SIGNAL MODELING AND PARAMETER ESTIMATION FOR 1/f PROCESSES USING SCALE STATIONARY MODELS*

Birsen Yazıcı
General Electric Corporate
Research and Development Center
P.O. Box 8, Schenectady NY 12301

Rangasami L. Kashyap
Purdue University
School of Electrical Engineering
W. Lafayette IN 47907

ABSTRACT

In our previous work, we proposed two classes of self-similar models for $1/f$ processes which we referred to as scale stationary and p-self similar models. We introduced a new mathematical framework and several new concepts, such as periodicity, autocorrelation, and spectral density function to analyze scale stationary and p-self similar processes. In particular, we introduced a family of finite parameter scale stationary models, similar in spirit to ARMA models by which any scale stationary processes can be approximated. In this work, we utilized the framework of scale stationary processes and introduced novel methods of $1/f$ signal modeling and parameter estimation techniques. These include a sampling theorem, a mathematically consistent estimator for the self-similarity parameter, an unbiased estimator for the scale autocorrelation function and a maximum likelihood estimator for scale stationary autoregressive models. Results from our study suggest that scale stationary processes provide a powerful framework for practical $1/f$ signal processing problems.

I. INTRODUCTION

In many signal processing problems, observations are assumed to be stationary because it immediately provides a set of mathematical tools and concepts, such as spectral density, autocorrelation function, efficient parameter estimation methods etc., by which one can approach to the problem. However there is a broad range of physical phenomena which do not exhibit statistical invariance with respect to time shifts. An important class of such physical phenomena is known as $1/f$ or fractal processes [1],[4]. Unlike the ordinary ARMA models, these processes exhibit statistical invariance with respect to time scales and long term correlations. Despite its wide spread occurrence, $1/f$ signal processing have received little attention in signal processing literature. This has been partly due to the mathematical intractability of the proposed fractal models. In [2] and [3], we proposed a class of mathematically simple, intuitive and practical self similar processes and an alternative mathematical framework to model and analyze $1/f$ phenomena. The foundations of the proposed class is based on the extensions of the basic concepts of classical time series analysis, in particular on the notion of stationarity with respect to time scales. We referred to these processes as *scale stationary* and *p-self similar* processes. We introduced a class of finite parameter self similar processes, similar in spirit to ARMA models, by which an arbitrary self similar process can be approximated. We introduced new tools and concepts, such as periodicity, autocorrelation, and spectral density function for scale stationary processes by which practical signal processing schemes can be developed.

In this paper, we utilize the mathematical framework of scale stationary processes and develop efficient signal modeling and parameter estimation methods for $1/f$ processes. The proposed signal modeling and parameter estimation methods involve a sampling theorem for scale stationary processes, estimation of the self similarity parameter, the autocorrelation function, and first order self similar autoregressive model parameters.

II. WIDE SENSE SCALE STATIONARY AND P-SELF SIMILAR PROCESSES

In this section, we shall briefly review the properties of p-self similar processes and give examples of practical interest. For a more complete discussion of the topic, we refer the reader to [2],[3].

Definition 2.1 : A random process $\{X(t), t>0\}$ is called wide sense p-self similar with parameter H if it satisfies the following conditions:

i) $E[X(t)] = \lambda^{-H} E[X(\lambda t)] \quad t, \lambda > 0.$

ii) $E[X^2(t)] < \infty \quad t > 0.$

iii) $E[X(t_1)X(t_2)] = \lambda^{-2H} E[X(\lambda t_1)X(\lambda t_2)] \quad t_1, t_2, \lambda > 0.$ □

For $H = 0$, we refer p-self similar processes as ***wide sense scale stationary*** processes. Unless otherwise stated for the rest of the paper, we shall use the terms p-self similarity and scale stationarity in the second order sense. Before pursuing further, we want to point out that there is an isometry relationship between p-self similar and scale stationary processes. Given any p-self similar process,

* This work was partly sponsored by the Office of Naval Research under contract ONR-N00014-91-J-4126.

0-7803-3192-3/96 $5.00©1996 IEEE

$\{X(t), t>0\}$, with parameter $H \neq 0$, there is a scale stationary process, $\{\tilde{X}(t), t>0\}$, such that

$$X(t) = t^H \tilde{X}(t), \quad t>0. \tag{2.1}$$

Thus $\{\tilde{X}(t), t>0\}$ is appropriately referred to as the *generating scale stationary process*, and t^H as the *trend term* of a p-self similar process $\{X(t), t>0\}$. It immediately follows from Definition 2.1 *iii)* that for $H=0$

$$E[X(t)X(\lambda t)] = R(\lambda), \quad t,\lambda>0 \tag{2.2}$$

where R is called the *S-autocorrelation* function of a scale stationary process. For $H \neq 0$, (2.2) becomes

$$E[X(t)X(\lambda t)] = t^{2H}\lambda^H R(\lambda), \quad t,\lambda>0 \tag{2.3}$$

where R is the S-autocorrelation of the underlying generating process and $\lambda^H R(\lambda)$ is referred to as the *basic autocorrelation* function of a p-self similar process. Both S-autocorrelation and basic autocorrelation functions have properties similar to the ordinary shift based autocorrelation function.

Example 1: The first example is the well-known fractional Brownian motion [4]. The basic autocorrelation function of fBm, $\{B_H(t), t>0\}$, is given by

$$E[B_H(t)B_H(\lambda t)] = t^{2H}\frac{\sigma^2}{2}\left\{1+\lambda^{2H}-|1-\lambda|^{2H}\right\}, \quad t,\lambda>0 \tag{2.4a}$$

where H is a parameter between 0 and 1 and σ^2 is a function of H. The S-autocorrelation function of the generating process, $\{\tilde{B}_H(t), t>0\}$, of fBm is given by

$$E[\tilde{B}_H(t)\tilde{B}_H(\lambda t)] = \sigma^2\left\{\cosh(H\ln\lambda) - |\sinh(1/2\ln\lambda)|^{2H}\right\}, \ \lambda>0.$$

Example 2: An important class of p-self similar processes is defined by the generalized Euler-Cauchy system. These processes are called self-similar autoregressive (SS-AR) processes. Symbolically, an Nth order SS-AR process with parameter H, $\{y(t), t>0\}$, can be represented by the following time varying, scale invariant ordinary differential equation:

$$\alpha_N t^N \frac{d^N}{dt^N}y(t)+\ldots+\alpha_1 t\frac{d}{dt}y(t)+\alpha_0 y(t) = \beta_0 t^H x(t), \quad t>0 \tag{2.5a}$$

where $\{x(t), t>0\}$ can be interpreted as a scale stationary white noise process. The basic autocorrelation function of an Nth order SS-AR process is given by

$$\Gamma(\lambda) = \begin{cases} \sum_{j=0}^{n}\sum_{i=0}^{m_j} a_{ij}^2(\ln\lambda^{-j})\lambda^{H+b_{ij}} & 0<\lambda\leq 1 \\ \sum_{j=0}^{n}\sum_{i=0}^{m_j} a_{ij}^2(\ln\lambda^{j})\lambda^{H-b_{ij}} & \lambda>1 \end{cases}. \tag{2.5b}$$

It was shown in [2] that under some regularity conditions, any p-self similar process can be approximated in some sense by a finite order SS-AR process.

III. SAMPLING THEOREM FOR SCALE STATIONARY PROCESSES

In practical applications, often times the continuous time data is not available. Therefore, it is important to develop sampling methods by which the statistical structure of the continuous data can be recovered from its discrete samples. For the so called band limited shift stationary processes, it is well known that the continuous process can be recovered from its discrete samples recorded at equally spaced intervals. In this section, we develop a similar sampling scheme for scale stationary processes. This result enables us to fit continuous time scale stationary models to discrete observations.

Theorem 3.1: Let $\{X(t), t>0\}$ be a scale stationary process satisfying

$$\int_0^\infty \lambda^{-j\omega-1}R(\lambda)d\lambda = 0 \text{ for some } |\omega|>\Omega, \text{ then}$$

$$R(\lambda) = \sum_{n=-\infty}^{\infty} R(s_0^n)\operatorname{sinc}(\Omega\ln(\lambda/s_0^n)), \quad \lambda>0 \tag{3.1a}$$

and

$$X(T\lambda) = X(T)\operatorname{sinc}(\Omega\ln\lambda) + \sum_{n=1}^{\infty} X(Ts_0^n)\left\{\operatorname{sinc}(\Omega\ln(\lambda/s_0^n)) + \operatorname{sinc}(\Omega\ln(\lambda/s_0^{-n}))\right\} \quad T,\lambda>0. \tag{3.1b}$$

where $\operatorname{sinc}(x) = \sin(x)/x$, $s_0 = e^{\pi/\Omega}$, and the limit in (3.1b) is in the mean square sense.

Proof : See [5]. □

The condition of the theorem can be interpreted as band limitedness for scale stationary processes. The formula (3.1) states that any band limited scale stationary process is completely determined by its sampled values, $X(Ts_0^n)$, at exponentially spaced sampling intervals $\{Ts_0^n, n=0,\pm1,\ldots\}$. As in the case of shift stationary processes, not all scale stationary processes are band limited. Nevertheless, many practical situations correspond to an effective band limitedness, since low pass filtering is a common procedure for signal conditioning.

IV. PARAMETER ESTIMATION

There are three major results on which we base our parameter estimation method: *i)* the Equation (2.1) which states that self-similar processes are trended scale stationary processes, *ii)* the exponential sampling scheme and the interpolation formula, and *iii)* the concept of S-autocorrelation function. In our framework, self-similarity parameter controls nothing but the trend of the process, while the generating process forms the stochastic part. Hence the idea is to first remove the trend term using (2.1), and next to analyze the underlying stochastic process using the concept of S-autocorrelation function.

Least Squares Estimation of the Self Similarity Parameter

Let $\{X(t), t>0\}$ be a wide sense p-self similar process, and let $\{\tilde{X}(t), t>0\}$ be its generating process with mean μ. Define

$$\overline{X}(t)=\frac{1}{\mu}\tilde{X}(t), \quad t>0. \tag{4.1a}$$

Thus, $\{X(t),t>0\}$ is scale stationary with unit mean. By Equation (2.1)

$$\log|X(t)|=H\log t+\log|\mu|+\log|\overline{X}(t)|, \quad t>0. \tag{4.1b}$$

To simplify the notation, let us rewrite (4.1b) in vector form.

$$Y(t)=\theta^T Z(t)+\log|\overline{X}(t)|, \quad t>0 \tag{4.1c}$$

where $Y(t)=\log|X(t)|$, $Z^T(t)=[\log(t),1]$, $\theta^T=[H,\log|\mu|]$. and $\gamma=\log|\mu|$. Now assume that N observations starting from time T at equally spaced intervals of length Δ are available. Define $y(k)=Y(T+k\Delta)$, $z(k)=Z(T+k\Delta)$, for $k=0,\ldots,N-1$. Then, the linear least squares estimate of the parameter θ is given by

$$\hat{\theta}=\left[\sum_{k=0}^{N-1}z(k)z^T(k)\right]^{-1}\left[\sum_{k=0}^{N-1}z(k)y(k)\right]. \tag{4.2}$$

Theorem 4.1 : Let $\{\tilde{X}(t),t>0\}$ be the generating process of a p-self similar process with mean μ. Assume that the initial observation time T is known. Then $\hat{H}$ is an unbiased estimate of H. $\hat{\gamma}$ is unbiased estimate of γ only if $E\left[\log|\tilde{X}(t)|\right]=\log|\mu|$. Moreover if $E\left[\left(\log|\tilde{X}(t)|\right)^2\right]=\beta<\infty$, both $\hat{H}$ and $\hat{\gamma}$ are consistent in the mean square sense.
Proof : See [5]. □

An alternative approach is to sample the data at exponentially spaced intervals at $\{Ts_0^k, k=0,1,\ldots,N-1\}$.

We synthesize 100 realizations of the fBm and the first order SS-AR process using the Gaussian number generator, the S-autocorrelation function and (2.1). Two sets of experiments with various noise levels and parameter values are performed. In the first set, the data is synthesized at uniform intervals with $\Delta=1.5$, and in the second set at exponentially varying intervals with $s_0=1.5$. The results are tabulated in Table I and II for each model. The method gives satisfactory results even for noisy observations. In general, exponential sampling approach appears to give better results than the uniform sampling approach.

Estimation of the S-Autocorrelation Function

Given the estimates $\hat{H}$ and $|\hat{\mu}|$, we form the following process:

$$\hat{X}(k)=\frac{\tilde{X}(T+k\Delta)}{(T+k\Delta)^{\hat{H}}|\hat{\mu}|}, \quad k=0,\ldots,N-1. \tag{4.3}$$

We assume that the resulting process (4.3) obeys a scale stationary parametric model and estimate its S-autocorrelation function. We propose three estimates for the S-autocorrelation function. The first estimate is based on exponential sampling, the second one is based on continuous time observations, and the last one is derived from the continuous time estimate, and based on uniform sampling.

$$\hat{R}_1(s_0^n)=\frac{1}{N-n}\sum_{k=0}^{N-1-n}X(Ts_0^k)X(Ts_0^{n+k}), \quad n=0,\ldots,N-1. \tag{4.4a}$$

Assuming that the process is band limited in scale, we can employ the interpolation formula, (3.1a), to obtain a continuous time of the S-autocorrelation function.

$$R_1(\lambda)=R_1(1)\mathrm{sinc}(\Omega\ln(\lambda))+\sum_{n=1}^{N-1}\hat{R}_1(s_0^n)\left\{\mathrm{sinc}(\Omega\ln(\lambda/s_0^n))+\mathrm{sinc}(\Omega\ln(\lambda/s_0^{-n}))\right\}, \quad \lambda\geq 1. \tag{4.4b}$$

Now, let us assume that continuous time observations are available from time T_1 to T_2. Consider the following estimate:

$$\hat{R}_2(\lambda)=\frac{1}{\ln(T_2/\lambda)-\ln(T_1)}\int_{T_1}^{T_2/\lambda}X(t)X(\lambda t)\frac{dt}{t}, \quad 1\leq\lambda\leq T_2/T_1. \tag{4.5}$$

To implement (4.5), the integral must be replaced by a sum through an appropriate approximation. In our numerical study, we use the following discrete approximation to (4.5):

$$\hat{R}_3(\lambda)=\frac{1}{\ln(T_2/\lambda)-\ln(T_1)}\sum_{k=0}^{\lfloor T_2/\lambda\Delta-T_1/\Delta\rfloor}X(T_1+k\Delta)X(\lambda(T_1+k\Delta))\ln\left(\frac{T_1+(k+1)\Delta}{T_1+k\Delta}\right)_{1\leq\lambda\leq T_2/T_1} \tag{4.6}$$

where $\lfloor x\rfloor$ stands for the largest integer smaller or equal to x. Note that λ has to satisfy $\lambda(T_1+k\Delta)=T_1+n\Delta$ for some integer $0\leq n\leq\lfloor T_2/T_1\rfloor$. This limits the range of λ on which the autocorrelation function can be estimated. One way to extend the range is to interpolate the observations using the Equation (3.1b) at the desired points.

It is easy to check that all three estimates are unbiased. To check the practical value of the proposed estimators, we estimated the S-autocorrelation function of the generating process of fBm. First, we computed $\hat{R}_1$ at exponentially spaced points based on a single realization of the model. Next, we interpolated $\hat{R}_1$ using the Equation (4.4b). The result is illustrated in Figure 2.

Estimation of the First Order SS-AR Model Parameters

In this section, we shall study estimating the parameters of the first order SS-AR model. We begin by deriving the likelihood functions.

Let X $N\times 1$ vector of samples of the generating process of the first order SS-AR taken at instances $T+\Delta(i-1)$, $i=1,\ldots,N$, $T>0$. Observe that the probability density function of X is given by

$$\wp(\Theta)=\frac{1}{(2\pi)^{N/2}|\Sigma(\Theta)|^{1/2}}\exp\left\{-\frac{1}{2}X^T\Sigma(\Theta)^{-1}X\right\} \tag{4.7}$$

where Θ denotes the parameters $[\sigma^2 \quad \nu]$ for the first order SS-AR model. $|\Sigma(\Theta)|$ denotes the determinant of the autocorrelation matrix $\Sigma(\Theta)$. The entries Σ_{ij}, $i,j=1,\ldots,N$ of $\Sigma(\Theta)$ are given by

$$\Sigma_{ij}=\begin{cases}\sigma^2\lambda_{ij}^{-\nu} & \lambda_{ij}\geq 1\\ \sigma^2\lambda_{ij}^{\nu} & else\end{cases} \tag{4.8}$$

where $\lambda_{ij}=(T+\Delta(i-1))/(T+\Delta(j-1))$. The equivalent log-likelihood function can be expressed as

$$\ell(\Theta) = -\frac{N}{2}\log(\sigma^2) - \frac{1}{2}\log\left|\tilde{\Sigma}(H)\right| - \frac{1}{2\sigma^2}X^T\tilde{\Sigma}(H)^{-1}X \quad (4.9)$$

where $\Sigma(\Theta) = \sigma^2\tilde{\Sigma}(H)$. The maximum likelihood estimate $\hat{\Theta}$ of Θ is given by $\hat{\Theta} = \text{Argmax}_{\Theta}\,\ell(\Theta)$. By taking the derivative of the likelihood function with respect to σ^2, setting it to zero and verifying that the second derivative is negative at the value of the estimate, we obtain the following estimate of σ^2:

$$\hat{\sigma}^2 = \frac{1}{N}X^T\tilde{\Sigma}^{-1}(H)X. \quad (4.11)$$

Inserting (4.11) into (4.9) yields the following equivalent likelihood function which has to be maximized over the parameter H:

$$\tilde{\ell}(H) = -\frac{N}{2}\log\left(X^T\tilde{\Sigma}^{-1}(H)X\right) - \frac{1}{2}\log\left|\tilde{\Sigma}(H)\right| \quad (4.12)$$

We performed a large number of experiments to demonstrate the basic functionality and the viability of the maximum likelihood estimator. A part of the results is tabulated in Table III. Estimation results from 30 trials are averaged to obtain estimate statistics. The length of the observations is chosen to be 64. We concluded that the empirical results agree with the theoretical properties of the maximum likelihood estimator. However, the computational complexity of covariance matrix inversion prohibits its use for long data records.

True H	Est H	Var H	Est |M|	Var |M|
0.1	0.0998	0.0000	3.7719	0.0491
0.3	0.2991	0.0000	3.9446	0.0345
0.5	0.5008	0.0002	3.6975	0.2084
0.9	0.9019	0.0001	4.3451	0.0555

Table I
Estimates of the self-similarity parameter of fBm using exponential sampling. $M = -4$, SNR = 10dB.

True ν	Est ν	Var ν	Est σ^2	Var σ^2
0.2	0.1850	0.0030	0.9986	0.0328
0.8	0.7700	0.0228	1.1581	0.0024
2.0	2.1008	0.0067	1.0926	0.0045

Table III
Maximum likelihood estimates of the 1st order SS-AR model. True value of σ^2 is equal to 1.

V. CONCLUSION

In this paper, we introduced a signal modeling and parameter estimation method for $1/f$ phenomena in the mathematical framework of scale stationary and p-self similar processes. We developed a sampling scheme for scale stationary processes, and outlined estimation techniques for the self similarity parameter and S-autocorrelation function. The experimental results show that proposed $1/f$ signal modeling methods are very powerful.

REFERENCES

[1] M.S. Keshner, "$1/f$ noise," Proceedings *of IEEE*, vol. 70, pp. 212-218.

[2] Birsen Yazıcı and R.L. Kashyap, "A Class of Second Order Self Similar Processes for $1/f$ Phenomena," in review . *Transactions of IEEE Signal Processing*.

[3] Birsen Yazıcı and R.L. Kashyap, "A Class of Second Order Self Similar Processes for $1/f$ Phenomena," *Proceedings of ICASSP 95*. Vol 2, pp. 1573-1576.

[4] B.B. Mandelbrot and H.W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, pp. 422-436.

[5] Birsen Yazıcı and R.L. Kashyap, "Signal Modeling and Parameter Estimation for $1/f$ Phenomena Using P-Self Similar Models," *manuscript in preparation*.

True H	Est H	Var H	Est |M|	Var |M|
-0.3	-0.2916	0.0192	9.4168	0.5417
0.5	0.4874	0.0145	10.6030	0.4154
0.7	0.7074	0.0127	9.5802	0.4014
3.0	2.9988	0.0163	10.1302	0.4695

Table II
Estimates of the self similarity parameter of 1st order SS-AR using uniform sampling. $M = 10$, SNR = 10dB

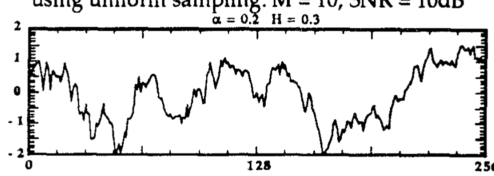

Figure 1
Sample paths of the first order self similar autoregressive process with $\sigma^2 = 1$.

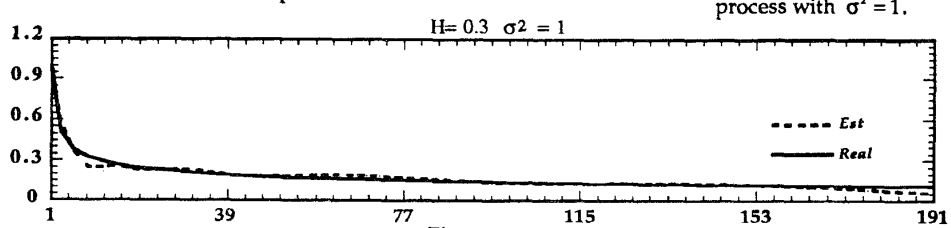

Figure 2
S-autocorrelation function estimate $\tilde{R}_1$ of the fBm process with $H = 0.7$.
The data length is 128, exponential sampling density is $s_0 = 1.5$ and the initial observation time T is 1.