STOCHASTIC DECONVOLUTION OVER GROUPS FOR INVERSE PROBLEMS IN IMAGING

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ABSTRACT

In this paper, we present a stochastic deconvolution method for a class of inverse problems that are naturally formulated as *group convolutions*. Examples of such problems include Radon transform inversion for tomography, radar and sonar imaging, as well as channel estimation in communications. Key components of our approach are group representation theory and the concept of group stationarity. We formulate a minimum mean square solution to the deconvolution problem in the presence of nonstationary measurement noise. Our approach incorporates a priori information about the noise and the unknown signal into the inversion problem, which leads to a natural regularized solution.

1. INTRODUCTION

This work addresses a class of inverse problems that are naturally formulated as *group convolutions*. The concept of group convolution arises naturally in an amazingly varied set of engineering scenarios. These include ambiguity functions in radar and sonar, Radon transform in tomographic image reconstruction, error correcting codes in communications, invariant template matching in pattern recognition, workspace estimation in robotics and texture analysis of solids in mechanics, just to name a few [1]-[10]. Group convolution operation can be viewed as a representation of the input-output relationship of a linear system, which has dynamics invariant under the group composition law. As such, it is the generalization of the classical convolution integral associated with the linear time invariant systems, in which the underlying structure is the additive group. Classical minimum mean square deconvolution techniques rely on the assumption of stationarity and time invariance, and utilize the Fourier transform to develop inverse filtering methods. In this work, we develop a stochastic inverse filtering technique based on the minimum mean square error criteria to solve the convolution integral equation for a class of locally compact groups of both commutative and noncommutative type. This class of groups includes finite, compact and algebraic Lie groups, separable locally compact commutative groups, and majority of well-behaved locally compact groups. Key components of our study are Fourier transforms on groups, and the concept of group stationarity.

The particular focus of our work utilizes noncommutative harmonic analysis over groups to solve convolution integrals in a probabilistic setting. To the best of our knowledge, there are limited number of studies in the literature on this specific topic. In [9] Naparst addressed the deconvolution over the affine group in the context of wideband target density estimation for sonar and radar applications. In [28] Chirikjian developed a regularized solution for a specific convolution integral equation over the Euclidean motion group and demonstrated its application into the kinematic design of binary manipulators [5]. Both of these works address a specific problem in a deterministic setting. Our work addresses the deconvolution problem in a probabilistic setting for a broad range of topological groups that arise naturally in engineering applications. Our minimum mean square formulation also provides a natural regularization to the inverse problem.

2. IMAGE RECONSTRUCTION PROBLEMS

In this section, we formulate radar and sonar inverse scattering problem and Radon transform inversion as deconvolution problems over groups. Apart from the inverse problems in imaging described here, group deconvolutions appear in variety of other engineering applications. For example, the echo model described in equation (2.3a) can be utilized to model wide band wireless communication channels, in which the reflectivity density function is interpreted as the unknown communication channel. In [5] and [24], it was shown that the deconvolution problem over the Euclidean motion group arises in kinematic design of binary robot manipulators and statistics of macromolecules. The results that are developed in our study are directly applicable to these problems.

2.1 Radar and Sonar Image Reconstruction by Wideband and Narrowband Processing

In radar and sonar imaging, the transmitter emits an electromagnetic signal. The signal is reflected off a target and detected by the transmitter/receiver as the echo signal. Assuming negligible acceleration of the reflector, the wideband model of the echo from a point reflector is given as the time delayed and time-scaled replica of the transmitted pulse [13]-[15]:

$$e(t) = \sqrt{s}\, f(st+\tau)\,, \tag{2.1}$$

where f is the transmitted pulse, τ is the time delay, and s is the time scale or Doppler stretch. The term $\sqrt{s}$ is needed if we require, the energy of the echo signal is to be conserved. It is given as $s=(c-\upsilon)/(c+\upsilon)$ where c is the speed of the transmitted signal propagating in a homogenous medium and υ is the radial velocity of the reflector. The narrowband model of the echo from a point reflector is given by

$$e(t) = f(t-\tau)e^{j\omega t}\;, \tag{2.2}$$

where f is the transmitted pulse, τ is the time delay, and ω is the frequency or Doppler shift.

It is often desirable to image a dense group of reflectors, which may be several objects or a single object distributed in size. This dense group of reflectors is then described by ***a reflectivity density function***. The received signal is modeled as a weighted average [7]-[9]. For wideband signals, it is given as

$$e_W(t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} S_W(s,\tau)\frac{1}{\sqrt{s}} f\left(\frac{t-\tau}{s}\right)\frac{ds}{s^2}d\tau, \qquad (2.3a)$$

where $S_W(s,\tau)$ is the wideband reflectivity density function associated with each time delayed and time scaled version of the transmitted signal. The narrowband model is given by

$$e_N(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_N(\omega,\tau) f(t-\tau)e^{j\omega t}d\tau d\omega, \qquad (2.3b)$$

where $S_N(\omega,\tau)$ is the narrowband reflectivity density function associated with each time delayed and frequency shifted version of the transmitted signal.

The goal in radar and sonar imaging is to estimate $S_W(s,\tau)$ and $S_N(\omega,\tau)$ given the transmitted and the received signals. Typically, the received echo in a radar or sonar system is very weak due to clutter and system noise. Therefore, the detection at the receiver side is performed by matched filtering, which amounts to correlating the received echo with the transmitted pulse. When the two echo models described in (2.3a) and (2.3b) are inserted into the narrowband and wideband correlation receivers, the resulting outputs are expressed as group convolution integrals. In the case of wideband processing, it is the *affine group* convolution, and in the case of narrowband processing, it is the *Heisenberg group* convolution. We will demonstrate that the deconvolution techniques described here provides a minimum mean square estimate of the wideband and narrowband reflectivity density functions.

2.2. Radon transform inversion for tomographic imaging

The Radon transform and its generalizations play an important role in the tomographic image reconstruction problems in fields as diverse as medical imaging, radar target shape estimation, and radio astronomy. This problem is equivalent to computing the inverse Radon transform. Here, we show that Radon transform inversion can be posed as a deconvolution problem over the Euclidean motion groups.

In X-ray computed tomography, an X-ray beam with known energy is sent trough the object and the attenuated X-ray is collected by an array of collimated detectors. The attenuation in the final X-ray beam provides a means of determining the integral of the mass density of the object along the path of the X-ray. In 2D, the relationship between the mass density along the path and the attenuation at angle θ, and radius r, is given by the following Radon transform:

$$p(r,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(r - x\cos\theta - y\sin\theta)dxdy, \qquad (2.4)$$

where δ is the Dirac delta function. Similarly, in PET, SPECT and synthetic aperture radar (SAR), the line projections and the attenuation coefficients are related by the Radon transform.

The Euclidean motion group is the semi direct product of the rotation group $SO(N)$ and the additive group in $\mathbb{R}^N$. Redefining, $\tilde{f}(R,r) = f(r)\delta(R)$, where R is the two dimensional rotation matrix, $r = [x,y]^T$, and $k(Q,\tau) = \delta(e\cdot\tau)$, and e is the vector in x direction, it can be shown that the Radon transforms and its generalizations can be written as an Euclidean group convolution in the following form:

$$p(R,r) = \int_{SE(2)} k(g\circ h^{-1})\tilde{f}(h)dh, \qquad g,h\in SE(2),$$

$$= \int_{SE(2)} k(RQ^{-1}, r - RQ^{-1}\tau)\tilde{f}(Q,\tau)d(Q,\tau),\ g=(R,r) \quad (2.5)$$

where $SE(2)$ denotes the two dimensional Euclidean motion group, $\circ$ denotes the group composition law and, $d(Q,\tau)$ denotes the invariant measure on the Euclidean motion group. Our approach provides a minimum mean square solution to this problem in the presence of nonstationary measurement noise.

3. THE CONCEPT OF CONVOLUTION AND FOURIER ANALYSIS ON GROUPS

We shall indicate a group by G and its elements by $g,h,\ldots$. The group composition law will be written by $g\circ h$, and we shall use e for the identity element, for which $e\circ g = g\circ e = g$ for all elements g of G. We shall indicate inverse elements by g^{-1} so that $g^{-1}\circ g = g\circ g^{-1} = e$ for all elements g of G.

Let $L^2(G,dg)$ denote the Hilbert space of all complex valued, square integrable functions on a group G, and let x and f be two finite energy signals, then the convolution of x and f are defined as [19]:

$$(x*f)(g) \equiv \int_G x(h)f(h^{-1}\circ g)dh. \qquad (3.1)$$

For a square integrable function on arbitrary Lie group, $f(h^{-1}\circ g)$ is called a translation in the same sense that $f(t-\tau)$ is a translation of a function defined on the real line. Note that in general, the left and right translations are not equal.

From the perspective of group representation theory, Fourier analysis deals with characterization of unitary representations as a direct sum (integral) of irreducible unitary group representations. This characterization is then utilized to define the Fourier transformation, which has the property of mapping convolution integral to a multiplication in the transform domain. However, for an arbitrary group, such a characterization is far from unique. Nonetheless, considerably satisfactory results can be obtained if some restrictions are imposed on the group structure. It was shown that if the group G is a separable, locally compact group of Type I, [20] unique characterizations of the unitary representations can be obtained in terms of the irreducible unitary representations of the group. Fortunately, most of the noncommutative groups of interest in engineering applications, such as the Euclidean motion groups, affine group, Heisenberg group fall into this category and admit unique Fourier decomposition of functions.

Let $U(g,\lambda)$ be the λ th irreducible unitary representation of a separable locally compact group of Type-I. Then, the operator valued Fourier transform on G maps each f in $L^2(G,dg)$ to the family $\{\hat{f}(\lambda)\}$ of bounded operators, where each $\hat{f}(\lambda)$ is defined by

$$\mathcal{F}(f)(\lambda)=\hat{f}(\lambda)=\int_G f(g)U(g^{-1},\lambda)dg\,, \tag{3.2}$$

The collection of all λ values is denoted by $\hat{G}$ and is called the dual of the group G. The collection of Fourier transforms $\{\hat{f}(\lambda)\}$ for all $\lambda\in\hat{G}$ is called the spectrum of the function f.

An important property of the operator valued Fourier transform, reminiscent of the classical Fourier transform over the reals, is that the group convolution becomes operator multiplication on the Fourier side, more precisely,

$$\mathcal{F}(f_1*f_2)(\lambda)=\mathcal{F}(f_2)(\lambda)\mathcal{F}(f_1)(\lambda)\,. \tag{3.3}$$

For locally compact commutative groups, all irreducible representations of the group are one-dimensional. Hence, the Fourier spectrum is scalar valued and appears similar to the classical Fourier transform. The inversion formula, in this case, is given by

$$f(g)=\int_{\hat{G}}\hat{f}(\lambda)U(g,\lambda)d\lambda\,. \tag{3.4}$$

4. GROUP STATIONARY PROCESSES

These processes are nonstationary in the classical sense but exhibit invariance under the right or left regular transformations of the group. The author demonstrated in her earlier work that the special cases of group stationary processes for the multiplicative and affine group form suitable mathematical frameworks for modeling and analysis of self-similar and multiscale processes [14]-[18].

Second order group stationarity is a weaker condition in which, only the second order statistics of the random process is required to be invariant under the right or left regular transformations of the group. Loosely speaking, second order group stationary processes obey the following structure [21]-[23]:

$$E\left[X(g)\overline{X(h)}\right]=\mathrm{R}(g\circ h^{-1}),\quad g,h\in G \tag{4.1}$$

where R is a positive definite function defined on the group

The central fact in the analysis of group stationary processes is the existence of spectral decomposition, which is facilitated by the Fourier theory on groups. For compact groups, left group stationary processes admit the following spectral decomposition:

$$X(g)=\sum_{\lambda\in\hat{G}} trace\big(U(g,\lambda)Z(\lambda)\big) \tag{4.2a}$$

and

$$R(g)=\sum_{\lambda\in\hat{G}} trace\big(U(g,\lambda)F(\lambda)\big)\ \text{ with }\ \sum_{\lambda\in\hat{G}} trace\big(F(\lambda)\big)<\infty \tag{4.2b}$$

where $R(\cdot)$ is the autocorrelation function of the process, $U(g,\lambda)$ is the λ th irreducible unitary representation of the group G with dimension $d(\lambda)$, $Z(\lambda)$ is a random matrix of dimension $d(\lambda)$ and $F(\lambda)$ is a bounded Hermitian positive definite operator over $\hat{G}$. Let

$$S(\lambda)\equiv\mathcal{F}(\mathrm{R})(\lambda)=\int_G dg\mathrm{R}(g)U(g,\lambda)\,. \tag{4.3b}$$

We shall refer to S as the *spectral density function* of a group stationary process. This is a natural generalization of the spectral density function defined for ordinary stationary processes.

5. WIENER FILTERING OVER GROUPS

In this section, we shall introduce a novel stochastic deconvolution method over groups based on the Fourier theory of topological groups. We shall pose the deconvolution problem within the framework of minimum mean square error prediction, and develop a Wiener filtering method to estimate unknown signals from noisy measurements. While our results will be stated for the locally compact groups of Type I, special cases of finite, compact, and commutative groups can be easily deduced from the main result.

Let the forward model that relates the measurements y and the unknown function x are given by the following convolution integral:

$$y(g)=\int_G x(h)f(h^{-1}\circ g)dh+n(g)\,, \tag{5.1}$$

where $f:G\to C$ is a known complex valued, square summable function, n is an additive noise indexed by the group G, taking values in the field of complex numbers C. Without loss of generality, we assume that $E[x(g)]=E[n(g)]=0$. Then the classical linear Wiener problem of recovering x from noisy measurements y can be posed as follows: Find the linear filter $W:G\times G\to C$ such that the least squares error variance

$$J(\varepsilon_w)=\int_G E\left[|\varepsilon_w(g)|^2\right]dg \tag{5.2a}$$

is minimized where

$$\varepsilon_w(g)=\int_G W(g,h)y(h)dh-x(g)\,, \tag{5.2b}$$

and dh is the left Haar measure on the group G. Note that it is implicit by the equation (5.2a) that the filter W is required to be doubly square summable. Then, the solution to the above linear least squares problem is provided by the following Wiener-Hopf type equation:

$$\int_G W(g,s)\mathrm{R}_{yy}(s,h)ds=\mathrm{R}_{xy}(g,h)\,, \tag{5.3}$$

where $\mathrm{R}_{yy}(s,h)=E\left[y(s)\overline{y(g)}\right]$ and $\mathrm{R}_{xy}(g,h)=E\left[x(g)\overline{y(h)}\right]$. Alternatively,

$$(\tilde{f}*\mathrm{R}_x)(g)=\int_G \mathrm{R}_y(p)\tilde{W}(p^{-1}\circ g)dp\,, \tag{5.4a}$$

where

$$\mathrm{R}_x(g)=\mathrm{R}_{xx}(g,e)=E\left[x(g)\overline{x(e)}\right]$$

$$\mathrm{R}_y(g)=\mathrm{R}_{yy}(g,e)=E\left[y(g)\overline{y(e)}\right], \tag{5.4b}$$

$$\tilde{W}(g,e)=W(g^{-1},e)\ \text{and}\ \tilde{f}(g)=\overline{f(g^{-1})}\,. \tag{5.4c}$$

The following theorem states an explicit solution for the Wiener-Hopf equation, which in turn leads to the linear least squares recovery of the signal x.

Theorem : Let G be a separable locally compact group of Type-I, and $x(g)$ and $n(g)$, g of G, be two zero mean left group stationary processes, referred to as signal and noise, respectively.

Assume that the measurements obey the following convolution integral and noise model:

$$y(g)=\int_G x(h)f(h^{-1}\circ g)dh+n(g) \tag{5.5a}$$

and

$$E\left[x(g)\overline{n(g)}\right]=0\,, \tag{5.5b}$$

where the filter f belongs to $L^2(G,dg)$. Then, the optimum linear least squares deconvolution filter W_{opt}, minimizing (5.2a) is left group invariant and the estimate of the signal is given as a convolution integral

$$\hat{x}(g)=\int_G y(h)W_{opt}(h^{-1}\circ g)dh\,. \tag{5.5c}$$

The Fourier transform of the optimal filter W_{opt} is given as follows:

$$\hat{W}_{opt}(\lambda)=S_x(\lambda)\hat{f}^{\dagger}(\lambda)\left[\hat{f}(\lambda)S_x(\lambda)\hat{f}^{\dagger}(\lambda)+S_n(\lambda)\right]^{-1} \tag{5.5d}$$

where $\lambda\in\hat{G}$. Here, $\hat{f}$ is the Fourier transform of the convolution filter f, and, $\hat{f}^{\dagger}$ denotes the adjoint of the operator $\hat{f}$. S_x and S_n are operator valued spectral density functions of the signal and noise, respectively. The spectral density function of the least square error between the signal and its filtered estimate is given as

$$S_e(\lambda)=\left(I-\hat{W}_{opt}(\lambda)\hat{f}(\lambda)\right)S_x(\lambda) \tag{5.5e}$$

where I denotes the identity operator.

Proof: See [25].

In [15], the author demonstrated the utility of proposed deconvolution method in designing a Wiener filter for self-similar processes in which the underlying group is the multiplicative group. Detailed numerical studies showed that proposed method is effective in signal recovery embedded in self-similar noise.

6. CONCLUSIONS

In this paper, we have shown how group representation theory can be utilized to solve a class of inverse problems formulated as group convolutions. Classical deconvolution problem in the linear time invariant systems and signals framework is a special case of the general deconvolution problem, in which the underlying structure is the additive group. We broaden the classical framework to include wide range of groups of both commutative and noncommutative type. These include finite, compact, and a large class of well-behaved locally compact groups that arise naturally in engineering applications. We developed a minimum mean square solution for the deconvolution problem using the group representation theory and the concept of group stationarity. The methodology described here can be implemented efficiently using the fast Fourier algorithms available for a variety of groups [26]-[27]. The research in this direction is on going and will be reported in the future.

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