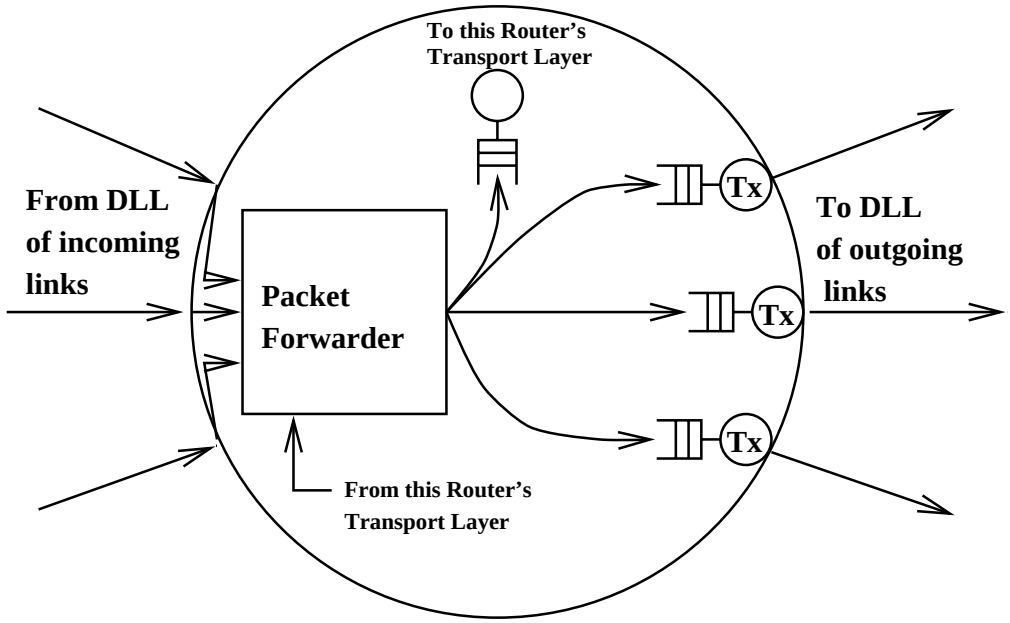
NETWORK LAYER PERFORMANCE MODELING AND ANALYSIS

- Three parts.
- Part I: Essentials of Probability.
- ⋆ Part II: Inside a Router.
- Part III: Network Analysis.

NETWORK LAYER PERFORMANCE MODELING AND ANALYSIS, PART II: INSIDE A ROUTER

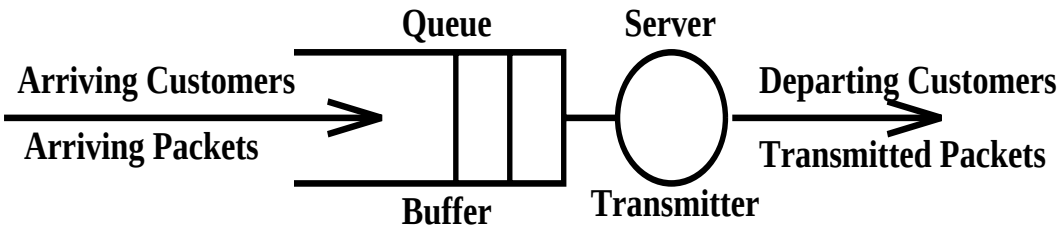
- Basic Single Queue Model
- Poisson Arrival Model
- The M/M/1 Queue
- Read any of the queueing theory references, e.g. Schwartz (Sections 2.1-3), Molloy, Kleinrock.

Queueing in the Network Layer at a Router



Basic Single Queue Model

- Classical queueing theory can be applied to an output link in a router.



- For example, a 56 $kbps$ transmission line can "serve" 1000-bit packets at a rate of

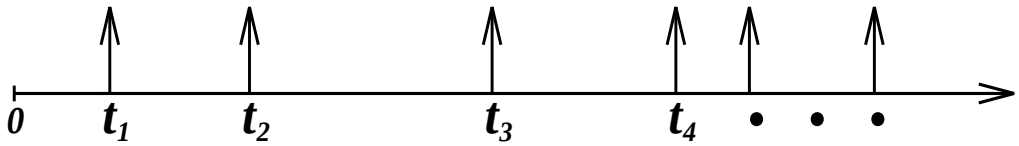
$$\frac{56,000\ bits/sec}{1000\ bits/packet} = 56\ packets/sec$$

Applications of Queueing Analysis Outside of Networking

- Checkout line in a supermarket.
- Waiting for a teller in a bank.
- Batch jobs waiting to be processed by the CPU.
- "That's the way the whole thing started,
  Silly but it's true,
  Thinking of a sweet romance
  Beginning in a queue."
  –G. Gouldman, "Bus Stop," The Hollies

The Poisson Arrival Model

- A <u>Poisson process</u> is a sequence of events "randomly spaced in time."



- Examples
  – Customers arriving to a bank.
  – Packets arriving to a buffer.
- The <u>rate</u> λ of a Poisson process is the average number of events per unit time (over a long time).

Properties of a Poisson Process

- For a length of time t, the probability of n arrivals in t units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

- For 2 disjoint (non-overlapping) intervals, (s_1, s_2) and (s_3, s_4), (i.e. $s_1 < s_2 \leq s_3 < s_4$), the number of arrivals in (s_1, s_2) is independent of the number of arrivals in (s_3, s_4).

Interarrival Times of a Poisson Process

- Pick an arbitrary starting point in time (call it 0).
- Let τ_1 = the time until the next arrival.

$$P(\tau_1 > t) = P_0(t) = e^{-\lambda t}$$

- So

$$F_{\tau_1}(t) = P(\tau_1 \leq t) = 1 - e^{-\lambda t} \quad \text{and} \quad f_{\tau_1}(t) = \lambda e^{-\lambda t}$$

τ_1, the time until the first arrival, has an exponential distribution!

Interarrival Times of a Poisson Process (cont.)

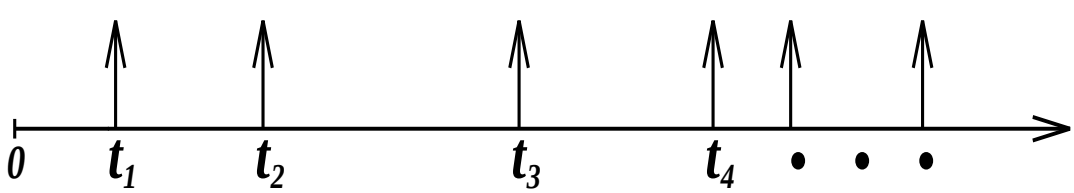
- Let τ_2 = the length of time <u>between</u> the first and second arrival.
- We can show that

$$P(\tau_2 > t \mid \tau_1 = s) = P(\tau_2 > t) = e^{-\lambda t} \quad \text{for any } s, t > 0$$

  i.e. τ_2 is exponential and independent of τ_1!
- Similarly define τ_3 as the time between the second and third arrival; τ_4 as the time between the third and fourth arrival; . . .
- The random variables $\tau_1, \tau_2, \tau_3, \ldots, \tau_n, \ldots$ are called the <u>interarrival times</u> of the Poisson process.

Interarrival Times of a Poisson Process (cont.)

- The interarrival time random variables, $\tau_1, \tau_2, \tau_3, \ldots$
  - Are (pair-wise) independent.
  - Each has an exponential distribution with mean $1/\lambda$.



The M/M/1 Queue

- An M/M/1 queue has

  1. Poisson arrivals (with rate λ).
  2. Exponential service times (with mean $1/\mu$, so μ is the "service rate").
  3. One (1) server.
  4. An infinite length buffer.

- The M/M/1 queue is the most basic and important queueing model.

Queueing Notation

"M/M/1" is a special case of more general (Kendall) notation: X/Y/m/k, where

- X is a symbol representing the interarrival process
  M = Poisson (exponential interarrival times τ).
  D = Deterministic (constant τ).
- Y is a symbol representing the service distribution
  M = exponential, D = deterministic.
  G = General (or arbitrary).
- m = number of servers.
- k = number of buffer slots (omitted when k = ∞).

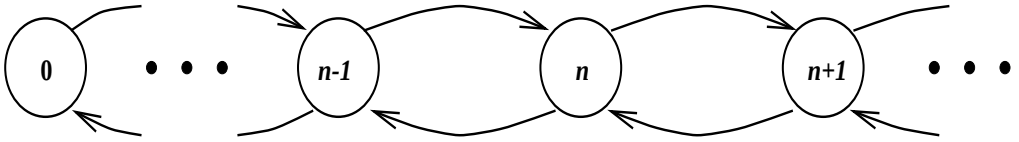
Aside: The D/D/1 Queue

- The D/D/1 queue has
  - Deterministic arrivals (periodic with period = $1/\lambda$).
  - Deterministic service times (each service takes exactly $1/\mu$).
  - As well as 1 server and an infinite length buffer.
- If $\lambda < \mu$ then there is <u>no waiting</u> in a D/D/1 queue.

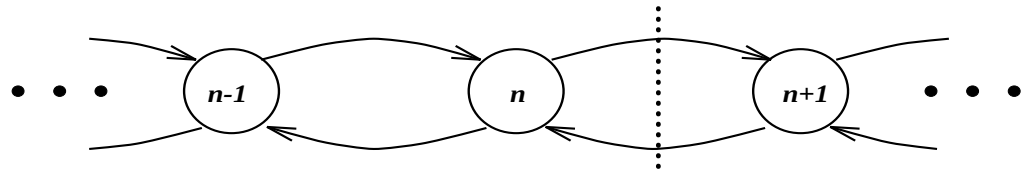
<u>Randomness</u> is a major cause of delay in a network node!

State Analysis of an M/M/1 Queue

- Let n be the state of the system = the number of packets in the system (including the server).
- Let p_n be the steady state probability of finding n customers waiting in the system (including the server).
- How to find p_n? The state diagram:



State Analysis of an M/M/1 Queue (cont.)



- If this system is stable (i.e. $p_n \neq 0$ for each n), then in steady state it will drift back and forth across the dotted line. So,
- the number of transitions from left to right
  = the number of transitions from right to left.
- Thus we obtain the <u>balance equations</u>

$$p_n \lambda = p_{n+1} \mu \qquad \text{for each } n \geq 0$$

State Analysis of an M/M/1 Queue (cont.)

- Lets solve the balance equations: $p_n \lambda = p_{n+1} \mu$
- For $n = 0$ we get
- If we let $\rho = \lambda / \mu$, this becomes

$$p_1 = \rho p_0$$

- Similarly

$$p_2 = \rho p_1 = \rho^2 p_0$$

- And in general

$$p_n =$$

State Analysis of an M/M/1 Queue (cont.)

- **We have** $p_n = \rho^n p_0$ **for** $n = 1, 2, 3, \ldots$
- **We need to solve for** p_0**, so we need one more equation. Use**

$$\sum_{n=0}^{\infty} p_n = 1$$

- **We obtain**

$$1 = \sum_{n=0}^{\infty} \rho^n p_0 = p_0 \sum_{n=0}^{\infty} \rho^n = \begin{cases} p_0 \left(\frac{1}{1-\rho}\right) & \text{for } \rho < 1 \\ \infty & \text{for } \rho \geq 1 \end{cases}$$

- **So we must have**

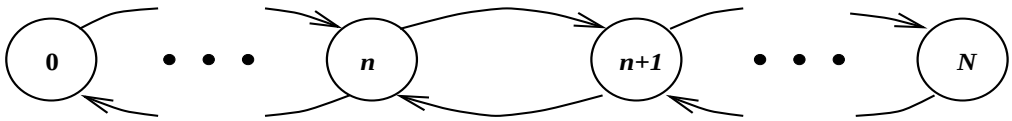
$$p_0 = 1 - \rho \qquad \textbf{and} \qquad p_n = (1-\rho)\rho^n \quad \textbf{for } n = 1, 2, 3, \ldots$$

State Analysis of an M/M/1 Queue (cont.)

- **Note that requiring** $\rho < 1$ **for stability (i.e.** $\lambda < \mu$**) makes intuitive sense.**
- **Also** $\rho = 1 - p_0$
  = probability that the queueing system is NOT empty
  = probability that the server is working
  So ρ **is sometimes called the "server utilization"**
- **Finally note that** $p_n = (1-\rho)\rho^n$, $n = 0, 1, 2, 3, \ldots$ **is a geometric distribution.**

The Finite Buffer Case: M/M/1/N

- Infinite buffer assumption is unrealistic in practice.
- N = total number of buffer slots (including server).
- New state diagram:



- Get the same balance equations $p_n\lambda = p_{n+1}\mu$ but now only for $n = 0, 1, 2, \ldots, N-1$ with $N < \infty$. So

$$p_n = \rho p_{n-1} = \rho^n p_0 \qquad \text{for} \quad n = 0, 1, 2, \ldots, N$$

as before, but we get a different p_0.

The Finite Buffer Case: M/M/1/N (cont.)

- From $p_n = \rho^n p_0$ for $n = 0, 1, 2, \ldots, N < \infty$ and $\sum_{n=0}^{N} p_n = 1$ we get

$$p_0 = 1 - \sum_{n=1}^{N} \rho^n p_0$$

- So

$$p_0 = \frac{1}{1 + \sum_{n=1}^{N} \rho^n} = \frac{1}{1 + \frac{\rho(1-\rho^N)}{(1-\rho)}} = \cdots = \frac{1-\rho}{1-\rho^{N+1}}$$

- Note that this holds <u>for any</u> $\rho \geq 0$. No need to assume $\rho < 1$. We always have stability in finite buffer case.

Blocking Probability and the Right Size Buffer

- So in the finite buffer case,

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}} \qquad \text{for} \quad n = 0, 1, 2, \ldots, N$$

- Note that p_N is the probability that the buffer is full at an arbitrary point in time.
- Since arrivals are independent of buffer state, we have $p_N = P_B =$ probability an arriving packet is turned away due to a full buffer.
- P_B is called the <u>blocking probability</u>.

Blocking Probability and Buffer Size (cont.)

- P_B is very important!
- We can use P_B to choose the correct buffer size.
- <u>Example</u>: For $\rho = 0.5$, $p_N > 10^{-6}$ for $N \leq 18$, while $p_N < 10^{-6}$ for $N \geq 19$.
- Thus, if we desire a blocking probability less than 10^{-6}, we need a buffer capable of holding 19 or more packets.

Throughput in the Finite Buffer Case

- The <u>throughput</u> γ of any queueing system is the rate at which customers successfully leave the system.
- For the M/M/1 infinite buffer case, $\gamma = \lambda$ if the system is stable. (Everything that arrives must eventually depart.)
- For the M/M/1/N finite buffer case, $\gamma = \lambda(1 - P_B)$. (Everything that arrives <u>and is not blocked</u> must eventually depart.)

Throughput in the Finite Buffer Case (cont.)

Alternate way to compute throughput of M/M/1/N: Look at the output side.

- P(server is busy) $= 1 - p_0$
- When the server is busy, the output rate $= \mu$
- When the server is idle, the output rate $= 0$

- So the average rate $= \gamma = \mu(1 - p_0) + 0p_0$

Aside: Derivation of $p_N = P_B$ Using Throughput

- Equating our two formulas for γ we get

$$\mu(1 - p_0) = \lambda(1 - P_B)$$

- Solving for P_B we get

$$P_B = 1 - \frac{1 - p_0}{\rho} = \cdots = \frac{(1-\rho)\rho^N}{1 - \rho^{N+1}} = p_N$$

- Isn't that neat?

Approximation of a Finite Buffer System by the Infinite Buffer Model

- For an infinite buffer, $p_n = (1 - \rho)\rho^n$
- For a finite buffer, $p_n = (1 - \rho)\rho^n / \left(1 - \rho^{N+1}\right)$
- For $\rho = 0.8$ and $N = 16$ packets, these probabilities differ by less than 2.3%.
- For $\rho = 0.8, N = 32$, the difference is only 0.06%.

The infinite buffer model is a very good approximation of a finite buffer system. Even for moderate buffer sizes!

How Long is That Line?

- Lets look again at the M/M/1 queueing system.
- n = the number in the system (including the server).
- So the average number in the system is

$$E(n) = \sum_{n=0}^{\infty} np_n = (1-\rho) \sum_{n=0}^{\infty} n\rho^n$$

$$= (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

Little's Formula and Queueing Delay

- Let T = time spent by a customer in a queueing system (waiting and being served).
- $E(T)$ = the average <u>delay</u> for a customer.
- <u>Little's Formula</u> says

$$\boxed{\lambda E(T) = E(n)}$$

where λ is the "arrival rate for customers eventually served" (which we had called γ).

- Little's Formula holds for very general queueing systems (not just M/M/1). Even whole networks!

Little's Formula and Queueing Delay (cont.)

- Little's Formula is either deep or obvious. <u>Intuition</u>:
- Pick a "typical customer."
- When it arrives to the queueing system, it should find $E(n)$ customers waiting.
- When it leaves the system, it has been in the system for $E(T)$. Thus, $\lambda E(T)$ customers should have arrived during its time in the system.
- In steady state, the average number of customers left behind on departure should equal the average number found on arrival, i.e. $\lambda E(T) = E(n)$

Little's Formula and Queueing Delay (cont.)

- Lets apply Little to the M/M/1 queue

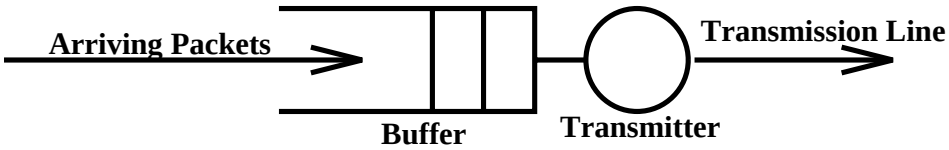
$$E(T) = \frac{E(n)}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu - \lambda}$$

- $E(T)$ is measured in units of time. Sometimes it is more convenient to consider

$$\mu E(T) = \frac{\mu E(n)}{\lambda} = \frac{\rho}{\rho(1-\rho)} = \frac{1}{1-\rho}$$

  which is unitless.
- Sometimes we consider the <u>waiting time</u> W, i.e. the time spent waiting in the queue (not in service). So, $E(W) = E(T) - \frac{1}{\mu}$

Single Link Example



- Poisson packet arrivals with rate $\lambda = 2000\ p/s$.
- Fixed link capacity $C = 1.544\ Mb/s$ (T1 Carrier rate).
- We approximate the packet length distribution by an exponential with mean $L = 515\ b/p$.
- Thus the service time is exponential with mean

$$\frac{1}{\mu} = \frac{L}{C} = \frac{515\ b/p}{1.544\ Mb/s} \approx 0.33\ ms/p$$

i.e. packets are served at a rate of $\mu = 3000\ p/s$.

Single Link Example (cont.)

- Using our formulas for an M/M/1 queue

$$\rho = \frac{\lambda}{\mu} = 0.67$$

So,

$$E(n) = \frac{\rho}{1-\rho} = 2.0\ packets$$

and

$$E(T) = \frac{E(n)}{\lambda} = 1.0\ ms$$

Other Queueing Models

- There are many other important queueing models which are useful in networking.
- M/M/k for k $>$ 1. Multiple servers.
  - Good model of a link which is made up of multiple channels, either physically or through multiplexing (e.g. a T1 Carrier is typically time division multiplexed with k = 24).
  - Has worse performance at lower loads than M/M/1 with same total capacity.

Other Queueing Models (cont.)

- M/M/k/k for k $\geq$ 1. One or more servers, no buffers (except one in each server).
  - Important model in circuit switched networks.
  - Models a trunk line with k circuits available.
  - Any customer (a call) which doesn't get a circuit is blocked (gets a busy signal).
  - Blocking probability is given by the Erlang B (or Erlang Loss) Formula

$$P_B = \frac{\rho^k/k!}{\sum_{i=0}^{k} \rho^i/i!} \qquad \rho \geq 0$$

Other Queueing Models (cont.)

- M/G/1. Arbitrary service (packet length) distribution.
  - Can still compute the mean number in the system via the Pollaczek-Khinchine (P-K) Formula

$$E(n) = \left(\frac{\rho}{1-\rho}\right)\left[1 - \frac{\rho}{2}(1-\mu^2\sigma^2)\right] \qquad \rho < 1$$

    where σ^2 is the variance of the service time distribution. Again, variability (randomness) causes delay.
  - Can apply Little's Formula to get the mean delay.

Other Queueing Models (cont.)

- M/D/1. Deterministic service times (packet length).
  - Special case of M/G/1 with $\sigma^2 = 0$

$$E(n) = \left(\frac{\rho}{1-\rho}\right)\left(1 - \frac{\rho}{2}\right) \qquad \rho < 1$$

  - Under heavy load ($\rho \approx 1$), M/D/1 has half the delay of an M/M/1.
  - This is one motivation for fixed-packet-length systems like ATM.

Other Queueing Models (cont.)

- Can also model and analyze other queueing systems
  - With priority.
  - With more general arrival processes.
  - With "vacations."
  - Many others.
- See Schwartz (Ch. 2), Kleinrock (Vol. I & II) or take ECSE-6820/DSES-6820, Queuing (sic) Systems & Applications.
- Queueing theory is also used in analysis of Operating Systems, e.g. in CSCI-6140.