# How many rays do we need to ensure a less-than-1% error in ray-tracing simulations?

To answer this question, first we need to know the definition of error. Here ***error*** is defined as the typical difference between two simulation results under identical simulation conditions. Thus, a less-than-1% error means the typical difference between two simulation results should be less than 1%. Mathematically the relative error of a statistical distribution can be written as

$$Error = \frac{\sigma}{\mu} \tag{1}$$

where σ is the ***standard deviation*** of the distribution and μ is the ***mean value*** of the distribution.

There are two common distributions to describe real-valued random distributions: the ***normal*** (or ***Gaussian***) ***distribution*** and the ***Poisson distribution***. The normal distribution is used when the sample size is relatively large and the Poisson distribution is used when the sample size is relatively small.

The normal distribution is described by a mathematical formula which is known as the Gaussian function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2}$$

The graph of the normal distribution is shown in Fig. 1.

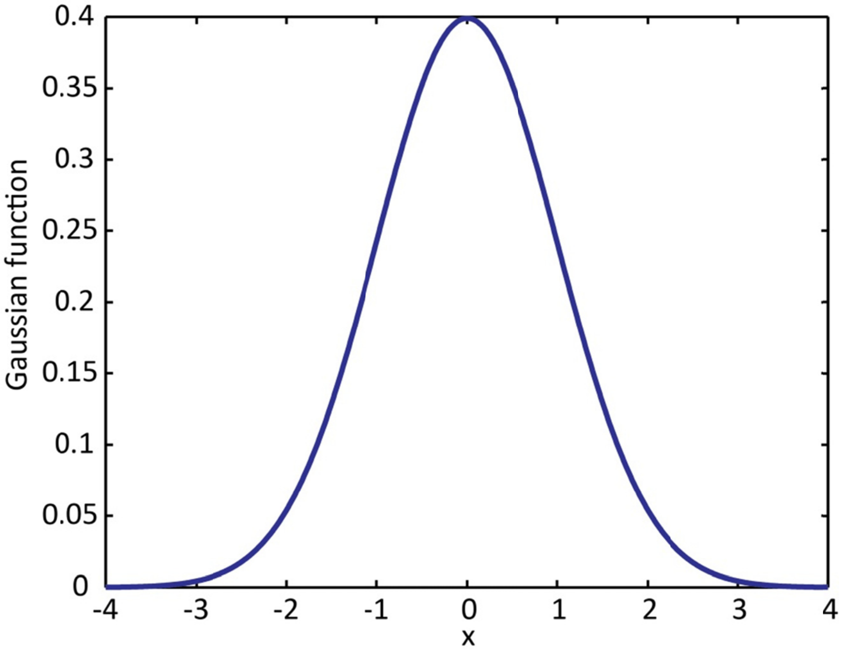

Fig. 1: The normal distribution for a mean value of μ = 0 and a standard deviation of σ = 1.

As shown in Fig. 1, there are several properties of the normal distribution:

1) The distribution is evenly symmetric around its mean value;
2) The distribution has its maximum at its mean value;
3) The distribution approaches zero at its limits ($x \to -\infty$ and $x \to +\infty$).

The Poisson distribution is mathematically described as

$$f(k,\lambda) \quad = \quad \frac{\lambda^k \, e^{-\lambda}}{k!} \tag{3}$$

where $k$ is the integer number of occurrences of an event, the probability of which is given by the function; and λ is an integer positive real number that is equal to the expected number of occurrences during a given interval; that is, λ is the ***mean value*** (λ = μ). The graph of the Poisson distribution is shown in Fig. 2.

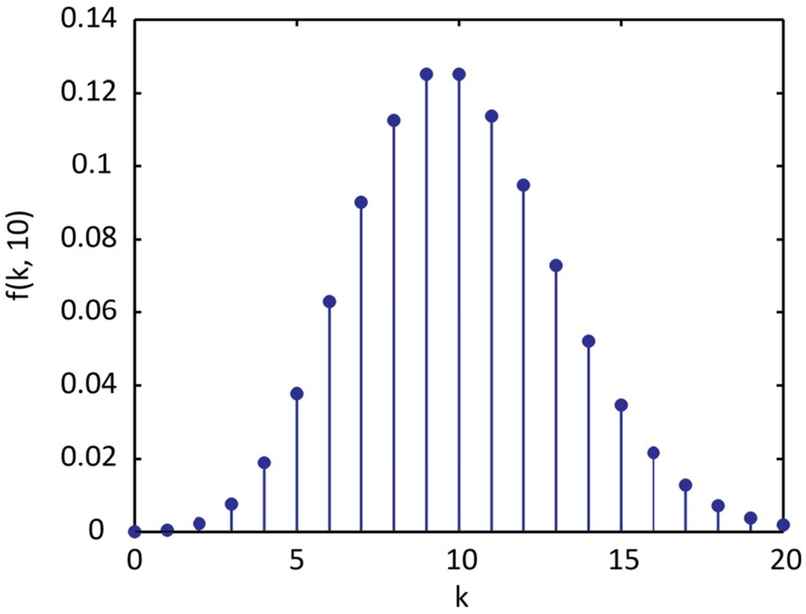

Fig. 2: The Poisson distribution for λ = 10.

Inspection of figure shows that the Poisson distribution has very similar properties as the normal distribution, except that the Poisson distribution does not have even symmetry especially when λ is very small. Besides, the Poisson distribution has a very unique property:

$$\sigma = \sqrt{\mu} \tag{4}$$

That is, ***the standard deviation of the Poisson distribution is equal to the square root of the mean value***. Inserting Eq. (4) into Eq. (1) we get

$$Error = \frac{\sigma}{\mu} = \frac{\sqrt{\mu}}{\mu} = \mu^{-\frac{1}{2}} \tag{4}$$

Thus, if we use the Poisson distribution to describe the rays in a ray-tracing simulation, the error can be represented by the reciprocal of the square root of the mean.

***Example***: In this example, we choose a certain number of rays that we would like to be received by a receiver of interest in a ray-tracing simulation. The receiver could be a sphere surrounding an LED, or it could be a particular spot on the road being illuminated by an automotive headlight.

If $\mu = 100$, then $\sigma = \sqrt{\mu} = 10$; thus $Error = \frac{\sigma}{\mu} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{100}} = 10\%$;

If $\mu = 1000$, then $\sigma = \sqrt{\mu} \approx 31.6$; thus $Error = \frac{\sigma}{\mu} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{1000}} \approx 3.2\%$;

If $\mu = 10\ 000$, then $\sigma = \sqrt{\mu} = 100$; thus $Error = \frac{\sigma}{\mu} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{10000}} = 1\%$.

This example shows that if we are satisfied with an error of 10%, we would need to perform ray tracing simulations until the receiver is hit by 100 rays. If we want our simulation to have an error of 1%, we would need to perform ray tracing simulations until the receiver is hit by 10 000 rays.

Now we can answer the first question of this Teaching Module: In order to have a less than 1% error in the ray-tracing simulation, we need at least 10 000 rays to be received by the receiver. If we use a far-field receiver to simulate a LED whose expected light-extraction efficiency is 25%, then in order to have 10 000 rays to be received by the receiver, we need at least a total of 40 000 rays to ensure a less than 1% error.

May 14, 2011
Ming Ma and E. Fred Schubert

References:
(1) Wikipedia, 2011