Thermal model of packaged LEDs and *RC* circuit analogue

The junction temperature is one of the main factors affecting the performance of LEDs. It's a critical parameter that affects internal quantum efficiency, external efficiency, output power, and reliability. An analogy between thermal dissipation and electrical charge dissipation can be made based on the following circuit analyses. We assume that the electrical input power is switched off at $t = 0$.

Basic equations for 1-stage electrical RC circuit

A 1-stage electrical *RC* circuit is shown in Fig. 1:

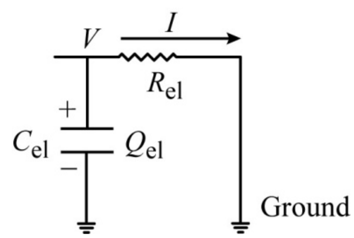

Fig. 1: 1-stage electrical *RC* circuit.

It's well known that

$$C_{\text{el}} = \frac{Q_{\text{el}}}{V} \quad \Rightarrow \quad Q_{\text{el}} = C_{\text{el}}\, V\,. \tag{1}$$

From Ohm's law and Kirchhoff's voltage law,

$$V = I\, R_{\text{el}}\,. \tag{2}$$

It's also known that

$$Q_{\text{el}} = Q_{\text{initial charge}} - \int_0^t I(t')\,\mathrm{d}t' \quad \Rightarrow \quad I(t) = -\frac{\mathrm{d}Q_{\text{el}}(t)}{\mathrm{d}t}\,. \tag{3}$$

Inserting Eq. (1) into Eq. (3) and then substituting I of Eq. (2) into Eq. (3) yields (note that C_{el} is time independent):

$$I(t) = -\frac{\mathrm{d}Q_{\text{el}}(t)}{\mathrm{d}t} = -C_{\text{el}}\frac{\mathrm{d}V(t)}{\mathrm{d}t} \quad \Rightarrow \quad V(t) = -R_{\text{el}}\, C_{\text{el}}\frac{\mathrm{d}V(t)}{\mathrm{d}t}\,, \tag{4}$$

So

$$V(t) + R_{\text{el}}\, C_{\text{el}}\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 0\,. \tag{5}$$

This differential equation is easy to solve. We define the time constant $\tau_{\text{el}} = R_{\text{el}}C_{\text{el}}$. So the solution is

$$V = V(0)\exp\left(-\frac{t}{\tau_{\text{el}}}\right), \tag{6}$$

where $V(0)$ is the voltage at $t = 0$.

In the steady state, we need to consider only the resistor:

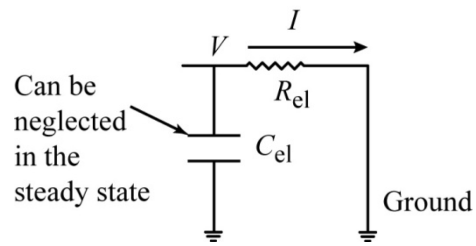

Fig. 2: Electrical *RC* circuit in the steady state.

Ohm's law in the steady state is:

$$V = I\,R_{el}\,. \tag{7}$$

Basic equations for the heat flow in an LED

To analyze the heat flow in an LED, we can begin with a simple case shown in Fig. 3, which shows an LED chip, an interface material, and the ambient (e.g. a large heat sink). The heat flow path always goes from the chip, crosses the thermal interface material and then goes to the ambient.

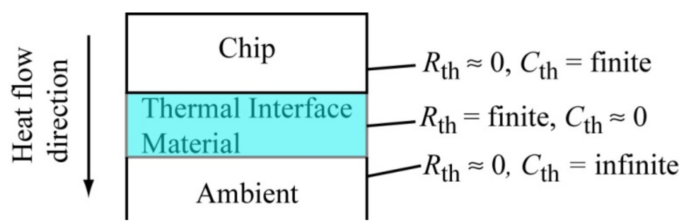

Fig. 3: A simple thermal structure.

The LED chip has a uniform temperature with a finite thermal capacitance. So in this context, the thermal resistance for the chip can be considered to be zero. The thermal interface material has a very small volume. From the relation $C_{th} = c\,\rho\,V$ (c is the specific heat, ρ is the mass density and V is the volume), we know that the interface material has $C_{th} \approx 0$. The ambient has a constant temperature, so it can be assumed to have zero thermal resistance and infinite thermal capacitance.

Based on the above approximations, we can draw a simplified diagram for this problem:

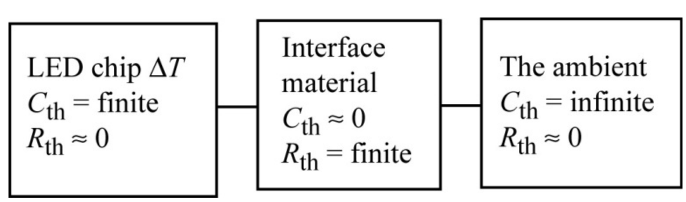

Fig. 4: Thermal circuit in the heat flow.

In the transient state, after the electrical power is switched off, the thermal energy flows through the thermal interface material to the ambient. The boundary condition of this problem is that the ambient is kept at a constant temperature. For $t \rightarrow \infty$, the junction temperature reaches the ambient temperature, that is, $\Delta T\,(t \rightarrow \infty) = 0$, where ΔT is the difference between chip temperature and the ambient temperature. Thus,

$$C_{th} = \frac{Q_{th}}{\Delta T} \quad \Rightarrow \quad Q_{th} = C_{th}\ \Delta T\,, \tag{8}$$

where Q_{th} is the thermal energy stored in the chip.

The lumped thermal resistance for the material is defined as

$$R_{th} = \frac{\Delta T}{P_{th}}\,, \tag{9}$$

where P_{th} is the thermal power transferred through the interfacial 1-D material and R_{th} is the thermal resistance of the interface material.

It's also known that,

$$Q_{th} = Q_{\text{initial heat}} - \int_0^t P_{th}(t')\,\mathrm{d}t' \quad \Rightarrow \quad P_{th}(t) = -\frac{\mathrm{d}Q_{th}(t)}{\mathrm{d}t}\,. \tag{10}$$

Similarly, inserting Eq. (8) into Eq. (10) and then substituting P_{th} of Eq. (9) into Eq. (10) yields (again, note that C_{th} is time independent):

$$P_{th}(t) = -\frac{\mathrm{d}Q_{th}(t)}{\mathrm{d}t} = -C_{th}\frac{\mathrm{d}\Delta T(t)}{\mathrm{d}t} \quad \Rightarrow \quad \Delta T(t) = -R_{th}\ C_{th}\frac{\mathrm{d}\Delta T(t)}{\mathrm{d}t}\,. \tag{11}$$

So

$$\Delta T(t) + R_{th}\ C_{th}\frac{\mathrm{d}\Delta T(t)}{\mathrm{d}t} = 0\,. \tag{12}$$

This differential equation has the solution

$$\Delta T = \Delta T(0)\exp\left(-\frac{t}{\tau_{th}}\right), \tag{13}$$

where $\tau_{th} = R_{th}C_{th}$ and $\Delta T\,(0)$ is the initial temperature of the chip.

In the steady state, the temperature of the chip does not increase but is constant (thus the thermal capacity is not a relevant factor). So in the steady state, the thermal power flowing through the interface material is constant with time. The thermal resistance of the interface is:

$$R_{th} = \frac{\Delta T}{P_{th}} \quad \Rightarrow \quad \Delta T = P_{th}\ R_{th}\,. \tag{14}$$

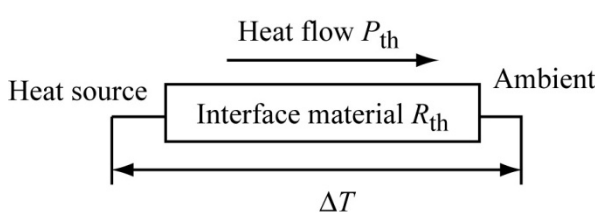

Fig. 5: Thermal circuit in the steady state.

Comparing Eq. (5) with Eq. (12), we find that the two equations are mathematically identical. Thus, we reach the conclusion that it is possible to associate properties of the electric circuit with properties of the thermal "circuit". So the electrical circuit representation of a thermal system is justified by the analogy between electrical and thermal quantities. Table 1 gives this analogue due to the comparison discussed above:

Electrical Quantities	Thermal Quantities
Electrical charge, Q_{el}	Thermal energy, Q_{th}
Voltage, V	Temperature, ΔT
Current, $I(t) = -\mathrm{d}Q_{el}(t) / \mathrm{d}t$	Thermal Power, $P_{th}(t) = -\mathrm{d}Q_{th}(t) / \mathrm{d}t$
Electrical capacitance, $C_{el} = \varepsilon S / d$	Thermal capacity, $C_{th} = c \rho V$
Electrical resistance, $R_{el} = l / (\sigma A)$	Thermal resistance, $R_{th} = l / (k_{th} A)$
Relation 1: $R_{el} = V / I$	Relation 1: $R_{th} = \Delta T / P$
Relation 2: $C_{el} = Q_{el} / V$	Relation 2: $C_{th} = Q_{th} / \Delta T$

Table 1: List of analogous quantities between thermal and electrical systems.

where $\varepsilon = \varepsilon_r \varepsilon_0$ is the dielectric permittivity, S is the area of the capacitor plate, d is the distance between the two plates, ρ is the mass density of the material, l is the length of the material, A is the cross area, σ is the electric conductivity of the material, and k_{th} is thermal conductivity of the interface material.

The equivalent thermal circuit can then be shown as follows:

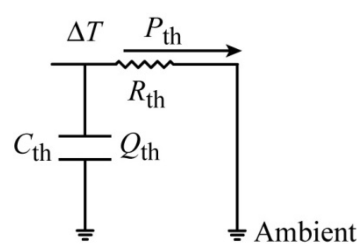

Fig. 6: Equivalent thermal circuit.

Basic equation for 2-stage thermal RC circuit

For a 2-stage *RC* circuit, we need to connect the two stages properly. From the conservation of energy, the correct model is shown as the following figure:

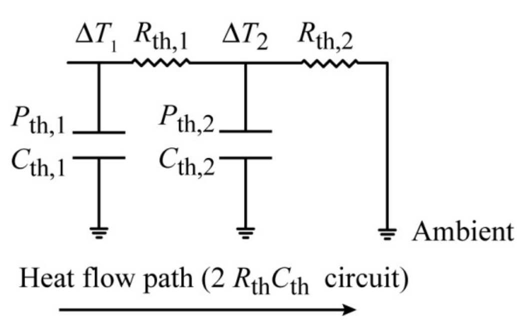

Fig. 7: 2-stage thermal *RC* circuit.

The corresponding physical structure is shown in Fig. 8:

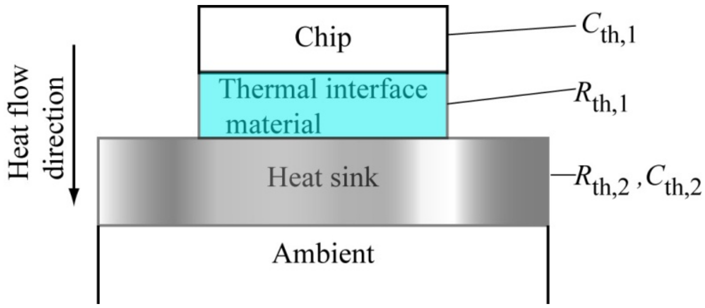

Fig. 8: 2-stage thermal structure.

As used in our 1-stage-circuit analysis, from the relation $C_{th} = c\,\rho\,V$, we conclude that since the heat sink is much larger than the chip, we have $C_{th,1} << C_{th,2}$. We also know that the heat transfer between the heat sink and the air ambient is by thermal convection, which is a low efficiency thermal transfer compared with thermal conduction. Then $R_{th,2}$ is the thermal "bottleneck" point of the whole thermal system. A well-designed LED should have $R_{th,1} \approx R_{th,2}$, so that there is no thermal "bottleneck".

Six equations describe the above thermal circuit (see Fig. 7):

$$\Delta T_2(t) = R_{th,2}\,[P_{th,1}(t) + P_{th,2}(t)] \tag{15}$$

$$\Delta T_1(t) - \Delta T_2(t) = R_{th,1}\;P_{th,1}(t) \tag{16}$$

$$P_{th,1}(t) = -\frac{dQ_{th,1}(t)}{dt} \tag{17}$$

$$P_{th,2}(t) = -\frac{dQ_{th,2}(t)}{dt} \tag{18}$$

$$C_{th,1}\;\Delta_1 T(t) = Q_{th,1}(t) \tag{19}$$

$$C_{th,2}\;\Delta_2 T(t) = Q_{th,2}(t)\,. \tag{20}$$

The initial conditions (at $t = 0$) are: $\Delta T_1\ (t = 0) = \Delta T_{10}$ and $\Delta T_2\ (t = 0) = \Delta T_{20}$.

We assume that the system is in steady state for $t < 0$, so that

$$\frac{\Delta T_{20}}{\Delta T_{10} - \Delta T_{20}} = \frac{R_{th,2}}{R_{th,1}}\,. \tag{21}$$

After mathematical manipulations of Eqs. (15) – (20), we get the following second-order differential equation for $\Delta T_1(t)$:

$$\frac{d^2\Delta T_1(t)}{dt^2} + \left(\frac{1}{R_{th,2}C_{th,2}} + \frac{1}{R_{th,1}C_{th,2}} + \frac{1}{R_{th,1}C_{th,1}}\right)\frac{d\Delta T_1(t)}{dt} + \frac{1}{R_{th,1}C_{th,1}R_{th,2}C_{th,2}}\,\Delta T_1(t) = 0\,. \tag{22}$$

It's a typical homogenous differential equation. The characteristic equation is:

$$\left(\frac{1}{\tau_{th}}\right)^2 - \left(\frac{1}{R_{th,2}C_{th,2}} + \frac{1}{R_{th,1}C_{th,2}} + \frac{1}{R_{th,1}C_{th,1}}\right)\frac{1}{\tau_{th}} + \frac{1}{R_{th,1}C_{th,1}R_{th,2}C_{th,2}} = 0\,. \tag{23}$$

The two solutions of this quadratic equation $\tau_{th,1}$ and $\tau_{th,2}$ have following relations:

$$\frac{1}{\tau_{\mathrm{th},1}}+\frac{1}{\tau_{\mathrm{th},2}}=\frac{1}{R_{\mathrm{th},2}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},1}}\,; \tag{24a}$$

$$\frac{1}{\tau_{\mathrm{th},1}}\,\frac{1}{\tau_{\mathrm{th},2}}=\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},1}R_{\mathrm{th},2}C_{\mathrm{th},2}}\,. \tag{24b}$$

As justified earlier, we use the approximations:

$$C_{\mathrm{th},1} << C_{\mathrm{th},2} \tag{25}$$

and

$$R_{\mathrm{th},1} \approx R_{\mathrm{th},2}\,. \tag{26}$$

Using these approximations, we find:

$$\begin{aligned}\left(\frac{1}{\tau_{\mathrm{th},1}}+\frac{1}{\tau_{\mathrm{th},2}}\right)^2 &= \left(\frac{1}{R_{\mathrm{th},2}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},1}}\right)^2 \\ &>> \frac{4}{R_{\mathrm{th},1}C_{\mathrm{th},1}R_{\mathrm{th},2}C_{\mathrm{th},2}} = 4\,\frac{1}{\tau_{\mathrm{th},1}}\,\frac{1}{\tau_{\mathrm{th},2}}\end{aligned}\,. \tag{27}$$

That means one solution is much larger than the other one. If we assume

$$\frac{1}{\tau_{\mathrm{th},1}} >> \frac{1}{\tau_{\mathrm{th},2}} \tag{28}$$

and use Eqs. (24a) and (24b), we find the following approximate solutions:

$$\frac{1}{\tau_{\mathrm{th},1}}=\frac{1}{R_{\mathrm{th},2}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},2}}+\frac{1}{R_{\mathrm{th},1}C_{\mathrm{th},1}} \tag{29a}$$

and

$$\frac{1}{\tau_{\mathrm{th},2}}=\frac{1}{R_{\mathrm{th},2}C_{\mathrm{th},2}+R_{\mathrm{th},1}C_{\mathrm{th},2}+R_{\mathrm{th},1}C_{\mathrm{th},1}}\,. \tag{29b}$$

Then, the solution of the differential equation is (see Eq. 22):

$$\Delta T_1(t) = A_1\,\exp\left(-\frac{t}{\tau_{\mathrm{th},1}}\right) + A_2\,\exp\left(-\frac{t}{\tau_{\mathrm{th},2}}\right). \tag{30a}$$

From Eq. (16), (17) and (20), we obtain:

$$\begin{aligned}\Delta T_2(t) &= \Delta T_1(t) + R_{\mathrm{th},1}C_{\mathrm{th},1}\,\frac{\mathrm{d}\Delta T_1(t)}{\mathrm{d}t} \\ &= \left(\frac{\tau_{\mathrm{th},1}-R_{\mathrm{th},1}C_{\mathrm{th},1}}{\tau_{\mathrm{th},1}}\right)A_1\,\exp\left(-\frac{t}{\tau_{\mathrm{th},1}}\right)+\left(\frac{\tau_{\mathrm{th},2}-R_{\mathrm{th},1}C_{\mathrm{th},1}}{\tau_{\mathrm{th},2}}\right)A_2\,\exp\left(-\frac{t}{\tau_{\mathrm{th},2}}\right)\end{aligned}. \tag{30b}$$

From the initial conditions, we get,

$$A_1 + A_2 = \Delta T_{10} \tag{31a}$$

and

$$\left(\frac{\tau_{\mathrm{th},1}-R_{\mathrm{th},1}C_{\mathrm{th},1}}{\tau_{\mathrm{th},1}}\right)A_1+\left(\frac{\tau_{\mathrm{th},2}-R_{\mathrm{th},1}C_{\mathrm{th},1}}{\tau_{\mathrm{th},2}}\right)A_2 = \Delta T_{20}\,. \tag{31b}$$

Then,

$$\Delta T_1(t) = \frac{\tau_{th,1}\, R_{th,2}C_{th,2}}{(\tau_{th,2} - R_{th,2}C_{th,2})(\tau_{th,2} - \tau_{th,1})}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,1}}\right) + \frac{\tau_{th,2}\,(\tau_{th,2} - \tau_{th,1} - R_{th,2}C_{th,2})}{(\tau_{th,2} - R_{th,2}C_{th,2})(\tau_{th,2} - \tau_{th,1})}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,2}}\right) \tag{32a}$$

$$\Delta T_2(t) = \frac{R_{th,2}C_{th,2}\,(\tau_{th,1} - R_{th,1}C_{th,1})}{(\tau_{th,2} - R_{th,2}C_{th,2})(\tau_{th,2} - \tau_{th,1})}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,1}}\right) + \frac{(\tau_{th,2} - R_{th,1}C_{th,1})\,(\tau_{th,2} - \tau_{th,1} - R_{th,2}C_{th,2})}{(\tau_{th,2} - R_{th,2}C_{th,2})(\tau_{th,2} - \tau_{th,1})}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,2}}\right). \tag{32b}$$

Known in thermal management and justified earlier, $R_{th,1}$ and $R_{th,2}$ are similar,

$$R_{th,1} \approx R_{th,2}$$

then, Eq. (29a) and Eq. (29b) can be written as:

$$\frac{1}{\tau_{th,1}} = \frac{1}{R_{th,2}C_{th,2}} + \frac{1}{R_{th,1}C_{th,2}} + \frac{1}{R_{th,1}C_{th,1}} \approx \frac{1}{R_{th,1}C_{th,1}} \tag{33a}$$

$$\frac{1}{\tau_{th,2}} = \frac{1}{R_{th,2}C_{th,2} + R_{th,1}C_{th,2} + R_{th,1}C_{th,1}} \approx \frac{1}{R_{th,2}C_{th,2}} \tag{33b}$$

In the same way, the solution can be written as

$$\begin{aligned}
\Delta T_1(t) &= A_1\,\exp\left(-\frac{t}{\tau_{th,1}}\right) + A_2\,\exp\left(-\frac{t}{\tau_{th,2}}\right)\\
&= \left(1 - \frac{\tau_{th,2}}{\tau_{th,2} - \tau_{th,1}}\frac{R_{th,2}}{R_{th,1} + R_{th,2}}\right)\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,1}}\right) + \frac{\tau_{th,2}}{\tau_{th,2} - \tau_{th,1}}\frac{R_{th,2}}{R_{th,1} + R_{th,2}}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,2}}\right)\\
&= \frac{R_{th,1}}{R_{th,1} + R_{th,2}}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,1}}\right) + \frac{R_{th,2}}{R_{th,1} + R_{th,2}}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,2}}\right)
\end{aligned} \tag{34a}$$

and

$$\Delta T_2(t) = \Delta T_{20}\,\exp\left(-\frac{t}{\tau_{th,2}}\right) = \frac{R_{th,2}}{R_{th,1} + R_{th,2}}\,\Delta T_{10}\,\exp\left(-\frac{t}{\tau_{th,2}}\right). \tag{34b}$$

From the above results, we can draw a decay profile of $\Delta T_1(t)$ shown as Fig. 9:

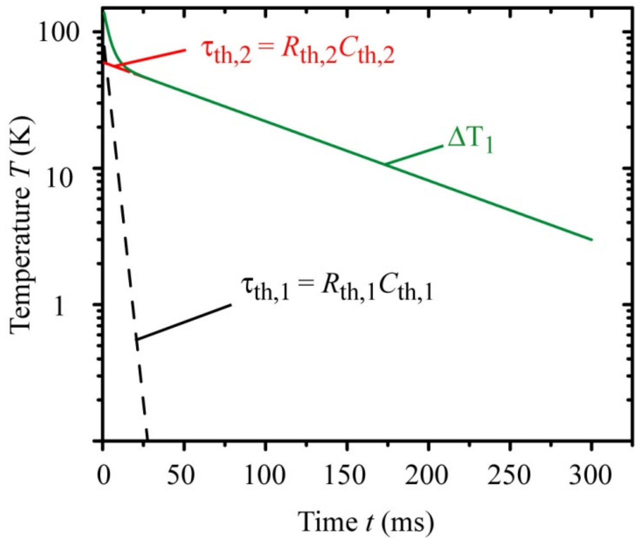

Fig. 9: A junction temperature decay example (2-stage thermal *RC* circuit).

Multi-stage thermal RC circuit

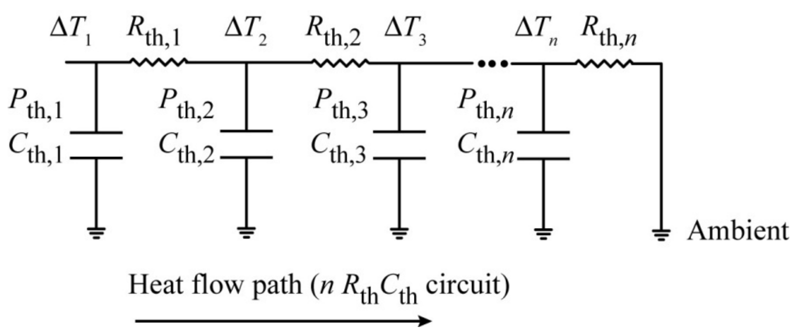

Fig: 10. Multi-stage thermal *RC* circuit.

Following the analyses in the previous sections, we can list the equations for an n-stage thermal *RC* circuit. $3n$ equations describe this network:

$$\Delta T_n(t) = R_{th,n}\,[P_{th,1}(t) + P_{th,2}(t) + \ldots + P_{th,n}(t)]$$

$$\Delta T_{n-1}(t) - \Delta T_n(t) = R_{th,n\text{-}1}\,[P_{th,1}(t) + \ldots + P_{th,n\text{-}1}(t)]$$

$$\ldots$$

$$P_{th,1}(t) = -\frac{\mathrm{d}Q_{th,1}(t)}{\mathrm{d}t}$$

$$P_{th,2}(t) = -\frac{\mathrm{d}Q_{th,2}(t)}{\mathrm{d}t}$$

$$\ldots$$

$$C_{th,1}\ \Delta T_1(t) = Q_{th,1}(t)$$

$$C_{th,2}\ \Delta T_2(t) = Q_{th,2}(t)$$

$$\ldots$$

Apparently, the differential equation for ΔT_1 (t) is an n^{th} order homogenous differential equation. Following on precious analyses and the results obtained, the solution is just the linear combination of n exponential decay terms. So,

$$\Delta T_1(t) = \sum_{i=1}^{n} A_i \exp\left(-\frac{t}{\tau_{th,i}}\right) \tag{35}$$

Due to the same reason analyzed in the last section, in the thermal path, thermal capacitance varies a lot, $C_{th,1} << C_{th,2} \ldots << C_{th,n}$. But the thermal resistance may not change much. Thus, the different time constants vary a lot. If we assume that $\tau_{th,1} << \tau_{th,2} \ldots << \tau_{th,n}$, then, we can get the approximate time constants: $\tau_{th,1} \approx R_{th,1}C_{th,1}$, $\tau_{th,2} \approx R_{th,2}C_{th,2}$, …, $\tau_{th,n} \approx R_{th,n}C_{th,n}$. Therefore, we draw an estimating decay curve example shown as Fig. 11:

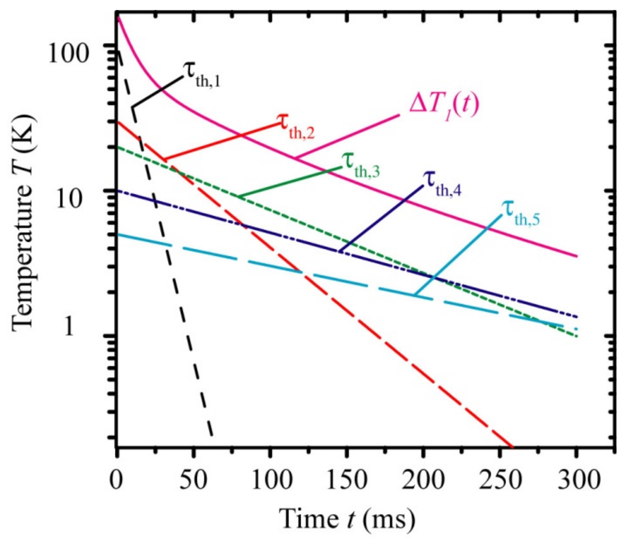

Fig. 11: A junction temperature decay example (multi-stage thermal *RC* circuit).