = 0.42MW$  for  $t = 1, 2, 3$ . Let assume PEV 1 starts updating its charging profile at odd steps  $\{2n+1\}_{n=0}^{\infty} = \{1, 3, \dots\}$  and PEV 2 updates its charging profile at the even steps  $\{2n\}_{n=1}^{\infty} = \{2, 4, \dots\}$ . For  $n \geq 1$ , the charging profiles are  $\mathbf{p}_1^{2n} = \mathbf{p}_1^{2n-1} = (\frac{2}{3} - \frac{2}{3 \times 4^n}, \frac{1}{3} + \frac{2}{3 \times 4^n}, 0)KW$  and  $\mathbf{p}_2^{2n} = \mathbf{p}_2^{2n+1} = (0, \frac{1}{3} - \frac{1}{3 \times 4^n}, \frac{2}{3} + \frac{1}{3 \times 4^n})KW$ . As  $n$  goes to infinity the charging profiles converge to  $\mathbf{p}_1^* = (\frac{2}{3}, \frac{1}{3}, 0)KW$  and  $\mathbf{p}_2^* = (0, \frac{1}{3}, \frac{2}{3})KW$ ; however we do not achieve  $\mathbf{p}^*$  in finite number of time steps.

## V. CHARGING PEVS IN A DISTRIBUTION GRID NETWORK WITH TRANSMISSION LINE CONSTRAINTS

In this section we revisit the price-driven PEV charging question in a distribution grid, taking into account the grid topology and limited capacity of electricity transmission lines. We model the distribution network as a tree rooted at the distribution substation. Each PEV is attached to one of the leaf nodes of the tree. Each PEV  $k$  is associated with a set of transmission lines (links in the tree graph),  $\beta_k$ , which transfer power from the distribution substation to PEV  $k$ , and each transmission line  $l$  carries power to a set  $\Gamma_l$  of PEVs, where  $\Gamma_l = \{k : l \in \beta_k\}$ . Let  $L$  be the total number of transmission lines (links) (indexed  $1, \dots, L$ ) in the tree topology. Let  $K_l = |\Gamma_l|$  denote the number of PEVs that receive electricity through the transmission line  $l$ ; obviously  $K_l \leq K$ . Let  $d_{max}$  denote the maximum path length (number of transmission lines) between the distribution substation and any PEV, or in other words, the maximum depth of the tree topology. Let  $l_k = |\beta_k|$  be the number of transmission lines that transfer power from the distribution substation to PEV  $k$ ; thus  $l_k \leq d_{max}$ .

We define  $P(t) = \sum_{k=1}^K p_k(t)$  as the aggregate PEV load

at time  $t$ . Let  $P_l(t) = \sum_{k \in \Gamma_l} p_k(t)$  denote the total load of PEVs fed from the  $l^{\text{th}}$  transmission line and  $d_l(t)$  denote the base demand load (non-PEV demand) transmitted through transmission line  $l$  at time  $t$ . As each transmission line  $l$  has a maximum transmission capacity,  $\rho_l$ , the maximum PEV demand it can support at time  $t$  is  $P_l^{\text{max}}(t) = \rho_l - d_l(t)$ . Thus for each transmission line,  $l$ , the *overload constraint* is as in the following:

$$P_l(t) \leq P_l^{\text{max}}(t). \quad (16)$$

We account transmission line capacity limitations by associating cost (penalty) functions with the constraints (16), and add them to the social cost function in (5) to create an *overload-discounted* objective function. More specifically, let each link  $l$  be associated with an overload cost function,  $C_l(x)$ , where  $C_l(x)$  is a continuous, non-negative and strictly convex function of  $x \in \mathbb{R}^+$ . The social cost function can then be *approximated* by minimizing an overload-discounted network objective function  $\mathcal{N}(\mathbf{p})$  over the set of all charging profiles  $\mathcal{D}$ , where

$$\mathcal{N}(\mathbf{p}) = \sum_{t=1}^T \left\{ \mathcal{V} \left( D(t) + \sum_{k=1}^K p_k(t) \right) + \sum_{l=1}^L C_l(P_l(t) - P_l^{\text{max}}(t)) \right\}. \quad (17)$$

Note that the difference between this approximate (overload-discounted minimum cost) problem and the original problem (minimizing (5) over  $\mathcal{D}$ ) subject to (16) can be made small (and therefore the optimum solutions for the two problems can be made close to each other) by defining the overload cost functions appropriately. Intuitively the cost function  $C_l(P_l(t) - P_l^{\text{max}}(t))$  should be “high” when link  $l$  is overloaded, i.e.,  $P_l(t) > P_l^{\text{max}}(t)$ , and “small” when it is underloaded,  $P_l(t) < P_l^{\text{max}}(t)$ . The above approach represents a classical penalty function method [18] (Chapter 3) of handling constraints in a constrained optimization problem. Intuitively, the problem derived using the penalty function method approaches the original problem as the penalty functions approach a “barrier” function.

In this scenario, each PEV  $k$  minimizes its own cost function,  $\mathcal{I}_k(\mathbf{p}_k, \mathbf{p}_{-k})$ ,  $\mathbf{p}_k \in \mathcal{D}_k$ , as in the following:

$$\begin{aligned} \mathcal{I}_k(\mathbf{p}_k, \mathbf{p}_{-k}) = & \sum_{t=1}^T \left[ \int_{x_k^t=0}^{p_k(t)} \left\{ \pi \left( D(t) + \sum_{j=1, j \neq k}^K p_j(t) + x_k^t \right) \right. \right. \\ & \left. \left. + \sum_{l \in \beta_k} c_l \left( \sum_{j \in \beta_k, j \neq k} p_j(t) + x_k^t - P_l^{\text{max}}(t) \right) \right\} dx_k^t \right], \end{aligned} \quad (18)$$

where the congestion price function,  $c_l(x)$  is a continuous and strictly increasing function of  $x$ .

If  $c_l(x)$  is defined as the derivative of  $C_l(x)$  then all the Nash equilibrium properties discussed in section III and

the best-response convergence result stated in section IV, extend to this scenario, where the PEV cost function  $\mathcal{J}_k$  and the social objective  $\mathcal{L}(\mathbf{p})$  are replaced by  $\mathcal{I}_k$  and  $\mathcal{N}(\mathbf{p})$ , respectively. Formal arguments on these will be provided in the full version of the paper, where we will also discuss distributed implementation of the best-response strategy of PEV  $k$  that minimizes (18). Reasonable conditions under which the Nash equilibrium is unique, which guarantees stronger convergence results in this model, will also be discussed.

## VI. NUMERICAL STUDIES

For the simulation study we implement and evaluate the performance of the best-response strategy method proposed in this paper. The hourly load demand data is obtained from [19]. The battery capacity of each vehicle is assumed to be  $W_k = 7$  KWh and it is assumed that the maximum charging rate is  $p_k^{\text{max}} = 1.96$  KW for all  $k, t$ . The charging starting and finish time is set to the hours  $t_k^s = 0$  and  $t_k^f = 24$  for all the PEVs. The PEVs update their charging profiles randomly; more specifically, at each time step, the PEV that updates its charging profile in that step is chosen uniformly at random. The variance function is set to the continuous and strictly convex function  $\mathcal{V}(x) = \frac{1}{2}x^2$  and the pricing function is set to the continuous and strictly increasing function  $\pi(x) = \mathcal{V}(x) = x$ . Let  $g_t(\mathbf{p}) = \sum_{k=1}^K p_k(t)$  denote the total PEV aggregate load at time  $t$  and the vector  $\mathbf{g}(\mathbf{p}) = (g_1(\mathbf{p}), \dots, g_T(\mathbf{p}))$  denote the total PEV aggregate load during the time  $1, \dots, T$ . Also, let the best-response strategy converge to the limit point  $\mathbf{p}^*$ . The normalized error at each step  $m$  is determined as the normalized convergence distance of the vector of the total PEV load at step  $m$  to  $\mathbf{p}^*$  as in the following:

$$\text{Normalized error} = \frac{\|\mathbf{g}(\mathbf{p}^m) - \mathbf{g}(\mathbf{p}^*)\|_2}{\|\mathbf{g}(\mathbf{p}^*)\|_2}. \quad (19)$$

Figure 1 shows the total load resulting after the convergence of the best-response method for  $K = 75$  PEVs, after a total of  $M = 4000$  updates. As expected, the converged charging profile is socially optimal as it minimizes the overall load variance (flattens the total demand “valley” as much as possible). The normalized error of the total load calculation in Fig. 1 at step  $m = 1200$  is less than  $10^{-7}$  as the limit point charging profile is set to  $\mathbf{p}^* = \mathbf{p}^{4000}$ .

Figure 2 demonstrates the normalized error vs the number of updates for  $K = 75$ ,  $K = 120$  and  $K = 200$  PEVs and Fig. 3 shows the number of updates to achieve the normalized error of  $10^{-5}$  with respect to the social optimum total load when the number of PEVs is increased. Figure 2 shows that the normalized error decreases exponentially with the number of updates run by PEVs. Figure 3 shows that the number of updates required to attain a pre-defined normalized error increases linearly with the number of PEVs.

## VII. CONCLUDING REMARKS

In this paper we analyzed the equilibrium resulting from a load-dependent pricing mechanism to encourage the PEVs

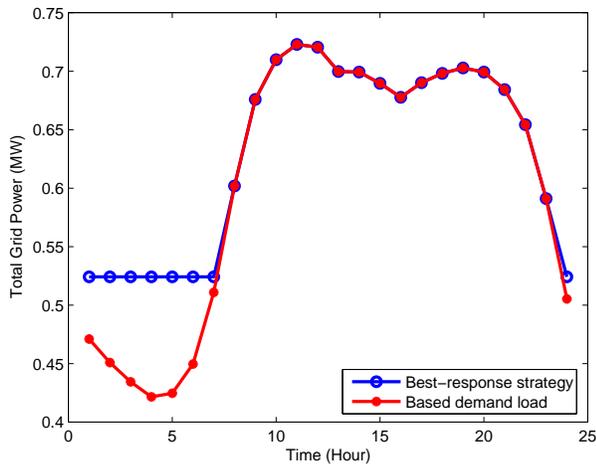


Fig. 1. Total load vs time in a daily 24 hour time zone with total energy charging of  $W_k = 0.007$  KWh for each PEV, total number of  $K = 75$  PEVs and total number of  $M = 4000$  PEV updates.

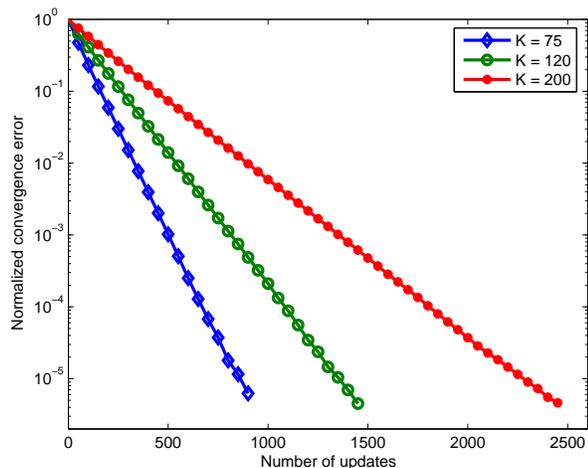


Fig. 2. Normalized error vs number of the updates for charging in the daily 24 hour time zone with total number of  $K = 75$ ,  $K = 120$  and  $K = 200$  PEVs and total energy charging of  $W_k = 7$  KWh for each PEV.

(acting in self-interest) to distribute their charging profile over time. The pricing mechanism incorporates the added load of the PEV at each time unit into the cost function of each PEV, and therefore prevents the PEV from charging when the grid is overloaded, as well as encourages it to spread its load over time. We established the social optimality of all equilibrium solutions, in that they minimize the social cost defined as the variance of the total load over the distribution grid. We showed that the best-response strategy converges to socially optimal solutions under asynchronous updates. We also briefly discussed how the model and results can be extended to take into account transmission line capacity constraints in the distribution grid. The best response strategy in this case is also implemented in a distributed manner, with the utility sending the time-dependent prices or load (congestion) information to the PEVs, and

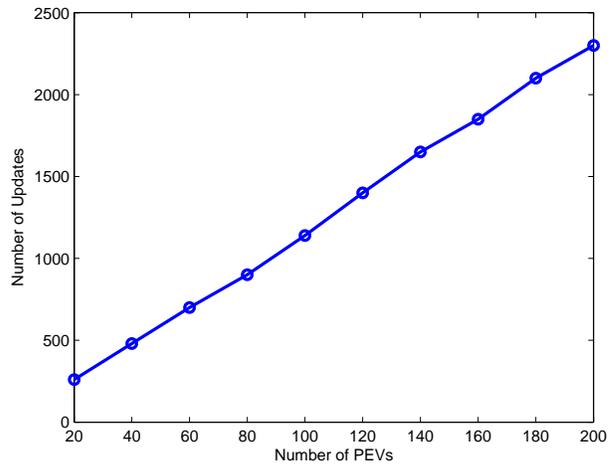


Fig. 3. Number of updates vs number of PEVs to achieve the normalized error of  $10^{-5}$  for charging in the daily 24 hour time zone with total energy of  $W_k = 7$  KWh for each PEV.

the PEVs updating their charging profiles accordingly, and communicating that back to the utility.

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