Chapter C-V-1

Image Formation and Waveform Design for Distributed Apertures in Multipath via Gram-Schmidt Orthogonalization

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1. Introduction

In this work, we consider a radar system where the antenna elements are arbitrarily distributed in space with several hundred wavelengths away. We call such a radar system *distributed aperture* radar system. We assume that the system is operating in multi-pathing environment, such as an urban area. We present methods for designing clutter rejecting waveforms and for reconstruction of reflectivity function. This work generalizes the monostatic radar waveform design method for range-Doppler imaging developed in [1, 2] to distributed aperture radar systems. Our approach utilizes Gram-Schmidt orthogonalization (GSO) procedure. The designed waveforms also lead to a filtered-backprojection type reconstruction of the reflectivity function which can be efficiently implemented in a parallel fashion.

2. Scattered Field in Multi-path

We model the antenna as a time-varying current density $j_{tr}(t, \mathbf{x})$ over an aperture. This is appropriate for a wide variety of antennas [3–5].

We assume that the electromagnetic waves emitted from the antenna travels in a known background and ignore polarization effects. Under these assumptions, the field emanating from the antenna satisfies the scalar wave equation

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2)u^{in}(t, \mathbf{x}) = -j_{tr}(t, \mathbf{x}), \qquad (1)$$

where $c_0(\mathbf{x})$ is the speed of light in the background.

Let $g_0(t, \mathbf{x}; \sigma, \mathbf{y})$ be the Green's function of (2)

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2) g_0(t, \mathbf{x}; \sigma; \mathbf{y}) = \delta(t - \sigma)\delta(x - y). \tag{2}$$

Then the incident field is given by

$$u^{in}(t, \mathbf{x}) = -\int g_0(t, \mathbf{x}; \sigma, \mathbf{y}) j_{tr}(\sigma, \mathbf{y}) d\sigma\, d\mathbf{y}. \tag{3}$$

The model we use for wave propagation, including the source, is

$$(\nabla^2 - c^{-2}(\mathbf{x})\partial_t^2) u(t, \mathbf{x}) = -j_{tr}(t, \mathbf{x}) \tag{4}$$

where $c(\mathbf{x})$ is the speed of light in the distorted medium. We write $u = u^{in} + u^{sc}$ in (4) and use (2) to obtain

$$(\nabla^2 - c_0^{-2}(\mathbf{x})\partial_t^2) u^{sc}(t, \mathbf{x}) = T(\mathbf{x})\partial_t^2 u(t, \mathbf{x}) \tag{5}$$

where

$$T(\mathbf{x}) = \frac{1}{c^2(\mathbf{x})} - \frac{1}{c_0^2(\mathbf{x})}. \tag{6}$$

The *reflectivity function* T contains all the information about how the scattering medium differs from given background. It is T, or at least its discontinuities and other singularities, that we want to recover.

We make the *single-scattering* approximation to the scattered field u^{sc} by replacing the full field u on the right side of (2) by the incident field u^{in}. Solution of the resulting differential equation leads to

$$u^{sc}(t, \mathbf{x}) \approx -\int g_0(t, \mathbf{x}; \sigma, \mathbf{z}) T(\mathbf{z}) \partial_\sigma^2 u^{in}(\sigma, \mathbf{z}) d\sigma d\mathbf{z}. \tag{7}$$

For the incident field (3), (7) becomes

$$u^{sc}(t, \mathbf{x}) \approx \int g_0(t, \mathbf{x}; \sigma, \mathbf{z}) T(\mathbf{z}) \partial_\sigma^2 \left(\int g_0(\sigma, \mathbf{z}; \tau, \mathbf{y}) j_{tr}(\tau, \mathbf{y}) d\tau d\mathbf{y} \right) d\sigma d\mathbf{z} \tag{8}$$

3. Measurement Model for Distributed Apertures in Multipath

We consider multiple source and receiver antennas. Without loss of generality, we assume that the transmitter is an isotropic, point antenna and focus on the model $j_{tr}(t, \mathbf{x}) = p(t) j(\mathbf{x}) \delta(\mathbf{x} - \mathbf{z}_j)$ for some transmitter location $\mathbf{z}_j$, where p is the transmitted waveform, which we assume to be real value, referred to as the pulse, and $j(\mathbf{x})$ is the waveguide.

For transmitter location $\mathbf{z}_j$, $j \geq 0$, receiver location $\mathbf{x}_i$, $i \geq 0$, and receiver antenna beam pattern $j_{rc}(t, \mathbf{x}_i)$, we model the measurements as

$$e(t, \mathbf{x}_i, \mathbf{z}_j) = [u^{sc}(\,.\,, \mathbf{x}_i, \mathbf{z}_j) *_t j_{rc}(\,.\,, \mathbf{x}_i)](t) \tag{9}$$

where $*_t$ denotes convolution in t, and

$$u^{sc}(t, \mathbf{x}_i, \mathbf{z}_j) = \int T(\mathbf{z}) g_0(t, \mathbf{x}_i; \sigma, \mathbf{z}) \partial_\sigma^2 g_0(\sigma, \mathbf{z}; \tau, \mathbf{z}_j) j(\mathbf{z}_j) p(\tau)\, d\sigma\, d\mathbf{z}\, d\tau. \tag{10}$$

Let $\lambda_{ij} = (\mathbf{x}_i, \mathbf{z}_j)$ and

$$\kappa(\mathbf{z}; t; \lambda_{ij}; \tau) = \int g_0(\tau', \mathbf{x}_i; \sigma, \mathbf{z}) \partial_\sigma^2 g_0(\sigma, \mathbf{z}; \tau, \mathbf{z}_j) j(\mathbf{z}_j) j_{rc}(t - \tau', \mathbf{x}_i)\, d\sigma\, d\tau'. \tag{11}$$

Then, (9) defines a bilinear integral operator $\mathcal{F}^{(\lambda_{ij})}$ acting on T and p with kernel κ as follows:

$$e(t, \lambda_{ij}) = [\mathcal{F}^{(\lambda_{ij})}(T, p)](t, \tau) = \int T(\mathbf{z}) \kappa(\mathbf{z}; t; \lambda_{ij}; \tau) p(\tau) d\mathbf{z} d\tau. \tag{12}$$

We study $\mathcal{F}^{(\lambda_{ij})}$ in two ways by defining the linear operators $\mathcal{H}^{(\lambda_{ij})}$ and $\mathcal{G}^{(\lambda_{ij})}$ that act on the scene T and the transmitted waveform p by

$$\begin{aligned} [\mathcal{H}^{(\lambda_{ij})}(T)](t, \tau) &= \int T(\mathbf{z}) \kappa(\mathbf{z}; t; \lambda_{ij}; \tau) d\mathbf{z} \\ &= \langle T(\,.\,), \kappa(\,.\,; t; \lambda_{ij}; \tau) \rangle, \end{aligned} \tag{13}$$

and

$$\begin{aligned} [\mathcal{G}^{(\lambda_{ij})} p](t, \mathbf{z}) &= \int \kappa(\mathbf{z}; t; \lambda_{ij}; \tau) p(\tau) d\tau \\ &= \langle \kappa(\mathbf{z}; t; \lambda_{ij}; \,.\,), p(\,.\,) \rangle, \end{aligned} \tag{14}$$

respectively. Thus, we rewrite (12) as

$$\begin{aligned} e(t, \lambda_{ij}) &= \int [\mathcal{H}^{(\lambda_{ij})}(T)](t, \tau) p(\tau) d\tau \\ &= \left\langle [\mathcal{H}^{(\lambda_{ij})}(T)](t, \,.\,), p(\,.\,) \right\rangle, \end{aligned} \tag{15}$$

and

$$\begin{aligned} e(t, \lambda_{ij}) &= \int T(\mathbf{z}) [\mathcal{G}^{(\lambda_{ij})} p](t, \mathbf{z}) d\mathbf{z} \\ &= \left\langle [\mathcal{G}^{(\lambda_{ij})} p](t, \,.\,), T(\,.\,) \right\rangle. \end{aligned} \tag{16}$$

We used $\langle \,.\,, \,.\, \rangle$ to denote the inner product of functions with respect to the t or $\mathbf{z}$ variable. The distinction of the usage would be clear from the context.

4. Design of a Waveform Preconditioner for Clutter Rejection

When the target is embedded in clutter, we model the environment to be imaged by $T+C$ and the corresponding measurements e_{c} by:

$$e_{\mathrm{c}}(t,\lambda_{ij}) = \left\langle [\mathcal{H}^{(\lambda_{ij})}(T+C)](t,.), p(.) \right\rangle \tag{17}$$

Without loss of generality, we assume that T and C are zero-mean random processes with autocorrelation functions:

$$R_T(\boldsymbol{z},\boldsymbol{z}') = E[T(\boldsymbol{z})T^*(\boldsymbol{z}')] \tag{18}$$

$$R_C(\boldsymbol{z},\boldsymbol{z}') = E[C(\boldsymbol{z})C^*(\boldsymbol{z}')]. \tag{19}$$

Furthermore, we assume that T and C are statistically uncorrelated.

A *preconditioner* is an operator applied to waveforms prior to transmission. Let Let W be a linear integral operator acting on the waveform space and let Wp be the filtered waveform transmitted between the ith receiver and jth transmitter. We define the mean square error between the measurements due to transmitted waveforms p and Wp as follows:

$$\Delta_W = \frac{E\left[\displaystyle\int \left| \left\langle [\mathcal{H}^{(\lambda_{ij})}(T+C)](t,.), Wp(.) \right\rangle - \left\langle [\mathcal{H}^{(\lambda_{ij})}(T)](t,.), p(.) \right\rangle \right|^2 dt \right]}{\|p\|^2}. \tag{20}$$

We define the *waveform preconditioner* $W^{(\lambda_{ij})}$ for the ith and jth receiver-transmitter pair as the W that minimizes the mean square (20) for any waveform p, i.e.,

$$W^{(\lambda_{ij})} = \min_W \Delta_W, \qquad \forall p. \tag{21}$$

Let b_n be an orthonormal basis for the waveform space. We define P to be the linear operator whose matrix elements are given by

$$P_{mn} = p_n\overline{p_m}$$

where

$$p_n = \langle p, b_n\rangle/\|p\|.$$

Thus, $\frac{p(t)\overline{p(t')}}{\|p\|^2} = \sum_{m,n} P_{mn} b_n(t) b_m(t')$.

Assuming T and C are statistically uncorrelated, (21) can be equivalently expressed by

$$W^{(\lambda_{ij})} = \min_W \operatorname{tr}\left[\left((W-I)^*\mathcal{K}_T^{(\lambda_{ij})}(W-I) + W^*\mathcal{K}_C^{(\lambda_{ij})}W\right)P\right], \quad \forall P, \tag{22}$$

where tr is the trace operator, and $\mathcal{K}_T^{(\lambda_{ij})}$ and $\mathcal{K}_C^{(\lambda_{ij})}$ are positive definite operators given by

$$\mathcal{K}_T^{(\lambda_{ij})} = E\left[\mathcal{H}^{(\lambda_{ij})}(T)^*\mathcal{H}^{(\lambda_{ij})}(T)\right], \tag{23}$$

$$\mathcal{K}_C^{(\lambda_{ij})} = E\left[\mathcal{H}^{(\lambda_{ij})}(C)^*\mathcal{H}^{(\lambda_{ij})}(C)\right]. \tag{24}$$

Note that the cross correlation terms vanish,

$$E[\mathcal{H}^{(\lambda_{ij})}(C)^*\mathcal{H}^{(\lambda_{ij})}(T)] = E[\mathcal{H}^{(\lambda_{ij})}(T)^*\mathcal{H}^{(\lambda_{ij})}(C)] = 0, \tag{25}$$

since $E[T(z)C(z')] = 0$.

We now express $\mathcal{K}_T^{(\lambda_{ij})}$ and $\mathcal{K}_C^{(\lambda_{ij})}$ in terms of R_T and R_C as follows:

$$\begin{aligned}\mathcal{K}_T^{(\lambda_{ij})}(t,\tau) &= \langle\kappa|R_T|\kappa^*\rangle(t,\tau)\\ &:= \int \kappa(\boldsymbol{z};t;\lambda_{ij},\tau)R_T(\boldsymbol{z},\boldsymbol{z}')\kappa^*(\boldsymbol{z}';t;\lambda_{ij};\tau)d\boldsymbol{z}\,d\boldsymbol{z}',\end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathcal{K}_C^{(\lambda_{ij})}(t,\tau) &= \langle\kappa|R_C|\kappa^*\rangle(t,\tau)\\ &:= \int \kappa(\boldsymbol{z};t;\lambda_{ij},\tau)R_C(\boldsymbol{z},\boldsymbol{z}')\kappa^*(\boldsymbol{z}';t;\lambda_{ij};\tau)d\boldsymbol{z}\,d\boldsymbol{z}'.\end{aligned} \tag{27}$$

We determine $W^{(\lambda_{ij})}$ by equating the variational derivative of (22) with respect to W to 0:

$$\mathrm{tr}\left[\left(\left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]W^{(\lambda_{ij})} - \mathcal{K}_T^{(\lambda_{ij})}\right)P\right] = 0. \tag{28}$$

In order for (28) to hold for any P, $W^{(\lambda_{ij})}$ must be

$$\begin{aligned}W^{(\lambda_{ij})} &= \left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]^{-1}\mathcal{K}_T^{(\lambda_{ij})}\\ &= I - \left[\mathcal{K}_T^{(\lambda_{ij})} + \mathcal{K}_C^{(\lambda_{ij})}\right]^{-1}\mathcal{K}_C^{(\lambda_{ij})},\end{aligned} \tag{29}$$

where $^{-1}$ denotes pseudo- or approximate-inverse.

5. Waveforms Design and Reconstruction of Reflectivity Function

For ease of exposition we first present our criterion for waveform design. Then, we present our method for a single transmitter-receiver pair case and finally generalize the method to distributed apertures.

5.1. *Waveform Design Criterion*

We assume that multiple waveforms, which are all real valued, are transmitted sequentially for each transmitter-receiver pair. Let $p_m^{(\lambda_{ij})}$ be the mth waveform transmitted for the ith and jth receiver-transmitter pair. We want to design the waveforms $p_m^{(\lambda_{ij})}$ such that given the preconditioner $W^{(\lambda_{ij})}$, when we matched filter the received echo $e_{\rm c}(t, \lambda_{ij})$ with the transmitted waveform $p_m(t)$, the target $T(\mathbf{z})$ is projected onto an orthonormal basis. In other words, we want to design p_m that satisfies

$$\begin{aligned}\langle e_{\rm c}(\,.\,, \lambda_{ij}), p_m(-\,.\,)\rangle &= \int [T + C](\mathbf{z}) \left[\int [\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} p_m^{(\lambda_{ij})}](t, \mathbf{z})\, p_n^{(\lambda_{ij})}(-t)\, dt \right] d\mathbf{z} \\ &\approx \langle T, \widehat{\Lambda}_{mn}^{(\lambda_{ij})} \rangle\end{aligned} \tag{30}$$

where

$$\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}) = \langle [\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} p_m^{(\lambda_{ij})}](\,.\,, \mathbf{z}), p_n^{(\lambda_{ij})}(-\,.\,)\rangle \tag{31}$$

form an orthonormal basis with respect to the inner product over $\mathbf{z}$, i.e.

$$\langle \widehat{\Lambda}_{mn}^{(\lambda_{ij})}, \widehat{\Lambda}_{kl}^{(\lambda_{i'j'})} \rangle = \delta_{mk}\delta_{nl}\delta_{ii'}\delta_{jj'}. \tag{32}$$

Here δ_{mn} is the Kronecker delta function, which is equal to one when $m = n$ and zero otherwise. We ignored the clutter related term in (30) by assuming optimally preconditioned waveforms suppresses the signal due to clutter in the received echo.

Note that since the waveforms are assumed to be real, performing inner product of a signal $q(t)$ with the time reversed waveform $p(t)$, i.e. $p(-t)$, by Plancherel theorem, corresponds to matching their corresponding Fourier transforms.

Thus, given the projection of $T(\mathbf{z})$ onto the orthonormal set $\{\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})\}_{m,\lambda_{ij}}$ the best approximation to $T(\mathbf{z})$ is

$$\widetilde{T}(\mathbf{z}) \approx \sum_{m,n,\lambda_{ij}} \alpha_{mn}^{(\lambda_{ij})} \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}), \tag{33}$$

where

$$\alpha_{mn}^{(\lambda_{ij})} = \langle T, \widehat{\Lambda}_{mn}^{(\lambda_{ij})} \rangle. \tag{34}$$

We can view (34) as a filtered-backprojection-type reconstruction formula [4,5], where the filtering takes place in two steps. The first step takes place transmission, where we filter waveforms p_m by $W^{(\lambda_{ij})}$. The second step is the matched filtering of the measurements with p_n in receive. Finally, we backproject the matched-filtered measurements via $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$. Note that with this particular choice of transmitted waveforms $p_m^{(\lambda_{ij})}$ the point spread function of the imaging process has the partitions of unity given in terms of the orthonormal basis $\{\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})\}_{m,n,\lambda_{ij}}$. If $\{\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})\}_{m,n,\lambda_{ij}}$ were not orthonormal then (35) should be replaced by a weighted sum, which would translate as more computations that should be performed in the image domain.

5.2. *Waveform Design and Reflectivity Reconstruction for Single Transmitter and Receiver Pair*

Let us consider only the single transmitter and receiver pair λ_{ij}. For the afore mentioned choice of transmitted waveforms $p_m^{(\lambda_{ij})}$, the best approximation to $T(\mathbf{z})$ is

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z}) \approx \sum_{m,n} \alpha_{mn}^{(\lambda_{ij})} \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}). \tag{35}$$

In the presence of clutter C, substituting (31) in (34),

$$\begin{aligned} \alpha_{mn}^{(\lambda_{ij})} &= \int [T+C](\mathbf{z}) \langle [\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} p_m^{(\lambda_{ij})}](\,.\,,\mathbf{z}), p_n^{(\lambda_{ij})}(-.\,)\rangle d\mathbf{z} \\ &= \langle e_{\mathrm{c}m}^{(\lambda_{ij})}(\,.\,,\lambda_{ij}), p_n^{(\lambda_{ij})}(-.\,)\rangle, \end{aligned} \tag{36}$$

where $e_{\mathrm{c}m}^{(\lambda_{ij})}$ denotes the scattered field from $T+C$ due to the transmitted waveform $W^{(\lambda_{ij})} p_m^{(\lambda_{ij})}$. Thus, by (36), (35) becomes

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z}) \approx \sum_{m,n} \langle e_{\mathrm{c}m}^{(\lambda_{ij})}(\,.\,,\lambda_{ij}), p_n^{(\lambda_{ij})}(-.\,)\rangle \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}). \tag{37}$$

5.2.1. *Design of Waveforms*

Let $\kappa_W(\mathbf{z}; t; \lambda_{ij}, \tau)$ denote the kernel of $\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})}$:

$$[\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} p](t,\mathbf{z}) = \int \kappa_W(\mathbf{z}; t; \lambda_{ij}; \tau) p(\tau) d\tau. \tag{38}$$

Then, given a set of orthonormal basis $\{q_m\}_m$, one can decompose κ_W as

$$\kappa_W(\mathbf{z}; t; \lambda_{ij}; \tau) = \sum_{m,n} q_n(-t) Q_{mn}^{(\lambda_{ij})}(\mathbf{z}) q_m(\tau), \tag{39}$$

where

$$Q_{mn}^{(\lambda_{ij})}(\mathbf{z}) = \int q_n(-t) \kappa_W(\mathbf{z}; t; \lambda_{ij}; \tau) q_m(\tau) d\tau\, dt. \tag{40}$$

We define $\{q_m^{(\lambda_{ij})}\}_m$ to be the anti-eigenfunctions of the operator $\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})}$, if

$$[\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} q_m^{(\lambda_{ij})}](t,\mathbf{z}) = Q_m^{(\lambda_{ij})}(\mathbf{z}) q_m^{(\lambda_{ij})}(-t), \tag{41}$$

with some ordering m. For convenience we will choose the ordering m such that as m increases $\|Q_m^{(\lambda_{ij})}\|_2^2$ decreases, i.e. $m = 0$ corresponds to the largest $\|Q_m^{(\lambda_{ij})}\|_2^2$. Here $\|f\|_2^2 = \int |f(\mathbf{z})|^2 d\mathbf{z}$. For example, if $\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})}$ were linear-time invariant, i.e.

both $\mathcal{G}^{(\lambda_{ij})}$ and $W^{(\lambda_{ij})}$ act as convolution in the time domain, and κ_W was periodic in t and τ with period 2π then $q_m^{(\lambda_{ij})}$ may be chosen to be $\exp[-\mathrm{i}mt]$. Then

$$\langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](.\,,\mathbf{z}),q_n^{(\lambda_{ij})}(-.\,)\rangle=\delta_{mn}Q_m^{(\lambda_{ij})}(\mathbf{z}),\tag{42}$$

where

$$Q_m^{(\lambda_{ij})}(\mathbf{z})=\langle[\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}q_m^{(\lambda_{ij})}](.\,,\mathbf{z}),q_m^{(\lambda_{ij})}(-.\,)\rangle.\tag{43}$$

Define

$$\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})=Q_m^{(\lambda_{ij})}(\mathbf{z})/\|Q_m^{(\lambda_{ij})}\|.\tag{44}$$

If the transmitted waveforms are $p_m^{(\lambda_{ij})}=q_m^{(\lambda_{ij})}$, then matching the measurement with $p_m^{(\lambda_{ij})}$ means the projection of $T+C$ onto $\widehat{Q}_m^{(\lambda_{ij})}$. However, $\{\widehat{Q}_m^{(\lambda_{ij})}\}_m$ does not necessarily form an orthonormal set in the image domain. In this regard, we want to design our waveforms from $\{p_m^{(\lambda_{ij})}\}_m$ such that the corresponding $\{\widehat{\Lambda}_m^{(\lambda_{ij})}(\mathbf{z})\}_m$ form an orthonormal set. Consequently, due to the linearity of the measurement model, designing transmitted waveforms is equivalent to form an orthonormal set $\{\widehat{\Lambda}_m^{(\lambda_{ij})}(\mathbf{z})\}_m$ from $\{\widehat{Q}_m^{(\lambda_{ij})}\}_m$, which we will present in the next section. Use of anti-eigenfunctions $\{q_m\}$ simplifies the decomposition (35) as

$$\widetilde{T}^{(\lambda_{ij})}(\mathbf{z})\approx\sum_m\alpha_{mm}^{(\lambda_{ij})}\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z}),\tag{45}$$

where

$$\alpha_{mm}^{(\lambda_{ij})}=\langle T,\widehat{\Lambda}_{mm}^{(\lambda_{ij})}\rangle=\langle e_m(.\,,\lambda_{ij}),p_m^{(\lambda_{ij})}(.\,)\rangle.\tag{46}$$

If we are to transmit a single pulse $p_0^{(\lambda_{ij})}$, then the minimum mean square error (MMSE) $E[\|T-\widetilde{T}^{(\lambda_{ij})}\|_2^2]$ is achieved when $p_0^{(\lambda_{ij})}$ is the anti-eigenfunction of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$ corresponding to the largest anti-eigenvalue, i.e. $p_0^{(\lambda_{ij})}=q_0^{(\lambda_{ij})}$. Similarly, if N pulses were transmitted then, the MMSE is achieved when $p_m^{(\lambda_{ij})}$ are chosen as the anti-eigenfunctions of the operator $\mathcal{G}^{(\lambda_{ij})}W^{(\lambda_{ij})}$ corresponding to the N largest eigenvalues, i.e., $p_i^{(\lambda_{ij})}=q_i^{(\lambda_{ij})}$ for $i=0,\dots,N-1$.

5.2.2. *Construction of* $\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z})$ *and* $p_m^{(\lambda_{ij})}(t)$

We form $\{\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z})\}_m$ by performing a Gram-Schmidt orthonormalization (GSO) process over $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_m$. Since the GSO depends on the choice of the order of $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m>0}$, in the light of the previous section, we choose $\{\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\}_{m>0}$ with the decreasing order as m increases to achieve MMSE. Let $\widehat{\Lambda}_{00}^{(\lambda_{ij})}(\mathbf{z})=\widehat{Q}_0^{(\lambda_{ij})}(\mathbf{z})$. Then, by GSO, we form

$$\widehat{\Lambda}_{mm}^{(\lambda_{ij})}(\mathbf{z})=\sum_{k=0}^{m-1}\beta_{mk}^{(\lambda_{ij})}\widehat{\Lambda}_{kk}^{(\lambda_{ij})}(\mathbf{z})+\beta_{mm}^{(\lambda_{ij})}\widehat{Q}_m^{(\lambda_{ij})}(\mathbf{z})\tag{47}$$

for some constants $\beta_{mn}^{(\lambda_{ij})}$ and $\gamma_{mn}^{(\lambda_{ij})}$ obtained via GSO.

Due to the linearity of each step of GSO, we can determine the transmitted waveforms along with the GSO of $\widehat{Q}_m^{(\lambda_{ij})}$ as follows:

Let $\{p_m^{(\lambda_{ij})}\}_m$ be the transmitted waveforms. In order to obtain $\widehat{\Lambda}_{mm}^{(\lambda_{ij})}$ after matching the mth measurement by $p_m^{(\lambda_{ij})}$, by GSO of $\widehat{Q}_m^{(\lambda_{ij})}$,

$$p_0^{(\lambda_{ij})}(t) = q_0^{(\lambda_{ij})}(t)/\sqrt{\|Q_0^{(\lambda_{ij})}\|}, \tag{48}$$

$$p_m^{(\lambda_{ij})}(t) = \sum_{k=0}^{m-1} \frac{q_k^{(\lambda_{ij})}(t)\sqrt{\gamma_{mk}^{(\lambda_{ij})}}}{\sqrt{\|Q_k^{(\lambda_{ij})}\|}} + \frac{q_m^{(\lambda_{ij})}(t)\sqrt{\beta_{mm}^{(\lambda_{ij})}}}{\sqrt{\|Q_m^{(\lambda_{ij})}\|}}, \tag{49}$$

where

$$\gamma_{mk}^{(\lambda_{ij})} = \sum_{n=k}^{m-1} \beta_{mn}^{(\lambda_{ij})}\gamma_{nk}^{(\lambda_{ij})}. \tag{50}$$

In real life application, we may not always have the luxury of transmitting the designed waveforms $\{p_m^{(\lambda_{ij})}\}$ but only a set of waveforms $s_m(t)$. In these cases, we can still use the GSO to form a basis $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}$ from

$$S_{mn}^{(\lambda_{ij})}(\mathbf{z}) = \langle [\mathcal{G}^{(\lambda_{ij})} W^{(\lambda_{ij})} s_m](\,.\,, \mathbf{z}), s_n(-\,.\,)\rangle. \tag{51}$$

Thus, for each $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$, we obtain a corresponding waveform $p_{mn}^{(\lambda_{ij})}(t)$.

5.3. Waveform Design and Reflectivity Reconstruction for Distributed Apertures

In the case of a distributed aperture, it is sufficient to generalize the GSO over $\{S_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$ by ordering the pairs $\{(\lambda_{ij}, m)\}$ with respect to a criterion.

For example, let the ordering on $\{(\lambda_{ij}, m)\}$ be defined by $(\lambda_{i'j'}, m') < (\lambda_{ij}, m)$ if $i' < i$, or if $i' = i$ and $j' < j$, or $\lambda_{i'j'} = \lambda_{ij}$ and $m' < m$. Then, given $\{S_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$, we determine $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$ by GSO such that $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$ is orthogonal to all $\widehat{\Lambda}_{m'n'}^{(\lambda_{i'j'})}(\mathbf{z})$, and hence to all $S_{m'n'}^{(\lambda_{i'j'})}(\mathbf{z})$, for $(\lambda_{i'j'}, m') < (\lambda_{ij}, m)$, with the initial condition $\widehat{\Lambda}_{00}^{(\lambda_{00})}(\mathbf{z}) = S_{00}^{(\lambda_{00})}(\mathbf{z})/\|S_{00}^{(\lambda_{00})}\|$.

The presented reconstruction method can be summarized by the following diagram:

$$\begin{tikzcd}[column sep=large]
T(\mathbf{z}) \arrow[r, "W^{(\lambda_{ij})}p_{mn}^{(\lambda_{ij})}(t)", "1"'] & e(t,\lambda_{ij}) \arrow[d, "2"', "{\langle \cdot, p_{mn}^{(\lambda_{ij})}(-.)\rangle}"] \\
\alpha_{mn}^{(\lambda_{ij})}\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z}) \arrow[u, "\sum\limits_{m,n,\lambda_{ij}}", "4"'] & \alpha_{mn}^{(\lambda_{ij})} \arrow[l, "3"', "\times\, \widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})"]
\end{tikzcd}$$

The first step is the preconditioning of the waveforms $\{p_{mn}^{(\lambda_{ij})}(t)\}$ by $W^{(\lambda_{ij})}$. The second step is the matching of the mth measurement due to the transmitted waveform $W^{(\lambda_{ij})}\{p_{mn}^{(\lambda_{ij})}(t)\}$ with $\{p_{mn}^{(\lambda_{ij})}(t)\}$. The third step is the backprojection of the matched signal by $\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})$. Finally, the image is formed by summing over all backprojected data.

Remark - Transmission vs Receive Instead of transmitting the waveforms $\{W^{(\lambda_{ij})}p_{mn}^{(\lambda_{ij})}(t)\}$, one can transmit an arbitrary set of waveforms $\{s_m(t)\}$. Since $W^{(\lambda_{ij})}$ is linear and Step 1 is assumed to be linear, the measurements due to transmitting $\{W^{(\lambda_{ij})}p_{mn}^{(\lambda_{ij})}(t)\}$ can be synthesized by weighted summation of the measurements due to $\{s_m(t)\}$. However, this results in increased computational load in receive.

6. Conclusion

We introduced a waveform design strategy and a complementary filtered backprojection type imaging method for distributed aperture radar systems.

The methods are inspired by the group theoretic statistical signal processing methods developed in [1, 2]. From a group theoretic perspective, (34) and (35) correspond to the "Fourier" and "inverse Fourier" transforms. In this regard, $\widehat{S}_{mn}^{(\lambda_{ij})}(\mathbf{z}) = S_{mn}^{(\lambda_{ij})}(\mathbf{z})/\|S_{mn}^{(\lambda_{ij})}\|$ can be viewed as the matrix elements of the irreducible unitary representations, where mn denotes the matrix entry and $\lambda_{ij} = (\mathbf{x}_i, \mathbf{z}_j)$ is the corresponding irreducible subspace. In this setting GSO is the analog of the square root of the linear operator known as the discrepancy operator for non-commutative groups, which forms the symmetric Fourier basis $\{\widehat{\Lambda}_{mn}^{(\lambda_{ij})}(\mathbf{z})\}$ from $\{\widehat{S}_{mn}^{(\lambda_{ij})}\}$. Finally, $W^{(\lambda_{ij})}$ corresponds to the Wiener filter developed in [1, 2].

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