Second Order Stationary Models for $1/f$ Processes

Birsen Yazıcı and Rangasami L. Kashyap

School of Electrical Engineering, Purdue University W. Lafayette, IN 47907, U.S.A.

Abstract - **A subclass of statistically self-similar processes with the correlation structure of the form $E[X(t)X(t\lambda)] = R(\lambda)$ is considered. Spectral decomposition theorem for such processes is stated. Based on the linear scale-invariant system theory, scale invariant autoregressive processes is developed. The proposed models are an intuitive, mathematically simple and practical candidates for modeling $1/f$ type of processes.**

I. INTRODUCTION

The properties of phenomena exhibiting $1/f$ spectrum can be best modeled by *statistically self-similar* processes. A random process is said to be statistically self similar with parameter H if for any $\alpha > 0$ it obeys the scaling relation $x(t) \overset{P}{=} a^{-H}x(at)$ where $\overset{P}{=}$ denotes the equivalence in the sense that all finite joint distributions are the same. In the Gaussian case, self-similarity could be expressed in terms of first and second order properties.

$$E[X(t)] = \alpha^{-H}E[X(\alpha t)]$$

$$E[X(t_1)X(t_2)] = \alpha^{-2H}E[X(\alpha t_1)X(\alpha t_2)] \text{ for any } \alpha > 0. \qquad (1)$$

For $0 < H < 1$, MandelBrot and Van Ness proposed the Fractional Brownian motion (fBm) [1]. Although, fBm has become one of the popular frameworks in modeling variety of $1/f$ processes [2], mathematical difficulty involved, makes the solution of many signal processing problems rather difficult. Since there is a wide range of $1/f$ type of physical phenomena whose self-similarity parameter H is close to zero [3], we proposed and studied the following class of stochastic processes.

$$E[X(t)] = E[X(\alpha t)] = \mu$$

$$E[X(t_1)X(t_2)] = E[X(\alpha t_1)X(\alpha t_2)] \text{ for any } t, t_1, t_2, \alpha \in R^+. \qquad (2)$$

We refer to these processes as *scale stationary* processes. Before pursuing further, we want to note that there is a natural isometry between scale stationary processes and classical wide sense stationary processes. A scale stationary process, $\{X(t)\}_{t\in R^+}$, becomes wide sense stationary under the following coordinate transformation.

$$t \to e^t \quad X(e^t) = \tilde{X}(t) \quad t \in R.$$

The key idea in our development is the definition of the autocorrelation function. It is immediate from (2) that a scale stationary process satisfies $E[X(t)X(t\lambda)] = R(\lambda)$ for any $t, \lambda > 0$. Such a definition of autocorrelation function facilitates the development of the analytical tools and therefore proves to be practically useful. The following theorem is a straightforward corollary of the isometry relation between classical wide sense stationary and scale stationary processes.

Theorem : A function $R(\cdot)$ defined on R^+ is the covariance function of a scale stationary process if and only if there exists a nonnegative measure F on R so that

$$R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda^{j\omega}dF(\omega) \qquad (3)$$

and the process has the following spectral representation

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}t^{j\omega}dZ(\omega) \qquad (4)$$

where the integral is defined in the mean square sense and $Z(\omega)$ is an orthogonal increment process defined on R. The spectral theorem stated above proves that Mellin transform "whitens" the scale stationary processes.

II. SCALE INVARIANT AUTOREGRESSIVE MODELS

In this section, we consider a special class of scale stationary processes to model $1/f$ type of phenomena. Scale stationary processes are intimately related to linear scale invariant system theory. The following ordinary differential equation has been proposed to represent the dynamics of a linear scale invariant system [4].

$$\gamma_N t^N \frac{d^N y(t)}{dt^N} + \gamma_{N-1}t^{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + \gamma_1 t\frac{dy(t)}{dt} + \gamma_0 y(t) = x(t). \qquad (5)$$

It is easy to show that the output of the above linear scale invariant system is a scale stationary process with the autocorrelation function

$$E[y(t)y(t\lambda)] = \sum_{j=1}^{N}\sum_{l=0}^{m_j}\beta_l^j(\ln\lambda)^l\lambda^{-\alpha_j} \qquad \lambda > 1, \qquad (6)$$

if it is driven by the process whose isometry is the classical wide sense stationary "white noise". We refer to these processes scale invariant autoregressive processes (SI-AR) of order N. This class of scale stationary processes are particularly well suited to model long term correlated data. Long term correlations are characterized in several ways : 1) by the empirical $1/f$ spectrum 2) by the sum of the correlations increasing without limit as the lag increases. For scale stationary processes, we shall say that the process is long term correlated if its autocorrelation function satisfies

$$\int_{0}^{\infty}|R(\lambda)|d\lambda \to \infty. \qquad (7)$$

For the SI-AR processes the sum of the correlations is unbounded whenever one of the parameters satifies $0 < \alpha_i < 1$. Synthesized sample paths of the Gaussian SI-AR process justifies the long term correlation criterion chosen for the scale stationary processes.

III. CONCLUSION

A class of self-similar process with parameter 0 is proposed. Basic properties are discussed. An example is presented. Long term correlation criterion is stated. The proposed processes have the advantage of having similar analysis framework as the classical wide sense stationary processes.

REFERENCES

[1] B.B. Mandelbrot and H.W.Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Rev.*, vol 10, pp. 422-36, Oct. 1968.

[2] A.P.Pentland, "Fractal based description of naturalscenes," *IEEE Trans. Patter Anal. Machine Intell.*, vol. PAMI-6, pp.661-674, Now. 1984.

[3] M.S. Keshner, "*1/f* noise," *Proc. IEEE*, vol. 70, pp. 212-218, March 1982.

[4] G.W. Womell,"*Synthesis, Analysis and Processing of Fractal signals, RLE Technical Report* ," No.566, Massachusetts Institude of Technology.

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