OPTIMAL WIENER FILTERING FOR SELF-SIMILAR PROCESSES

Birsen Yazıcı[†], *Meltem Izzetoğlu*[†], *Banu Onaral*[‡] *and Nihat Bilgutay*[†]

[†] Drexel University, Electrical and Computer Eng. Dept., Philadelphia, PA, 19104
[‡] Drexel University, Sch. of Biomed. Eng., Sci. and Health Sys., Philadelphia, PA, 19104

ABSTRACT

In this paper, we formulate an optimal Wiener filter for the restoration of self-similar processes from time-varying and nonstationary distortions. Fourier techniques and classical Wiener filtering are not applicable for the elimination of such distortions. We derive an optimal Wiener filtering algorithm by modeling time-varying degradation within the scale-invariant system theory and nonstationary, long-term correlated noise within the scale stationarity framework. Additionally, we develop fast and efficient estimation techniques for *scale spectra* in Mellin domain for the implementation of the proposed Wiener filter. The utility of the proposed techniques are demonstrated on simulation data.

1. INTRODUCTION

$1/f$ noise has been observed in many natural phenomena such as in electronic device noises, biomedical systems, and communication networks [1, 2, 3]. $1/f$ type of processes, defined by their empirical spectra $S(w) = \frac{\sigma^2}{|w|^\gamma}$, are characterized by statistical self-similarity and long range dependence .

In many engineering applications involving $1/f$ phenomena, the measurements are subject to time-varying degradation and additive nonstationary noise. For example in wireless communication channels, the received signal is typically degraded by time-varying fading channels, and nonstationary, long-term correlated environmental noise [4]. Estimating the transmitted $1/f$ signal from the degraded measurement is a typical inverse filtering problem. However, Fourier based inverse filtering techniques, i.e. the classical Wiener filtering is not optimal for this scenario since it is applicable only for time-invariant degradations and ordinary stationary noise. Hence, there is a need for the efficient restoration of $1/f$ processes distorted by time-varying linear degradations and nonstationary, long term correlated noises.

There have been a number of studies in the literature to develop Wiener filtering technique for the restoration of $1/f$ processes [4, 5, 6]. These studies utilize multiresolutional techniques based on the assumption that the orthogonal wavelet transform "whitens" the $1/f$ processes. In [5, 6] the proposed Wiener filter is utilized as a matched filter to eliminate nonstationary noise. In [4], fractional Brownian motion (fBm) [7] along with multiresolutional techniques are used to develop a Wiener filter technique for the restoration of $1/f$ signals for the time-varying fading channels embedded in additive, nonstationary noise.

Here, we present an alternative approach in developing an optimal Wiener filter for the restoration of $1/f$ phenomena from time-varying linear degradations and nonstationary additive noise. Our approach is based on the mathematical framework of "*scale stationarity*" [1]. This framework offers a number of advantages over the traditional fBm and wavelet based approaches for developing engineering oriented techniques. Scale stationarity framework is based on a powerful notion of stationarity defined as $E[X(t)X(t\lambda)] = R(\lambda)$, $t, \lambda > 0$, in the second order sense. This notion leads to a rich theory and techniques that are parallel to the powerful methods of classical time series analysis, such as ARMA models [1], Kalman filtering techniques [8] and efficient estimation techniques for self-similar processes [9]. It was shown that there is a one-to-one correspondence between the self-similar and scale stationary processes. Furthermore, the central fact in the analysis of scale stationary processes is the existence of a spectral decomposition theorem for the self-similar processes via the Mellin Transform. Our Wiener filtering technique is a direct result of this spectral decomposition theorem. For the implementation of the proposed filter, we developed a non-parametric power spectrum estimation algorithm in the Mellin domain. We show that under optimal data acquisition conditions the proposed filter can be implemented via Fast Fourier Transform (FFT). We discuss a number of issues associated with the implementation of the proposed Wiener filter. These include filter performance in non-optimal sampling and present numerical results.

2. BACKGROUND ON SCALE STATIONARY MODEL

In [1], a linear system satisfying self-similarity property $\tilde{S}\{x(t\lambda)\} = \lambda^{-H}y(t\lambda)$ where $x(t)$ is the input to the system and $y(t)$ is the output of the system is called a Linear Self-Similar (LSS) system with parameter H. When $H = 0$ the linear system is called Linear Scale-Invariant (LSI) system. These systems can be thought of an analog of the LTI systems through the isometry relationship between the additive and multiplicative groups.

In LSS systems, the input-output relationship is defined by the multiplicative convolution operation, $*$, as:

$$y(t) = \tilde{h}(t) * x(t) = t^H \int_0^\infty \tilde{h}(\frac{t}{\lambda})x(\lambda)dln\lambda, \qquad t > 0 \qquad (1)$$

where $\tilde{h}(t)$ is the response of the system to the *unit driving force*, $\tilde{\delta}(t)$ whose properties are given in [10].

It is shown in [1] that Mellin Transform or the Generalized Scale Transform (GST) which is the evaluation of Mellin Transform on the imaginary axis is the appropriate processing technique for LSS systems since it decomposes any signal into logarithmically damped scale-periodic sinusoids given by:

$$\begin{aligned} X(c) &= \int_0^\infty x(t)e^{(-H-jc)lnt}dlnt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^\infty X(c)e^{(H+jc)lnt}dc \end{aligned} \qquad (2)$$

where c is the scale parameter and H is the self-similarity parameter. The GST is the generalized version of the original Scale Transform (ST) introduced in [3].

Statistically self-similar processes with parameter H is defined as follows [7]:

$$x(t) \equiv a^{-H}x(at), \qquad -\infty < t < \infty, \ for\ any\ a > 0 \quad (3)$$

where $\equiv$ denotes equality in finite dimensional probability distributions. A stochastic process is strictly scale stationary if it is self-similar with $H = 0$, i.e. $\tilde{x}(t) \equiv \tilde{x}(at), \quad t > 0$, for any $a > 0$. It is shown in [1] that self-similar processes can be constructed using scale stationary processes as $x(t) = t^H\tilde{x}(t)$ in which case the scale stationary process $\tilde{x}(t)$ is referred as the generating scale stationary process.

Wide Sense Self-similarity (WSS) is defined as [1]:

$$\begin{aligned} &i)E[x(t)] = \lambda^{-H}E[x(\lambda t)], \quad for\ all\ t > 0,\\ &ii)E[x(t)^2] < \infty, \quad for\ all\ t > 0,\\ &iii)E[x(t_1)x(t_2)] = \lambda^{-2H}E[x(\lambda t_1)x(\lambda t_2)]\\ &\qquad\qquad for\ all\ t_1, t_2, \lambda > 0. \end{aligned} \quad (4)$$

The scale autocorrelation function for wide sense self-similar processes is defined as follows

$$E[x(t)x(\lambda t)] = R(\lambda) \quad for\ all\ t, \lambda > 0 \quad (5)$$

where $R(\lambda) = \lambda^H t^{2H} E[\tilde{x}(t)\tilde{x}(\lambda t)] = t^{2H}\Gamma(\lambda)$ and $\Gamma(\lambda)$ is referred as the basic autocorrelation function of the wide-sense self-similar process $x(t)$ satisfying $\Gamma(\lambda) = \lambda^H E[\tilde{x}(t)\tilde{x}(\lambda t)]$ [1].

The scale power spectral density $P(c)$ of a self-similar process is defined by the GST of the basic autocorrelation function:

$$P(c) = \int_0^\infty \Gamma(\lambda)\lambda^{-H-jc} dln\lambda \quad (6)$$

3. WIENER FILTER FOR THE RESTORATION OF SELF-SIMILAR SIGNALS

In this section, we formulate a minimum mean square Wiener filter based on the concepts of basic autocorrelation and the scale power spectral density function. We model the distorted self-similar measurement $y(t)$ with self-similarity parameter H as:

$$y(t) = t^H \int_0^\infty \tilde{h}(t/\lambda)x(\lambda)dln\lambda + w(t) \quad (7)$$

where the linear time-varying distortion is captured via a scale-invariant filter $\tilde{h}(t)$. The self-similar signal $x(t)$ and the additive self-similar noise $w(t)$ are assumed to be uncorrelated with the self-similarity parameters H_x and H, respectively.

The basic cross-correlation $\Gamma_{x\tilde{y}}(\lambda)$ and autocorrelation $\Gamma_{\tilde{y}\tilde{y}}(\lambda)$ functions satisfy the following relationships:

$$\begin{aligned} \Gamma_{x\tilde{y}}(\lambda) &= \tilde{h}(1/\lambda) * \Gamma_{xx}(\lambda)\\ \Gamma_{\tilde{y}\tilde{y}}(\lambda) &= \tilde{h}(\lambda) * \tilde{h}(1/\lambda) * \Gamma_{xx}(\lambda) + \Gamma_{\tilde{w}\tilde{w}}(\lambda) \end{aligned} \quad (8)$$

where $*$ is the multiplicative convolution operation. Then, the scale PSD of the received signal in GST domain is given by

$$P_{\tilde{y}}(c) = |\tilde{H}(c)|^2 P_x(c) + P_{\tilde{w}}(c) \quad (9)$$

Our objective is to develop a linear scale-invariant filter $\tilde{g}(t)$ so that the minimum mean square error between the actual input signal $x(t)$ and the estimated input signal $\hat{x}(t) = \int_0^\infty \tilde{g}(t/\lambda)\tilde{y}(\lambda)dln\lambda$ is minimized, i.e.

$$E[\epsilon(t)] = E[e(t)^2] = E[(x(t) - \hat{x}(t))^2] = minimum \quad (10)$$

Such an optimum filter that minimizes the mean square error can be obtained by using the orthogonality principle [11]

$$E[e(t)\tilde{y}(\lambda)] = E[(x(t) - \int_0^\infty \tilde{g}(t/\lambda')\tilde{y}(\lambda')dln\lambda')\tilde{y}(\lambda)] = 0 \quad (11)$$

which is simply

$$\Gamma_{x\tilde{y}}(\lambda) = \int_{-\infty}^\infty \tilde{g}(\lambda')\Gamma_{\tilde{y}\tilde{y}}(\lambda/\lambda')dln\lambda' \quad (12)$$

Therefore, in terms of the scale PSDs the optimal Wiener filter $\tilde{G}(c)$ in the GST domain can be expressed as:

$$\tilde{G}(c) = \frac{\tilde{H}^*(c)P_x(c)}{|\tilde{H}(c)|^2P_x(c) + P_{\tilde{w}}(c)} \quad (13)$$

It is not very surprising that the optimal Wiener filter in the GST domain for self-similar processes has the same form as the classical Wiener filter derived for the ordinary stationary processes in the Fourier domain. This is due to the duality between the concept of scale stationarity and ordinary stationarity.

Note that for most practical applications, the scale PSD function of the input signal $x(t)$ and the noise $w(t)$ are not available a priori and must be estimated from measurements. In the next section, we will discuss efficient estimation of scale autocorrelation and PSD function.

4. IMPLEMENTATION OF FAST AND EFFICIENT WIENER FILTER VIA FFT

Before developing scale PSD estimation techniques, we need to impose ergodicity conditions on scale stationary processes so that the ensemble average can be replaced by sample (time) averages. By definition, a stochastic process $x(t)$ is ergodic with probability 1, if all its statistical averages can be obtained from a single sample (time) function of the process. Under this assumption, we propose a basic autocorrelation estimate $\hat{\Gamma}_M(\lambda)$ based on the following time average

$$\hat{\Gamma}_M(\lambda) = \frac{1}{2lnM}\lambda^H \int_{1/M}^M \tilde{x}(t)\tilde{x}(t\lambda)dlnt \quad for\ M > 1 \quad (14)$$

where $\tilde{x}(t)$ is the wide sense scale stationary generating function of $x(t)$ [9]. This estimate has the following properties:

* It is unbiased, i.e. $E\{\hat{\Gamma}_M(\lambda)\} = \Gamma(\lambda)$.
* It is consistent with variance $\lim_{M\to\infty} var\{\hat{\Gamma}_M(\lambda)\} \to 0$.

A Shannon-type optimal sampling procedure for self-similar processes was introduced in [9] and [10]. This procedure suggests that sampling intervals must be exponentially spaced in order to recover a scale-band limited continuous signal from its discrete samples [9]. Thus, the continuous time basic autocorrelation estimate in (14) can be expressed as follows:

$$\hat{\Gamma}(s_oT^n) = \frac{1}{N}s_oT^{Hn}\sum_{k=0}^{N-1}\tilde{x}(s_oT^k)\tilde{x}(s_oT^{n+k}) \quad (15)$$

where exponential sampling period is T, initial sampling time is s_o, $N = lnM$ and $\tilde{x}(s_oT^k) = T^{-Hk}x(s_oT^k)$.

Based on (15) and Discrete Time Generalized Scale Transform (DTGST) [10], the periodogram-like estimate of the power spectral density can be developed as follows:

$$\hat{P}_{PER}(C) = \frac{1}{N}|\sum_{n=0}^{N-1}\tilde{x}(s_oT^n)e^{-jCn}|^2 \quad (16)$$

The discussion on the properties of the proposed periodogram-like estimate follows from the results of periodogram based spectral density estimation for the ordinary stationary processes. For example, the bias decreases with increasing data length but the variance does not change and the statistical properties of the periodogram-like estimate can be improved by averaging a set of periodogram-like estimates, in other words by using averaged periodogram-like estimates.

Furthermore, the periodogram-like estimate in (16) can be generalized for an arbitrary window $W(s_oT^n)$ of length N as:

$$\hat{P}_W(C) = \frac{1}{N}\left|\sum_{n=0}^{N-1} \tilde{x}(s_oT^n)W(s_oT^n)e^{-jCn}\right|^2 \qquad (17)$$

The windowing function $W(s_oT^n)$ can be chosen to achieve a balance between variance and bias problems.

Clearly, the Wiener filter in DTGST domain given in (13) and the scale PSD estimate in (16) after exponential sampling are in the exact same forms of the classical Wiener filter in Fourier domain and the periodogram estimate for ordinary stationary processes. Thus, the Wiener filter lends itself to a fast implementation via Fast Fourier Transform (FFT). In other words, fast and efficient processing of the proposed Wiener filter can be easily achieved using FFT techniques as long as the exponential samples of the measured data is obtained. The main problem in the implementation of the proposed Wiener filter lies in the determination of the exponential data samples since in practical applications, most digital data acquisition set-ups are designed based on Fourier transform and uniform sampling. Therefore, for practical applications of the proposed Wiener filter the exponentially sampled discrete data must be obtained from the uniformly sampled measurements.

The Shannon-type or ideal interpolator is derived for ordinary stationary processes. Since we are dealing with processes that are nonstationary in the ordinary sense, we cannot employ ideal interpolator to obtain exponential samples. Here, we use linear interpolation with delay [12] in order to obtain exponential samples from the uniformly sampled ones. In this interpolation technique successive sample points are connected by straight lines and then exponential samples within the appropriate uniform sample intervals are calculated on this line as:

$$\hat{x}(s_oT^n) = x(s_o + (k-1)T_u) + \frac{x(s_o + kT_u) - x(s_o + (k-1)T_u)}{T_u}(s_oT^n - s_o - kT_u)$$
$$for\ s_o + (k-1)T_u \le s_oT^n < s_o + kT_u \qquad (18)$$

where T_u and T are uniform and exponential sampling periods, respectively.

It is easy to show that this interpolator gives unbiased estimate of the true exponential samples where $E[\hat{\tilde{x}}(s_oT^n)] = E[\tilde{x}(s_oT^n)]$. Thus, the mean of the interpolation error is $E[\tilde{x}(s_oT^k) - \hat{\tilde{x}}(s_oT^k)] = 0$.

The variance of the estimation error can be found as:

$$E[(\tilde{x}(s_oT^n) - \hat{\tilde{x}}(s_oT^n))^2] = (2\alpha_{k,n}^2 - 2\alpha_{k,n} + 2)R(1) + 2(\alpha_{k,n} - \alpha_{k,n}^2)R(\frac{s_o + kT_u}{s_o + (k-1)T_u}) - 2\alpha_{k,n}R(\frac{s_o + kT_u}{s_oT^n}) - 2(1 - \alpha_{k,n})R(\frac{s_o + (k-1)T_u}{s_oT^n}) \qquad (19)$$

where $\alpha_{k,n} = (s_oT^n - s_o - kT_u)/T_u$. When $T_u \to 0$ and $T \to 1$ then from the above equation $E[(\tilde{x}(s_oT^n) - \hat{\tilde{x}}(s_oT^n))^2] \to 0$.

5. RESULTS AND DISCUSSION

In this section, we present numerical results of the proposed Wiener filter using a simulation example. We created the wide sense self-similar data for the input signal using a first order Euler-Cauchy system [1] as:

$$t\frac{d}{dt}x(t) + a_1x(t) = t^{H_x}u(t) \qquad (20)$$

where $a_1 = 1$, $H_x = -0.3$ and $u(t)$ is white noise. The additive noise is assumed to be white with $H = -0.2$. The uniformly sampled input signal $x(s_o + kT_u)$ is as given in Figure 1(a). The linear scale-invariant system with self-similarity parameter $H = -0.2$ is represented with unit driving force response $\tilde{h}(s_o + kT_u)$ as given in Figure 1(b). The measured signal $\tilde{y}(s_o + kT_u)$ after linear scale-invariant filter $\tilde{h}(s_o + kT_u)$ and additive white noise $\tilde{w}(s_o + kT_u)$ distortions is shown in Figure 1(c). We created this signal by stretching the time axis exponentially in time [13] and using regular convolution procedure and then sampling the signal uniformly. Here SNR is selected as 15dB. In this simulation example the initial time and uniform sampling period are selected as $s_o = 1$, $T_u = 0.005$. (Let us also mention here that, we assume the signals we are working with in this simulation example are scale band-limited in order to eliminate scale domain aliasing in DTGST domain after exponential sampling [10].) We assumed

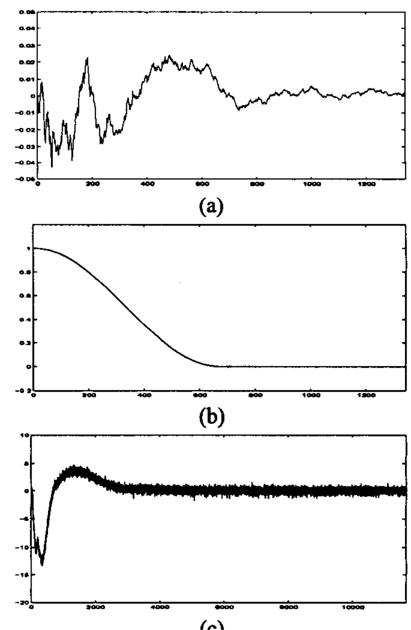

Fig. 1. Uniformly sampled simulated data a) input signal; b) unit driving force response of the scale invariant system; c) measured signal.

that the impulse response is known a priori and the power spectrum estimate for the input signal can be modeled by a first order auto regressive model given in (20). After exponential sampling the power spectrum of the input signal is estimated by taking the absolute value of the FFT of its basic autocorrelation function which is $\Gamma_x(s_oT^m) = \frac{s_oT^{(H_x-a_1)m}}{N^3}\sum_{k=0}^{N-1} s_oT^{2a_1k}\sum_{n=0}^{N-1} s_oT^{(2H_x-2a_1)n}$. Since the additive noise in the measured signal is assumed to be white its power spectrum estimate is $P_{\tilde{w}}(C) = \sigma_{\tilde{w}}^2$ where $\sigma_{\tilde{w}}$ is assumed to be known and selected as 1.

The estimated Wiener filter in DTGST domain is given in Figure 2(a). The exponentially sampled true (solid line) and estimated (dashed line) input signal is presented in Figure 2(b). Here, the linear interpolation technique is used in order to obtain the exponential samples from the uniform ones where the exponential sampling period is selected as $T = 1.004$.

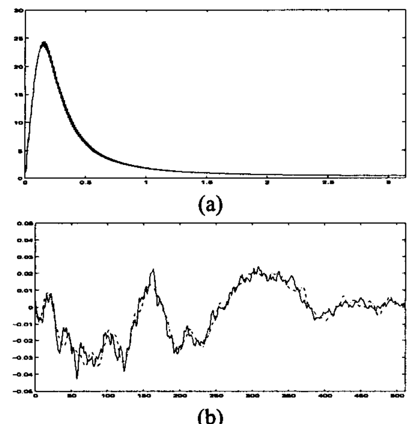

Fig. 2. For exponentially sampled data with known SNR a) estimated Wiener filter in DTGST domain; b) original (solid line) and the estimated (dashed line) input signal.

One possible application area for the proposed Wiener filtering technique can be in matched filtering for communication applications where the environmental noise is usually modeled as nonstationary [4]. When the transmitted signal is self-similar then classical Wiener filter in Fourier domain can not be applied for matched filtering. However, if the long-term correlated nonstationary environmental noise is modeled using the scale stationarity framework, the proposed Wiener filtering can be efficiently used for the denoising of the received signals in communication applications. Another possible application is in the restoration of self-similar textured images from time-varying blur and nonstationary noise distortions. It is shown before that some textured images exhibit self-similar characteristic [14]. To the best of our knowledge, other than estimating the self-similarity parameter for the classification of those self-similar images, there are no other studies for the solution of typical image processing problems, such as their deblurring and denoising. However, these images can be distorted by time-varying blur and nonstationary equipment noise which makes the restoration of such images difficult since classical techniques cannot be employed. In such image deblurring and denoising problems, by formulating the proposed Wiener filtering technique for 2D signals, self-similar images can be efficiently restored.

6. CONCLUSION

In this work, we present a fast and optimal Wiener filtering technique for the restoration of self-similar signals that are distorted by time-varying filters and nonstationary additive noise using the scale stationarity framework and minimum mean squares techniques. An efficient and non-parametric scale power spectral density estimation technique in DTGST domain is formulated. It is also shown here that, with the optimal sampling procedure which is the exponential sampling, the algorithm can be discretized and a fast processing of the proposed Wiener filtering can be achieved using fast Fourier transform techniques. An important implementational issue such as interpolation of the exponential samples of the data from its uniform samples is discussed and an unbiased interpolator is presented. The performance of the proposed Wiener filter is shown on a simulation example.

7. REFERENCES

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