Effect of Discretization on the Accuracy of Diffuse Optical Imaging

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ABSTRACT

In this paper, we analyze the error resulting from the discretization of the forward and inverse problems in simultaneously reconstructed optical absorption and scattering images. Our analysis indicates the mutual dependence of the forward and inverse problems, the number of sources and detectors, their configuration and the location of optical heterogeneities with respect to sources and detectors affect the extent of the error in the reconstructed optical images resulting from discretization. One important implication of the error analysis is that poor discretization of one optical coefficient results in error in the other, resulting in inter-parameter "cross-talk" due entirely to discretization.

Keywords: discretization, error, analysis, optical imaging, cross-talk

1. INTRODUCTION

Imaging in Diffuse Optical Tomography (DOT) is comprised of two interdependent stages which seek solutions to the forward and inverse problems. The forward problem is associated with describing the Near Infrared (NIR) light propagation, while the objective of the inverse problem is to estimate the unknown optical parameters from boundary measurements.[1] In this work, we model the forward problem by the diffusion equation in the frequency domain and the associated adjoint problem. For the inverse problem, we consider the simultaneous estimation of the optical diffusion and absorption coefficients.

A number of factors affect the accuracy of the DOT imaging: model accuracy (dependent on the light propagation model and/or linearization of the inverse problem), measurement noise, discretization, numerical algorithm efficiency, and inverse problem formulation. In this work, we focus on the effect of discretization on the accuracy of simultaneously reconstructed optical absorption and diffusion coefficients. In this respect, we extend our work in.[2,3] First, we show the effect of forward problem discretization. Next, we show the effect of discretization of the inverse problem whose formulation uses the numerical solutions of the forward problem. Finally, we use the error analysis to devise novel adaptive mesh generation algorithms that reduce the error in the reconstructed optical images due to discretization for a given number of unknowns (i.e. for a given number of nodes in the adaptive meshes).

There has been extensive research on the estimation of discretization error in the solutions of partial differential equations (PDEs).[4–6] A somewhat different approach is followed in[7,8] where error in quantities of interest is related to the discretization of the second order elliptic partial differential equation. In the area of parameter estimation problems governed by PDEs, relatively little has been published. See for example[9] for an *a posteriori* error estimate for the Lagrangian in the inverse scattering problem for the time-dependent acoustic wave equation and[10] for a similar approach, and[11] for *a posteriori* error estimates for distributed elliptic optimal control problems. In the area of DOT, it was numerically shown that the approximation errors resulting from the discretization of the forward problem can lead to significant errors in the reconstructed optical images.[12] However, an analysis regarding the error in the reconstructed optical images resulting from discretization has not been reported so far.

In this work, we model the forward problem by the frequency-domain diffusion equation. For the inverse problem, we focus on the simultaneous estimation of the absorption and diffusion coefficients. We consider the linear integral equation resulting from the iterative linearization of the inverse problem based on Born approximation and use zeroth order Tikhonov regularization to address the ill-posedness of the resulting integral

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equation. We use finite elements with first order Lagrange basis functions to discretize the forward and inverse problems and analyze the effect of the discretization on the reconstructed optical absorption and diffusion images. Then, we analyze the error in the simultaneously reconstructed optical absorption and diffusion coefficients resulting from the discretization of the forward and inverse problems.

In our analysis, we first consider the effect of the forward problem discretization when there is no discretization of the inverse problem, and provide a bound for the resulting error in the reconstructed optical images. Next, we analyze the effect of the discretization of the inverse problem whose formulation is based on the numerical (finite element) solutions of the forward problem and we obtain another bound for the resulting error in the reconstructed optical images. Our analysis shows that the error in the reconstructed optical images due to discretization depends on the configuration of the source and detectors, the positions of the sources and detectors with respect to locations of absorptive and diffusive heterogeneities, and on the regularization parameter(s). In addition, we notice that the error in the reconstruction of one optical parameter depends on how well the other optical parameter is discretized. As a result of this last implication, the error analysis provides an insight into the so-called "inter-parameter crosstalk"[13] that originates entirely from discretization.

Our analysis presents two important error estimates that can be employed to design new adaptive mesh generation algorithms. Furthermore, the analysis provides a means to identify and analyze the error in the simultaneously reconstructed optical images resulting from the linearization of the Lippmann-Schwinger type equations[14] using Born approximation, which will be an extension to our recent work.[15] Furthermore, the error analysis introduced in this paper is not limited to DOT, and can easily be extended for use in similar two-parameter inverse problems.

Table 1. Definition of function spaces and norms.

Notation	Explanation
$\overline{f}$	The complex conjugate of the function f
$C(\overline{\Omega})$	Space of continuous complex-valued functions on $\Omega \cup \partial\Omega$
$L^\infty(\Omega)$	$L^\infty(\Omega) = \{f \vert \ ess\sup_\Omega \vert f(\mathbf{x})\vert < \infty\ \}$
$L^p(\Omega)$	$L^p(\Omega) = \{f \vert \ (\int_\Omega \vert f(\mathbf{x})\vert^p d\mathbf{x})^{1/p} < \infty\ \}$, $p \in [1, \infty)$
$D_w^z f$	z^{th} weak derivative of f
$H^p(\Omega)$	$H^p(\Omega) = \{f \vert \ (\sum_{\vert z\vert \leq p} \Vert D_w^z f\Vert_0^2)^{1/2} < \infty\ \}$, $p \in [1, \infty)$
$W_\infty^1(\Omega)$	$W_\infty^1(\Omega) = \{f \in L_{loc}^1(\Omega) \vert \ \max_{\vert z\vert \leq 1} \Vert D_w^z f\Vert_\infty \leq \infty\}$
$\Vert f\Vert_0$	The $L^2(\Omega)$ norm of f
$\Vert f\Vert_p$	The $H^p(\Omega)$ norm of f
$\Vert f\Vert_\infty$	The $L^\infty(\Omega)$ norm of f
$\Vert f\Vert_{0,m}$	The L^2 norm of f over the m^{th} finite element Ω_m
$\Vert f\Vert_{p,m}$	The H^p norm of f over the m^{th} finite element Ω_m
$\Vert f\Vert_{\infty,m}$	The L^∞ norm of f over the m^{th} finite element Ω_m

2. FORWARD AND INVERSE PROBLEMS

In this section, we describe the model for NIR light propagation and define the forward and inverse DOT problems. Table 1 provides the definition of function spaces and norms used throughout the paper. We note that we use calligraphic letters to denote the operators, e.g. $\mathcal{A}_a$, $\mathcal{I}$, $\mathcal{K}$ etc. The subscripts or superscripts a and b will be used respectively to denote the relevance to absorption and diffusion coefficients, which will be defined where they appear. The superscript $*$ denotes the adjoint and "tilde" will be used to denote a function or an operator is an approximation to its accurate counterpart. Vector quantities are in bold characters, such as $\mathbf{x}$.

2.1. Forward Problem

We use the following boundary value problem to model the NIR light propagation in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$[1,16]:

$$-\nabla \cdot D(\mathbf{x})\nabla g_j(\mathbf{x}) + \left(\mu_a(\mathbf{x}) + \frac{i\omega}{c}\right) g_j(\mathbf{x}) = Q_j(\mathbf{x}) \quad \mathbf{x} \in \Omega, \tag{1}$$

$$g_j(\mathbf{x}) + 2aD(\mathbf{x})\frac{\partial g_j}{\partial n}(\mathbf{x}) = 0 \quad \mathbf{x} \in \partial\Omega, \tag{2}$$

where $g_j(\mathbf{x})$ is the photon density at $\mathbf{x} \in \Omega \cup \partial\Omega$ with frequency ω, Q_j is the j^{th} point source located at $\mathbf{x}_s^j$, $j = 1, \cdots, N_s$, where N_s is the number of sources. $D(\mathbf{x})$ is the diffusion coefficient and $\mu_a(\mathbf{x})$ is the absorption coefficient at $\mathbf{x}$, $i = \sqrt{-1}$, ω is the modulation frequency of the source, c is the speed of the light, $a = (1+R)/(1-R)$ where R is a parameter governing the internal reflection at the boundary $\partial\Omega$, and $\partial \cdot /\partial n$ denotes the directional derivative along the unit normal vector on the boundary. Note that we assume the diffusion coefficient is independent of the absorption coefficient and is isotropic. For the general anisotropic material, see.[17]

The adjoint problem[1] associated with (1)-(2) is given by the following boundary value problem:

$$-\nabla \cdot D(\mathbf{x})\nabla g_i^*(\mathbf{x}) + \left(\mu_a(\mathbf{x}) - \frac{i\omega}{c}\right) g_i^*(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \tag{3}$$

$$g_i^*(\mathbf{x}) + 2aD(\mathbf{x})\frac{\partial g_i^*}{\partial n}(\mathbf{x}) = Q_i^*(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega, \tag{4}$$

where Q_i^* is the adjoint source located at the i^{th} detector $\mathbf{x}_d^i$, $i = 1, \cdots, N_d$, where N_d is the number of detectors. We note that we approximate the point source Q_j in (1) and the adjoint source Q_i^* in (4) by Gaussian functions with sufficiently low variance, whose centers are located at $\mathbf{x}_s^j$ and $\mathbf{x}_d^i$, respectively. Note also that for any source $Q_j \in H^1(\Omega)$, our error analysis is valid. In this work, we consider the finite-element approximations of the solutions of the forward problem. Hence, before we discretize the forward problem (see section 3.1), we consider the variational formulations of (1)-(2) and (3)-(4).[2] We note that a unique solution exists for each of these variational problems. Furthermore, the following holds[18,19]:

- In addition to above conditions, noting $Q_j, Q_i^* \in C(\overline{\Omega})$; the solutions g_j, g_i^* satisfy[20]

$$g_j, g_i^* \in W_1^\infty(\Omega). \tag{5}$$

2.2. Inverse Problem

In this work, the objective of the inverse problem is to determine the unknown optical absorption and diffusion coefficients of a bounded optical domain. To address the nonlinear nature of the inverse DOT problem, we consider an iterative algorithm based on repetitive linearization of the inverse problem using first order Born approximation.[1] As a result, at each linearization step, the following linear integral equation relates the differential optical measurements to unknown small perturbations α and β on the absorption coefficient μ_a and the diffusion coefficient D, respectively, assuming $\beta = 0$ on $\mathbf{x} \in \partial\Omega$:

$$\begin{aligned} \Gamma_{i,j} &= -\int_\Omega \left[\overline{g_i^*(\mathbf{x})} g_j(\mathbf{x})\alpha(\mathbf{x}) + \overline{\nabla g_i^*(\mathbf{x})} \cdot \nabla g_j(\mathbf{x})\beta(\mathbf{x})\right] d\mathbf{x} && (6) \\ &:= \int_\Omega \left[H_{i,j}^a(\mathbf{x})\alpha(\mathbf{x}) + H_{i,j}^b(\mathbf{x})\beta(\mathbf{x})\right] d\mathbf{x} \\ &:= \left(\begin{bmatrix} \mathcal{A}_a & \mathcal{B}_b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right)_{i,j} \\ &:= (\mathcal{K}_{ab}\sigma)_{i,j}\,, && (7) \end{aligned}$$

where $\sigma = [\alpha\ \beta]^T \in L^2(\Omega) \times L^2(\Omega)$, $\mathcal{K}_{ab} = [\mathcal{A}_a\ \mathcal{B}_b] : L^2(\Omega) \times L^2(\Omega) \to \mathbb{C}^{N_d \times N_s}$, $H_{i,j}^a(\mathbf{x}) = -\overline{g_i^*(\mathbf{x})} g_j(\mathbf{x})$ is the (i,j)th kernel of the matrix valued operator $\mathcal{A}_a : L^2(\Omega) \to \mathbb{C}^{N_d \times N_s}$ at $\mathbf{x}$, and $H_{i,j}^b(\mathbf{x}) = -\overline{\nabla g_i^*(\mathbf{x})} \cdot \nabla g_j(\mathbf{x})$ is (i,j)th kernel of the matrix-valued operator $\mathcal{B}_b : L^2(\Omega) \to \mathbb{C}^{N_d \times N_s}$ at $\mathbf{x}$. g_j is the weak solution of (1)-(2) and $g_i^*(\mathbf{x})$ is weak solution of (3)-(4), and $\Gamma_{i,j}$ is the (i,j)th entry in the vector $\Gamma \in \mathbb{C}^{N_d \times N_s}$, which represents the differential measurement at the i^{th} detector due to the j^{th} source.[1,2] Thus,

$$\Gamma = K_{ab}\sigma. \tag{8}$$

Note that approximating Q_i^* in (4) by a Gaussian function centered at $\mathbf{x}_d^i$ implies that $\Gamma_{i,j}$ corresponds to the scattered optical field evaluated at $\mathbf{x}_d^i$, after filtering it by that Gaussian function.[2] Thus, the Gaussian approximation of the adjoint source models the finite size of the detectors. Similarly, approximating Q_j in (1) by a Gaussian function models the finite beam of the point source. We note the boundedness of the operators $\mathcal{A}_a$ and $\mathcal{B}_b$ are bounded, which leads to the boundedness of $\mathcal{K}_{ab}$. Furthermore, the operators $\mathcal{A}_a$ and $\mathcal{B}_b$ are compact.[2,21] Thus, for the given solution space $L^2(\Omega)$ for both α and β, (8) is ill-posed. To address the ill-posedness of (6), we regularize (8) with a zeroth order Tikhonov regularization.

2.3. Regularization of the inverse problem

Let $\mathcal{A}_a^* : \mathbb{C}^{N_d \times N_s} \to L^2(\Omega)$ and $\mathcal{B}_b^* : \mathbb{C}^{N_d \times N_s} \to L^2(\Omega)$ be the adjoint of the operators $\mathcal{A}_a$ and $\mathcal{B}_b$ defined respectively by

$$(\mathcal{A}_a^*\Theta)(\mathbf{x}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{a*}(\mathbf{x})\Theta_{i,j} := \sum_{i,j}^{N_d,N_s} -g_i^*(\mathbf{x})\overline{g_j(\mathbf{x})}\Theta_{i,j}, \tag{9}$$

$$(\mathcal{B}_a^*\Theta)(\mathbf{x}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{b*}(\mathbf{x})\Theta_{i,j} := \sum_{i,j}^{N_d,N_s} -\nabla g_i^*(\mathbf{x})\cdot\nabla\overline{g_j(\mathbf{x})}\Theta_{i,j}, \tag{10}$$

for all $\Theta \in \mathbb{C}^{N_d \times N_s}$, where $H_{i,j}^{a*}$ and $H_{i,j}^{b*}$ are the (i,j)th kernels of $\mathcal{A}_a^*$ and $\mathcal{B}_b^*$, respectively. Then we define

$$\mathcal{K}_{ab}^* := \left[\begin{array}{cc} \mathcal{A}_a^* & \mathcal{B}_b^* \end{array}\right].$$

Let $\mathcal{A} := \mathcal{A}_a^*\mathcal{A}_a{:}L^2(\Omega) \to L^2(\Omega)$, $\mathcal{B} := \mathcal{B}_b^*\mathcal{B}_b{:}L^2(\Omega) \to L^2(\Omega)$, $\mathcal{A}_B := \mathcal{A}_a^*\mathcal{B}{:}L^2(\Omega) \to L^2(\Omega)$, and $\mathcal{B}_A := \mathcal{B}_b^*\mathcal{A}_a{:}L^2(\Omega) \to L^2(\Omega)$. Then,

$$(\mathcal{A}\theta)(\mathbf{x}) := \int_\Omega \kappa_A(\mathbf{x},\acute{\mathbf{x}})\theta(\acute{\mathbf{x}})d\acute{\mathbf{x}}, \text{ where } \kappa_A(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{a*}(\mathbf{x})H_{i,j}^{a}(\acute{\mathbf{x}})$$

$$(\mathcal{B}\theta)(\mathbf{x}) := \int_\Omega \kappa_B(\mathbf{x},\acute{\mathbf{x}})\theta(\acute{\mathbf{x}})d\acute{\mathbf{x}}, \text{ where } \kappa_B(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{b*}(\mathbf{x})H_{i,j}^{b}(\acute{\mathbf{x}})$$

$$(\mathcal{A}_B\theta)(\mathbf{x}) := \int_\Omega \kappa_{AB}(\mathbf{x},\acute{\mathbf{x}})\theta(\acute{\mathbf{x}})d\acute{\mathbf{x}}, \text{ where } \kappa_{AB}(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{a*}(\mathbf{x})H_{i,j}^{b}(\acute{\mathbf{x}})$$

$$(\mathcal{B}_A\theta)(\mathbf{x}) := \int_\Omega \kappa_{BA}(\mathbf{x},\acute{\mathbf{x}})\theta(\acute{\mathbf{x}})d\acute{\mathbf{x}}, \text{ where } \kappa_{BA}(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} H_{i,j}^{b*}(\mathbf{x})H_{i,j}^{a}(\acute{\mathbf{x}})$$

for all $\theta \in L^2(\Omega)$. Note that $\kappa_{BA}(\mathbf{x},\acute{\mathbf{x}}) = \overline{\kappa_{AB}(\mathbf{x},\acute{\mathbf{x}})}$.

Let

$$\mathcal{K} := \left[\begin{array}{cc} \mathcal{A} & \mathcal{A}_B \\ \mathcal{B}_A & \mathcal{B} \end{array}\right], \text{ and } \gamma := \left[\begin{array}{c} \gamma_a \\ \gamma_b \end{array}\right] := \left[\begin{array}{c} \mathcal{A}_a^*\Gamma \\ \mathcal{B}_b^*\Gamma \end{array}\right], \tag{11}$$

where $\mathcal{K} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$. Then, using a zeroth order Tikhonov regularization, the following equation defines the inverse problem at each linearization step:

$$\gamma = (\mathcal{K} + \mathcal{L})\,\sigma^\lambda := \mathcal{T}\sigma^\lambda, \tag{12}$$

where $\mathcal{T} := (\mathcal{K} + \mathcal{L}) : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$, $\sigma^\lambda = [\alpha^\lambda\ \beta^\lambda]^T$ are approximations to α and β, respectively, and $\mathcal{L} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is given by

$$\mathcal{L} := \left[\begin{array}{cc} \lambda_a\mathcal{I} & 0 \\ 0 & \lambda_b\mathcal{I} \end{array}\right], \tag{13}$$

where $\lambda_a, \lambda_b > 0$ and $\mathcal{I}$ is the identity operator. We finally note that a bound for $\mathcal{T}$ can be given by $\|\mathcal{T}\|_{L^2(\Omega)\times L^2(\Omega)\to L^2(\Omega)\times L^2(\Omega)} \le \|\mathcal{K}_{ab}\|^2_{L^2(\Omega)\times L^2(\Omega)\to l^1} + \max(\lambda_a, \lambda_b)$.

2.4. Existence and boundedness of the inverse operator

Consider the inverse problem formulation (12). Owing to the regularization term, the inverse operator $\mathcal{T}^{-1} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ exists and by Lax-Milgram lemma[16] it is bounded by

$$\|\mathcal{T}^{-1}\|_{L^2(\Omega)\times L^2(\Omega)\to L^2(\Omega)\times L^2(\Omega)} \leq \frac{1}{\min(\lambda_a, \lambda_b)}.$$

In particular, the operator $\mathcal{T}^{-1}$ can be viewed as a 2×2 matrix of operators $\mathcal{T}_{ij}^{-1} : L^2(\Omega) \to L^2(\Omega)$, $i, j = 1, 2$, i.e.:

$$\mathcal{T}^{-1} := \begin{pmatrix} \mathcal{T}_{11}^{-1} & \mathcal{T}_{12}^{-1} \\ \mathcal{T}_{21}^{-1} & \mathcal{T}_{22}^{-1} \end{pmatrix}. \tag{14}$$

We remark that the boundedness of $\mathcal{T}^{-1}$ is a result of the boundedness of the operators $\mathcal{T}_{ij}^{-1}$:

$$\|\mathcal{T}_{ij}^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq \chi_{ij} \tag{15}$$

for some scalar value $\chi_{ij} > 0$, for $i, j = 1, 2$.

3. DISCRETIZATION OF THE FORWARD AND INVERSE PROBLEMS

In this section, we first consider the variational formulations of (1)-(2) and (3)-(4), and discuss the finite-element discretization of the forward problem. Next, we describe the discretization of the inverse problem (12).

3.1. Forward Problem Discretization

In this section, we consider the finite element discretization of the forward problem defined by the variational formulations of (1)-(2) and (3)-(4), and use their solutions to approximate the kernels $\kappa_A, \kappa_B, \kappa_{AB}$, and κ_{BA}. As a result, we obtain finite dimensional approximations to $\mathcal{K}$ and γ in (11) respectively.

Let L_k be the first order Lagrange basis functions, and $Y_j \subset H^1(\Omega)$ be the finite-dimensional subspace spanned by L_k, $k = 1, \ldots, N_j$, where N_j is the dimension of the finite-dimensional subspace for the jth source, $j = 1, \ldots, N_s$. Similarly, we define $Y_i^* \subset H^1(\Omega)$ as the finite-dimensional subspace spanned by L_k, for $k = 1, \ldots, N_i^*$, where N_i^* is the dimension of the finite-dimensional subspace for the ith detector, $i = 1, \ldots, N_d$. In this representation, N_j and N_i^* denote that for each source and detector, the dimension of the finite-dimensional subspace can be different. Replacing ϕ and g_j with their finite-dimensional counterparts $\Phi_j(\mathbf{x}) = \sum_{k=1}^{N_j} p_k L_k(\mathbf{x})$, $G_j(\mathbf{x}) = \sum_{k=1}^{N_j} c_k L_k(\mathbf{x})$; and replacing ϕ and g_i^* with $\Phi_i^*(\mathbf{x}) = \sum_{k=1}^{N_i^*} q_k L_k(\mathbf{x})$, $G_i^*(\mathbf{x}) = \sum_{k=1}^{N_i^*} d_k L_k(\mathbf{x})$ yields the matrix equations:

$$\mathbf{S}\mathbf{c}_j = \mathbf{q}_j, \text{ and } \mathbf{S}^*\mathbf{d}_i = \mathbf{q}_i^*, \tag{16}$$

for $\mathbf{c}_j = [c_1, c_2, \cdots, c_{N_j}]^T$ and $\mathbf{d}_i = [d_1, d_2, \cdots, d_{N_i}]^T$. Here $\mathbf{S}$ and $\mathbf{S}^*$ are the finite element matrices and $\mathbf{q}_j$ and $\mathbf{q}_i^*$ are the load vectors resulting from the finite element discretization of **??**) and (**??**.

The $H^1(\Omega)$ boundedness of the solutions g_j and g_i^* implies that the discretization error e_j and e_i^* in the finite element solutions G_j and G_i^* is bounded. Let $\{\Omega_{mj}\}$ denote the set of linear elements used to discretize Ω for $m = 1, \ldots, N_\Delta^j$; such that $\bigcup_m^{N_\Delta^j} \Omega_{mj} = \Omega$ for all $j = 1, \ldots N_s$. Similarly, let $\{\Omega_{ni}\}$ denote the set of linear elements used to discretize Ω for $n = 1, \ldots, N_\Delta^{*i}$; such that $\bigcup_n^{N_\Delta^{*i}} \Omega_{ni} = \Omega$ for all $i = 1, \ldots N_d$. Assuming the solutions g_j and g_i^* for all $j = 1, \cdots, N_s$ $i = 1, \cdots, N_d$ also satisfy $g_j, g_i^* \in H^2(\Omega)$, a bound for e_j and e_i^* on each finite element can be found by using the discretization error estimates[22]:

$$\|e_j\|_{0,mj} \leq C\|g_j\|_{2,mj} h_{mj}^2, \tag{17}$$

$$\|e_i^*\|_{0,ni} \leq C\|g_i^*\|_{2,ni} h_{ni}^2, \tag{18}$$

where C is a positive constant, $\|\cdot\|_{0,mj}$ ($\|\cdot\|_{0,ni}$) and $\|\cdot\|_{2,mj}$ ($\|\cdot\|_{2,ni}$) are respectively the L^2 and H^1 norms on Ω_{mj} (Ω_{ni}), and h_{mj} (h_{ni}) is the diameter of the smallest ball containing the finite element Ω_{mj} (Ω_{ni}) in the solution G_j (G_i^*). Similarly,

$$\|\nabla e_j\|_{0,mj} \leq C\|g_j\|_{2,mj}h_{mj}, \tag{19}$$
$$\|\nabla e_i^*\|_{0,ni} \leq C\|g_i^*\|_{2,ni}h_{ni}. \tag{20}$$

3.2. Approximation of $\mathcal{T}$ and γ with finite element solutions G_j and G_i^*

Following the discretization of the forward problem and the solution of the resulting discrete forward problem, we can rewrite the inverse problem formulation (12) by replacing g_j and g_i^* with G_j and G_i^* in $\mathcal{T}$ and γ. Consequently, we get the following inverse problem formulation, which is an approximation to the regularized inverse problem formulation in (12):

$$\tilde{\mathcal{T}}\tilde{\sigma}^\lambda = \tilde{\gamma}, \tag{21}$$

where

$$\tilde{\mathcal{T}} := \tilde{\mathcal{K}} + \mathcal{L} := \begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{A}}_B \\ \tilde{\mathcal{B}}_A & \tilde{\mathcal{B}} \end{bmatrix} + \begin{bmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{bmatrix} \tag{22}$$

$$\tilde{\gamma} := \begin{bmatrix} \tilde{\gamma}_a \\ \tilde{\gamma}_b \end{bmatrix} := \begin{bmatrix} \tilde{\mathcal{A}}_a^*\Gamma \\ \tilde{\mathcal{B}}_b^*\Gamma \end{bmatrix}. \tag{23}$$

are the approximations to $\mathcal{T}$ and γ, respectively, and $\tilde{\sigma}^\lambda$ is an approximation to the solution σ^λ as a result of the forward problem discretization: $\tilde{\sigma}^\lambda = \left[\tilde{\alpha}^\lambda\ \tilde{\beta}^\lambda\right]^T$. In (22), the kernels of the integral operators $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{A}}_B$, and $\tilde{\mathcal{B}}_A$ are given respectively by

$$\tilde{\kappa}_A(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} \tilde{H}_{i,j}^{a*}(\mathbf{x})\tilde{H}_{i,j}^{a}(\acute{\mathbf{x}}), \quad \tilde{\kappa}_B(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} \tilde{H}_{i,j}^{b*}(\mathbf{x})\tilde{H}_{i,j}^{b}(\acute{\mathbf{x}}),$$
$$\tilde{\kappa}_{AB}(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} \tilde{H}_{i,j}^{a*}(\mathbf{x})\tilde{H}_{i,j}^{b}(\acute{\mathbf{x}}), \quad \tilde{\kappa}_{BA}(\mathbf{x},\acute{\mathbf{x}}) := \sum_{i,j}^{N_d,N_s} \tilde{H}_{i,j}^{b*}(\mathbf{x})\tilde{H}_{i,j}^{a}(\acute{\mathbf{x}}),$$

where $\tilde{H}_{i,j}^a(\mathbf{x}) = -\overline{G_i^*(\mathbf{x})}G_j(\mathbf{x})$ and $\tilde{H}_{i,j}^b(\mathbf{x}) = -\nabla\overline{G_i^*(\mathbf{x})}\cdot\nabla G_j(\mathbf{x})$. Note that $\tilde{H}_{i,j}^{a*} = \overline{\tilde{H}_{i,j}^a}$ and $\tilde{H}_{i,j}^{b*} = \overline{\tilde{H}_{i,j}^b}$. The operator $\mathcal{T}^{-1}$, can be interpreted similar to $\tilde{\mathcal{T}}^{-1}$ as a 2 by 2 matrix of operators $\tilde{\mathcal{T}}_{ij}^{-1}$, each of which is bounded by $\tilde{\chi}_{ij}$ for some scalar value $\tilde{\chi}_{ij} > 0$, for $i,j = 1,2$:

In the following section, we describe the discretization of the inverse problem (21) which uses the finite element approximations G_j and G_i^* of g_j and g_i^* in its formulation.

3.3. Discretization of the inverse problem

For the discretization of the inverse problem (21), we use projection by the Galerkin method. Below, we give the details of the Galerkin method.

Let $X^a, X^b \subset L^2(\Omega)$ denote the finite-dimensional subspaces spanned by first order Lagrange polynomials $\{L_1,\ldots,L_{N^a}\}$ and $\{L_1,\ldots,L_{N^b}\}$, associated with vertices located at x_p^a $p = 1,\cdots,N^a$ and x_r^b $r = 1,\cdots,N^b$, respectively, where N^a and N^b are the dimensions of X^a and X^b. Note that X^a and X^b are not necessarily identical.

Let $\{\Omega_t\}$, $t = 1,\cdots,N_\Delta^a$ denote a set of linear finite elements such that $\bigcup_t^{N_\Delta^a}\Omega_t = \Omega$ and $\{\Omega_u\}$ be a set of linear finite elements used for $u = 1,\cdots,N_\Delta^b$ such that $\bigcup_u^{N_\Delta^b}\Omega_u = \Omega$. Then, we express $\tilde{\sigma}_{n,m}^\lambda = [\tilde{\alpha}_n^\lambda\ \tilde{\beta}_m^\lambda]^T$ on these finite elements as

$$\tilde{\alpha}_n^\lambda(\mathbf{x}) = \sum_{k=1}^{N^a} a_k L_k(\mathbf{x}), \text{ and } \tilde{\beta}_m^\lambda(\mathbf{x}) = \sum_{l=1}^{N^b} b_l L_l(\mathbf{x}). \tag{24}$$

Next consider the test function $\zeta = [\zeta_a \ \zeta_b]^T \in X^a \times X^b$ given by

$$\zeta_a(\mathbf{x}) = \sum_{k=1}^{N^a} c_k^a L_k(\mathbf{x}), \text{ and } \zeta_b(\mathbf{x}) = \sum_{l=1}^{N^b} c_l^b L_l(\mathbf{x}).$$

Then, the Galerkin method approximates the solution of (21) by an element $\tilde{\sigma}_{n,m}^{\lambda} = [\tilde{\alpha}_n^{\lambda} \ \tilde{\beta}_m^{\lambda}]^T \in X^a \times X^b$, which satisfies

$$\left(\tilde{\mathcal{T}}\sigma_{n,m}^{\lambda}, \zeta\right) = (\tilde{\gamma}, \zeta) \tag{25}$$

for all $\zeta \in (X^a \times X^b)$. We note that by Lax-Milgram theorem, a unique solution $\sigma_{n,m}^{\lambda} \in (X^a \times X^b)$ exists for (25) owing to the regularization which results in the positive-definiteness of the operator $\tilde{\mathcal{T}}$.[16, 21] Equivalently, (25) can be interpreted as follows:

$$\mathcal{P}_{a,b}\tilde{\mathcal{T}}\sigma_{n,m}^{\lambda} = \mathcal{P}_{a,b}\tilde{\gamma}, \tag{26}$$

where $\mathcal{P}_{n,m}$ is the matrix of orthogonal projection operators

$$\mathcal{P}_{a,b} = \begin{pmatrix} \mathcal{P}_a & 0 \\ 0 & \mathcal{P}_b \end{pmatrix} \tag{27}$$

where $\mathcal{P}_a : L^2(\Omega) \to X^a$ and $\mathcal{P}_b : L^2(\Omega) \to X^b$ are the orthogonal projection operators.[21] We note that the following condition holds for $(\mathcal{P}_{a,b}\tilde{\mathcal{T}}) : X^a \times X^b \to X^a \times X^b$ (see proof of theorem 13.27 in[21]):

$$\|(\mathcal{P}_{a,b}\tilde{\mathcal{T}})^{-1}\mathcal{P}_{a,b}\|_{L^2(\Omega)\times L^2(\Omega)\to X^a\times X^b} \leq \frac{1}{\min(\lambda_a, \lambda_b)}. \tag{28}$$

3.4. Summary: The inverse problem and its approximations

In this work, we consider the regularized inverse problem in (12) as the baseline for the error analysis. In this respect, we first consider the effect of discretization of the forward problem on the optical imaging accuracy, thus consider the inverse problem (21), i.e.

$$\tilde{\mathcal{T}}\tilde{\sigma}^{\lambda} = \tilde{\gamma},$$

Next, to show the effect of inverse problem discretization, we project the above equation on the finite-dimensional subspaces $X^a \times X^b$, and consider the resulting inverse problem formulation: (25)

$$\mathcal{P}_{a,b}\tilde{\mathcal{T}}\tilde{\sigma}_{n,m}^{\lambda} = \mathcal{P}_{a,b}\tilde{\gamma}.$$

Thus, we have three different inverse problem formulations:

1. The exact inverse problem formulation (12),
2. the inverse problem formulation (21) with the degenerate kernels, and
3. the full discrete inverse problem (25), which is the projection of (21) onto the finite dimensional subspaces $(X^a \times X^b) \subset (L^2(\Omega) \times L^2(\Omega))$ using Galerkin method.

4. DISCRETIZATION-BASED ERROR ANALYSIS

As a result of operator approximation and discretization of the inverse problem, the reconstructed images $\tilde{\sigma}_n^{\lambda} = [\tilde{\alpha}_n^{\lambda} \ \tilde{\beta}_n^{\lambda}]^T$ are approximations to the actual images $\sigma^{\lambda} = [\alpha^{\lambda} \ \beta^{\lambda}]$. Projecting the inverse problem onto finite-dimensional sub-spaces X^a and X^b and the discretization error in the solutions of the forward problem result in error in the reconstructed images. Therefore, the accuracy of the reconstructed image is challenged by the discretization schemes followed in the numerical solutions of the forward and inverse problems.

The error in the solution $\tilde{\sigma}_{n,m}^{\lambda}$ of (25) with respect to the actual solution σ^{λ} of (12) has two contributors: We write $\tilde{\sigma}_{n,m}^{\lambda} = \tilde{\sigma}^{\lambda} - e_{n,m}$, where $e_{n,m} = [e_n^a \ e_m^b]^T$ is the error resulting from projection of the inverse problem with the operator approximation and denote $\tilde{\sigma}^\lambda = \sigma^\lambda - \tilde{e}$, where $\tilde{e} = [\tilde{e}^a \ \tilde{e}^b]^T$ is the error due to forward problem discretization. As a result, we arrive at the following conclusion:

$$\tilde{\sigma}^\lambda_{n,m} = \tilde{\sigma}^\lambda - e_{n,m} = \sigma^\lambda - e_{n,m} - \tilde{e}. \tag{29}$$

Therefore, we can write an upper bound for the error $\sigma^\lambda - \tilde{\sigma}^\lambda_{n,m}$ as follows:

$$\|\sigma^\lambda - \tilde{\sigma}^\lambda_{n,m}\| = \|\sigma^\lambda - \tilde{\sigma}^\lambda + \tilde{\sigma}^\lambda - \tilde{\sigma}^\lambda_{n,m}\| = \|\tilde{e} + e_{n,m}\| \leq \|\tilde{e}\| + \|e_{n,m}\|. \tag{30}$$

4.1. Effect of forward problem discretization

The following theorem presents a bound for the $L^2(\Omega)$ norm of the error between the solution $\tilde{\sigma}^\lambda$ of (21) and the solution σ^λ of (12).

Theorem 1:

Let $\{\Omega_{mj}\}$ denote the set of linear elements used to discretize Ω for $m = 1, \ldots, N^j_\Delta$; such that $\bigcup_m^{N^j_\Delta} \Omega_{mj} = \Omega$, and h_{mj} be the diameter of the smallest ball that contains the element Ω_{mj} in the solution G_j, for all $j = 1, \ldots N_s$. Similarly, let $\{\Omega_{ni}\}$ denote the set of linear elements used to discretize Ω for $n = 1, \ldots, N^{*i}_\Delta$; such that $\bigcup_n^{N^{*i}_\Delta} \Omega_{ni} = \Omega$, and h_{ni} be the diameter of the smallest ball that contains the element Ω_{ni} in the solution G^*_i, for all $i = 1, \ldots N_d$. Assume further that the solutions g_j and g^*_i admit smoothness such that $g_j, g^*_i \in H^2(\Omega)$ and σ^λ is boundeds. Let

$$\begin{aligned} a(j,m) &:= \sum_{i=1}^{N_d} \|g^*_i \alpha^\lambda\|_{0,mj} \|g_j\|_{2,mj} \qquad b(j,m) := \frac{\|\alpha\|_0 + \|\beta\|_0}{2} \sum_{i=1}^{N_d} \|g^*_i\|_{\infty,mj} \|g_j\|_{2,mj}, \\ c(j,m) &:= \sum_{i=1}^{N_d} \left\| |\nabla g^*_i| \beta^\lambda \right\|_{0,mj} \|g_j\|_{2,mj} \qquad d(j,m) := \frac{\|\alpha\|_0 + \|\beta\|_0}{2} \sum_{i=1}^{N_d} \|\nabla g^*_i\|_{\infty,mj} \|g_j\|_{2,mj}, \end{aligned}$$

and

$$\begin{aligned} a^*(i,n) &:= \sum_{j=1}^{N_s} \|g_j \alpha^\lambda\|_{0,ni} \|g^*_i\|_{2,ni} \qquad b^*(i,n) := \frac{\|\alpha\|_0 + \|\beta\|_0}{2} \sum_{j=1}^{N_s} \|g_j\|_{\infty,ni} \|g^*_i\|_{2,ni}, \\ c^*(i,n) &:= \sum_{j=1}^{N_s} \left\| |\nabla g_j| \beta^\lambda \right\|_{0,ni} \|g^*_i\|_{2,ni} \qquad d^*(i,n) := \frac{\|\alpha\|_0 + \|\beta\|_0}{2} \sum_{j=1}^{N_s} \|\nabla g_j\|_{\infty,ni} \|g^*_i\|_{2,ni}. \end{aligned}$$

Given the *a priori* discretization error estimates (17)-(18) and a generic constant $C > 0$, a bound for the error between the solution α^λ and the solution $\tilde{\alpha}^\lambda$ of (12) due to the approximations $\tilde{\mathcal{T}}$ and $\tilde{\gamma}$ is given by:

$$\begin{aligned} &\|\alpha^\lambda - \tilde{\alpha}^\lambda\|_0 \leq 2C \max_{i,j} \|g^*_i g_j\|_1 \\ &\times \left(\sum_{j=1}^{N_s} \sum_{m=1}^{N^j_\Delta} \left[(\tilde{\chi}_{11} + \tilde{\chi}_{12}) a(j,m) + \tilde{\chi}_{12} b(j,m) \right] h^2_{mj} + \left[(\tilde{\chi}_{11} + \tilde{\chi}_{12}) c(j,m) + \tilde{\chi}_{12} d(j,m) \right] h_{mj} \right. \\ &\left. + \sum_{i=1}^{N_d} \sum_{n=1}^{N^{*i}_\Delta} \left[(\tilde{\chi}_{11} + \tilde{\chi}_{12}) a^*(i,n) + \tilde{\chi}_{12} b^*(i,n) \right] h^2_{ni} + \left[(\tilde{\chi}_{11} + \tilde{\chi}_{12}) c^*(i,n) + \tilde{\chi}_{12} d^*(i,n) \right] h_{ni} \right) \end{aligned}$$

and a bound for the error between the solution β^λ and the solution $\tilde{\beta}^\lambda$ of (12) due to the approximations $\tilde{\mathcal{T}}$ and $\tilde{\gamma}$ is given

$$\|\beta^\lambda - \tilde{\beta}^\lambda\|_0 \leq 2C \max_{i,j} \|g^*_i g_j\|_1$$

$$\times\left(\sum_{j=1}^{N_s}\sum_{m=1}^{N_\Delta^j}\left[(\tilde{\chi}_{21}+\tilde{\chi}_{22})a(j,m)+\tilde{\chi}_{22}b(j,m)\right]h_{mj}^2+\left[(\tilde{\chi}_{21}+\tilde{\chi}_{22})c(j,m)+\tilde{\chi}_{22}d(j,m)\right]h_{mj}\right.$$

$$\left.+\sum_{i=1}^{N_d}\sum_{n=1}^{N_\Delta^{*i}}\left[(\tilde{\chi}_{21}+\tilde{\chi}_{22})a^*(i,n)+\tilde{\chi}_{22}b^*(i,n)\right]h_{ni}^2+\left[(\tilde{\chi}_{21}+\tilde{\chi}_{22})c^*(i,n)+\tilde{\chi}_{22}d^*(i,n)\right]h_{ni}\right)$$

Theorem 1 shows that the error in the reconstructed absorption image $\tilde{\alpha}^\lambda$ depends on the diffusive heterogeneity and the solutions of the forward problem. Similarly, the error in the reconstructed diffusion image $\tilde{\beta}^\lambda$ depends on the absorptive heterogeneity and the solutions of the forward problem. With these observations, theorem 1 suggests the use of meshes designed individually for the solutions G_j, $j=1,\cdots,N_s$ and G_i^*, $i=1,\cdots,N_d$. Note also that the position of the detectors with respect to the sources is another factor that affects the error bound in theorem 1.

Note that the conventional interpolation error estimates given in (17)-(18) and (19)-(20) depend on only the smoothness and support of g_j and g_i^*.[16] On the other hand, the error estimates in Theorem 1 show that the accuracy of the reconstructed images $\tilde{\alpha}\lambda$ and $\tilde{\beta}\lambda$ depend on the location of the absorptive and diffusive heterogeneities with respect to the sources and detectors, as well as on the bounds (17)-(18) and (19)-(20).

The parameters $\tilde{\chi}_{ij}$ $i,j=1,2$ affect the bounds on $\|\alpha^\lambda-\tilde{\alpha}^\lambda\|_0$ and $\|\beta^\lambda-\tilde{\beta}^\lambda\|_0$. Note that the parameters $\tilde{\chi}_{ij}$, $i,j=1,2$ depend on the regularization parameters λ_a,λ_b and on the kernels of the operator $\mathcal{T}$. We also note that the kernels of $\mathcal{T}$ can be scaled to make $\tilde{\chi}_{ij}$ almost identical for all $i,j=1,2$.[23] Otherwise, the effect of forward problem discretization may be greater on one of the reconstructed optical coefficients as compared to the other one. Finally we note that increasing the number of sources and detectors increases the bounds on $\|\alpha^\lambda-\tilde{\alpha}^\lambda\|_0$ and $\|\beta^\lambda-\tilde{\beta}^\lambda\|_0$.

4.2. Effect of inverse problem discretization

In this section, we show the effect of inverse problem discretization on the optical imaging accuracy. In the analysis, we consider the inverse problem formulation and derive a bound for the $L^2(\Omega)$ norm of the error $e^{n,m}$ between the solution of (21) and the solution of (26).

Theorem 2:

Let $\{\Omega_t\}$ denote the set of linear elements used to discretize Ω for $t=1,\ldots,N_\Delta^a$; such that $\bigcup_t^{N_\Delta^a}\Omega_t=\Omega$, and h_{ta} be the diameter of the smallest ball that contains the element Ω_t in the solution. Similarly, let $\{\Omega_u\}$ denote the set of linear elements used to discretize Ω for $u=1,\ldots,N_\Delta^b$; such that $\bigcup_u^{N_\Delta^b}\Omega_u=\Omega$, and h_{ub} be the diameter of the smallest ball that contains the element Ω_u. Assume that the solutions $\tilde{\alpha}^\lambda$ and $\tilde{\beta}^\lambda$ are sufficiently smooth such that

$$\tilde{\alpha}^\lambda,\tilde{\beta}^\lambda\in H^1(\Omega).$$

Then,

$$\|\tilde{\alpha}^\lambda-\tilde{\alpha}_n^\lambda\|_0\le C(1+\lambda_a\pi_{11})\sum_{t=1}^{N_a^\Delta}\|\tilde{\alpha}^\lambda\|_{1,ta}h_{ta}+C\lambda_b\pi_{12}\sum_{u=1}^{N_b^\Delta}\|\tilde{\beta}^\lambda\|_{1,ub}h_{ub}$$
$$+C(\pi_{11}+\pi_{12})\max_{i,j}\|G_i^*G_j\|_1$$
$$\times\left(\sum_{t=1}^{N_a^\Delta}\sum_{i,j}^{N_d,N_s}\|G_i^*G_j\|_{0,ta}\|\tilde{\alpha}^\lambda\|_{1,ta}h_{ta}+\sum_{u=1}^{N_b^\Delta}\sum_{i,j}^{N_d,N_s}\|\nabla G_i^*\cdot\nabla G_j\|_{0,ub}\|\tilde{\beta}^\lambda\|_{1,ub}h_{ub}\right).$$

and

$$\|\tilde{\beta}^\lambda - \tilde{\beta}^\lambda_m\|_0 \leq C(1+\lambda_b\pi_{22})\sum_{u=1}^{N_b^\Delta}\|\tilde{\beta}^\lambda\|_{1,ub}h_{ub} + C\lambda_a\pi_{21}\sum_{u=1}^{N_a^\Delta}\|\tilde{\alpha}^\lambda\|_{1,ta}h_{ta} + C(\pi_{21}+\pi_{22})\max_{i,j}\|G_i^*G_j\|_1 \times\left(\sum_{t=1}^{N_a^\Delta}\sum_{i,j}^{N_d,N_s}\|G_i^*G_j\|_{0,ta}\|\tilde{\alpha}^\lambda\|_{1,ta}h_{ta} + \sum_{u=1}^{N_b^\Delta}\sum_{i,j}^{N_d,N_s}\|\nabla G_i^*\cdot\nabla G_j\|_{0,ub}\|\tilde{\beta}^\lambda\|_{1,ub}h_{ub}\right).$$

The theorem shows that the accuracy of the reconstructed image $\tilde{\alpha}^\lambda_n$ depends on the discretization scheme followed to discretize $\tilde{\beta}^\lambda$ as well as the discretization scheme followed to discretize $\tilde{\alpha}^\lambda$ itself. Similarly, the theorem shows that the accuracy of the reconstructed image $\tilde{\beta}^\lambda_m$ depends on the discretization scheme followed to discretize $\tilde{\alpha}^\lambda$ as well as the discretization scheme followed to discretize $\tilde{\beta}^\lambda$ itself.

Theorem 2 shows the spatial dependence of the inverse problem discretization on the forward problem solution. The position of the detectors with respect to the sources is another factor that determines the extent of the error bound in theorem 2.

The parameters π_{ij} $i,j=1,2$ affect the bounds on $\|\tilde{\alpha}^\lambda - \tilde{\alpha}^\lambda_n\|_0$ and $\|\tilde{\beta}^\lambda - \tilde{\beta}^\lambda_m\|_0$. Note that, similar to $\tilde{\chi}_{ij}$, the parameters π_{ij}, $i,j=1,2$ depend on the regularization parameters λ_a, λ_b and on the kernels of the operator $\tilde{\mathcal{T}}$. Similar to the kernels of $\mathcal{T}$, the kernels of $\tilde{\mathcal{T}}$ can be scaled to make π_{ij} almost equal.[23, 24]

5. CONCLUSION

In this work, we presented an error analysis to show the relationship between the error in the simultaneously reconstructed optical absorption and diffusion coefficient images and the discretization of the forward and inverse problems.

We summarized the results of the error analysis in two theorems which provide an insight into the effect of forward and inverse problem discretizations on the accuracy of diffuse optical imaging. These theorems show that the error in the reconstructed optical images due to the discretization of each problem is bounded by roughly the multiplication of the discretization error in the corresponding solution and the solution of the other problem. One important implication of the error bounds is the dependence of the error in the reconstruction of one optical parameter (say the absorption coefficient) on the discretization of the other optical parameter (say the diffusion coefficient).

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