A Class of Second-Order Stationary Self-Similar Processes for $1/f$ Phenomena

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***Abstract*—We propose a class of statistically self-similar processes and outline an alternative mathematical framework for the modeling and analysis of $1/f$ phenomena. The foundation of the proposed class is based on the extensions of the basic concepts of classical time series analysis, in particular, on the notion of stationarity. We consider a class of stochastic processes whose second-order structure is invariant with respect to time scales, i.e., $E[X(t)X(\lambda t)] = t^{2H}\lambda^{H}R(\lambda)$, $\lambda, t > 0$ for some $-\infty < H < \infty$. For $H = 0$, we refer to these processes as wide sense scale stationary. We show that any self-similar process can be generated from scale stationary processes. We establish a relationship between linear scale-invariant system theory and the proposed class that leads to a concrete analysis framework. We introduce new concepts, such as periodicity, autocorrelation, and spectral density functions, by which practical signal processing schemes can be developed. We give several examples of scale stationary processes including Gaussian, non-Gaussian, covariance, and generative models, as well as fractional Brownian motion as a special case. In particular, we introduce a class of finite parameter self-similar models that are similar in spirit to the ordinary ARMA models by which an arbitrary self-similar process can be approximated. Results from our study suggest that the proposed self-similar processes and the mathematical formulation provide an intuitive, general, and mathematically simple approach to $1/f$ signal processing.**

I. Introduction

$1/f$ PROCESSES have a widespread occurrence in variety of science and engineering data. Typical examples in electrical systems include noises in electrical devices, burst error in communication channels, frequency variations in music, texture variations in natural terrain images, cloud formations, and medical imagery, to mention a few [1]–[5]. $1/f$ processes have two important characterizations. The first one is the strong interdependence between far-apart observations. Such a strong correlation is a result of high energy content of $1/f$ spectral density at low frequencies. In statistics literature, the correlation structure of the $1/f$ processes is taken as a basis, and they are referred to as long term correlated or long memory processes [6], [7]. The empirical correlation function of the long-term correlated processes decays hyperbolically, i.e., $\tau^{-\alpha}$, $\alpha > 0$, as the lag $\tau \rightarrow \infty$, as opposed to the fast exponential decay of the short-term correlated processes, such as autoregressive moving average processes (ARMA). As a result, ordinary time series models do not offer parsimonious representations for $1/f$ processes. A second characterization of $1/f$ processes is the statistical self-similarity, that is, the statistical properties of the process remain invariant to or within an amplitude factor under arbitrary scalings of the time axis. Statistical self-similarity of $1/f$ processes can be viewed as a manifestation of their fractal nature.

The sharp contrast between the properties of the $1/f$ processes and ordinary time series models have given rise to many proposals about the best way of modeling and analyzing $1/f$ processes. Generally speaking, most of the proposals are based on either the concept of self-similarity, infinite-order ARMA modeling, or physical origins of the $1/f$ processes. Engineering literature contains various models based on the physical nature of some specific $1/f$ phenomena [2]–[5]. While this class of models is very helpful in understanding the mechanism of the physical phenomena by which $1/f$ processes are generated, they lack mathematical tractability to be useful in typical signal modeling tasks. Much of the published research has argued in favor of either the models based on ARMA processes or statistical self-similarity. The fractional differencing model, proposed independently by Hosking and by Granger and Joyeux is an example of the former one [7], [8]. It is a parsimonious, infinite-order, discrete autoregressive process whose coefficients are functions of a single parameter. Fractional Brownian motion (fBm) proposed by Mandelbrot and Van Ness is an example of the later one [6]. It is a continuous time moving average process in which the past increments are weighted by the kernel $(t-s)^{H-1/2}$, where the parameter H is associated with the degree of self-similarity. Although fBm became a popular framework for various signal processing and pattern recognition problems involving $1/f$ phenomena [9], [10], the mathematical structure of the model makes the solutions of many practical problems rather difficult. In particular, due to the nonstationarity of the model, traditional time series analysis techniques cannot be employed [11].

In this paper, we explore an alternative framework to the fractional Brownian and fractional differencing models to model and analyze $1/f$ phenomena. Our approach is based on a new concept of stationarity. We propose a class of nonstationary processes invariant in distribution with respect to time scales, i.e., $P_{t_1,\ldots,t_N}(\cdot) = P_{\lambda t_1,\ldots,\lambda t_N}(\cdot)$, $t_i, \lambda > 0$. We call these processes *scale stationary.* These processes have many properties similar in spirit to the ordinary stationary processes. We show that a rich class of self-similar processes can be generated by scale stationary processes. We illustrate the practical utility of the proposed class by several examples,

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which include Gaussian, non-Gaussian, Markovian, covariance, and generative models, as well as fractional Brownian motion as a special case. Our development of the theory of scale stationary processes is motivated by the linear scale-invariant systems and signals theory [12], [13]. Such a system theory perspective facilitates, first, the development of a class of statistically self-similar processes that are rich enough to embrace a variety of $1/f$ phenomena and, second, to develop a mathematical framework by which efficient signal processing tasks, such as spectral analysis and parameter estimation, can be performed with minimum mathematical difficulty. Results from our study suggest that there would appear to be many additional applications for the self-similar processes and the formulation we introduce in this work.

The rest of the paper is organized as follows. In Section II, we review the theory of linear scale-invariant systems and signals from a perspective relevant to our subsequent developments. In Section III, we introduce scale stationary processes and a class of self-similar processes generated by the scale stationary processes. We give several illustrative examples. We define a concept of autocorrelation, develop a spectral representation theorem, and discuss filtered self-similar processes as a prelude to the next section. In Section IV, we introduce a class of finite parameter self-similar processes that we refer to as self-similar autoregressive processes. In Section V, we discuss the long-term correlation properties of the proposed self-similar processes. Finally, in Section VI, we discuss briefly several further items of interest and conclude our discussion.

II. Generalized Linear Scale-Invariant Systems

In order to motivate our discussion, let us first state the relationship between linear time-invariant system theory and the theory of wide sense stationary processes from our perspective. We can view wide sense stationary processes as a generalization of the deterministic time or shift invariant signals in the sense that both are invariant with respect to the same translation, namely, the time shift. Given any finite energy, time-invariant signal, the energy of the system remains invariant under arbitrary shifts of the origin, i.e., $f, g \in L^2(R, dt)$ and $\tau \in R$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(t+\tau)\overline{g(t+\tau)}\,dt. \tag{2.1}$$

We could replace the Lebesgue measure, which is invariant with respect to time shifts, with a probability measure exhibiting a similar invariance property and obtain the well-known wide sense stationary processes. The statistical energy of wide sense stationary processes remains invariant under arbitrary time shifts of the origin, i.e.

$$\langle f(t_1), f(t_2)\rangle = \int_{-\infty}^{\infty} f(t_1)\overline{f(t_2)}\,dP = \int_{-\infty}^{\infty} f(t_1+\tau)\overline{f(t_2+\tau)}\,dP \tag{2.2a}$$

or

$$E[f(t_1)\overline{f(t_2)}] = E[f(t_1+\tau)\overline{f(t_2+\tau)}]. \tag{2.2b}$$

By the generalization described above, one can map the linear time-invariant system theory from deterministic to probabilistic setting and obtain a framework for the analysis of wide sense stationary processes. In addition, a broad range of statistical techniques and models in the realm of wide sense stationary processes can be viewed as stochastic counterparts of the deterministic methods developed in the framework of linear time-invariant systems and $L^2(R, dt)$ signals. For example, the linear least squares prediction can be viewed as a counterpart of the minimum norm approximation technique, and ARMA models can be viewed as a counterpart of the constant coefficient ordinary differential equations that are used to represent the dynamics of linear time-invariant systems. Thus, linear time-invariant system theory can be utilized to develop a statistical tool box, as well as a concrete physical understanding, for the wide sense stationary processes.

In our subsequent discussion, we shall consider signal processing frameworks that are invariant with respect to another type of translation, namely, *scale*. In order to unify the theory of stationary process and our proposed models on the basis of invariance, for the rest of the paper, we shall refer to the ordinary stationarity as shift stationarity and linear time-invariant systems and signals as linear shift invariant systems and signals. First, we shall sketch the framework of scale-invariant deterministic signals and linear systems. Later, using the steps of this section, we shall extend our discussion to the probabilistic setting.

Linear scale-invariant systems have been studied in signal processing and pattern recognition in connection with scale representations and scale-invariant filtering [12]–[15]. In our discussion, we will review and modify basic results and redefine some of the concepts from a perspective that is relevant for our subsequent development.

Let f be a signal on the positive real axis satisfying the following invariance property:

$$\int_0^{\infty} |f(t)|^2\,d\ln t = \int_0^{\infty} |f(\lambda t)|^2\,d\ln t < \infty, \quad \text{for any } \lambda > 0. \tag{2.3a}$$

Then, we shall call f a *scale-invariant*, finite energy signal and denote the signal space by $L^2(R^+, d\ln t)$. Note that the energy of the signal remains invariant under arbitrary scalings of the time axis. We can extend the class of scale-invariant signals by allowing a change in energy proportional with the scaling factor, i.e.

$$\int_0^{\infty} |\tilde{f}(t)|^2 \frac{1}{t^{2H}}\,d\ln t = \lambda^{-2H}\int_0^{\infty} |\tilde{f}(\lambda t)|^2 \frac{1}{t^{2H}}\,d\ln t < \infty, \quad \lambda > 0 \quad -\infty < H < \infty. \tag{2.3b}$$

We shall call $\tilde{f}$ a finite energy *self-similar* signal with parameter H. Note that a scale-invariant signal is self-similar with parameter 0. It is straightforward to show that given f, $\tilde{f}$, and $H \neq 0$ satisfying

$$f(t) = t^{-H}\tilde{f}(t), \quad \text{for all } t > 0 \tag{2.4}$$

the signal f is scale-invariant if and only if $\tilde{f}$ is self-similar with parameter H. Moreover, both signals have the same energy. Hence, self-similar and scale-invariant signals are in one-to-one correspondence.

Suppose $y(t)$ is the response of a linear system to an arbitrary self-similar signal $x(t)$. Then, we shall call the system $S\{\cdot\}$ *linear self-similar* (LSS) with parameter H whenever

$$S\{x(t\lambda)\} = \lambda^{-H} y(t\lambda), \qquad \text{for all } t,\, \lambda > 0. \tag{2.5}$$

For $H = 0$, we shall call the system linear scale-invariant (LSI). As is well known, the most general input-output relationship for a 1-D linear system is given by

$$y(t) = \int_0^\infty K(t, \lambda) x(\lambda)\, d\lambda, \qquad t > 0 \tag{2.6}$$

provided that the integral is well defined. Under the scale-invariance property defined in (2.5), the kernel $K(\cdot, \cdot)$ of the a self-similar system satisfies the following property:

$$K(t, \lambda) = \alpha^{1-H} K(\alpha t, \alpha\lambda), \quad \text{for any } \alpha,\, \lambda,\, t > 0. \tag{2.7}$$

Setting $\alpha = 1/\lambda$ and $h_H(t) = t^{-H} K(t, 1)$, the input output relation of an LSS system becomes

$$y(t) = t^H \int_0^\infty h_H\left(\frac{t}{\lambda}\right) x(\lambda)\, d\ln\lambda, \qquad t > 0. \tag{2.8a}$$

By a simple change of variables, (2.8a) can be alternatively expressed as

$$y(t) = t^H \int_0^\infty h_H(\lambda) x\left(\frac{t}{\lambda}\right) d\ln\lambda, \qquad t > 0. \tag{2.8b}$$

Note that for $H = 0$, the input–output relationship given in (2.8a) is nothing but the convolution operation with respect to multiplication. It has the same algebraic properties, such as commutativity and associativity, as the ordinary convolution operation. We shall refer to h_H as *pseudo impulse response* function of an LSS system [21]. We can interpret the pseudo impulse response function physically. To do so, let us introduce the following generalized signal as a "unit driving force" to an LSS system:

$$\tilde{\delta}\left(\frac{t_1}{t_2}\right) = \begin{cases} 1 & t_1 = t_2 \\ 0 & t_1 \neq t_2 \end{cases}, \qquad 0 < t_1,\, t_2 < 1. \tag{2.9a}$$

The signal can be defined rigorously, but for our development, we only need the following analog of the sifting property:

$$\begin{aligned} f(t) &= \int_0^\infty f(\lambda) \tilde{\delta}\left(\frac{t}{\lambda}\right) d\ln\lambda \\ &= \int_0^\infty f\left(\frac{t}{\lambda}\right) \tilde{\delta}(\lambda)\, d\ln\lambda, \qquad t > 0. \end{aligned} \tag{2.9b}$$

Using the superposition principle of the linear systems together with (2.9a) and (2.9b), we can verify that

$$S\{\tilde{\delta}(t)\} = t^H h_H(t). \tag{2.10}$$

Hence, the extra term t^H in (2.10) explains the choice of the word pseudo. As we shall see, pseudo impulse response has many properties similar to the properties of the impulse response function of linear shift-invariant systems. From (2.7) and (2.10), it is clear that linear self-similar and scale-invariant systems are in one-to-one correspondence.

Obviously, the system is not physically realizable for $\lambda > t$ in (2.8a) or $\lambda < 1$ in (2.8b). We shall say that an LSS system is causal if $h_H(t) = 0$ for $0 < t < 1$. Hence, for a causal system, the input–output relations given in (2.8a) and (2.8b) reduce to

$$\begin{aligned} y(t) &= t^H \int_0^t h_H\left(\frac{t}{\lambda}\right) x(\lambda)\, d\ln\lambda \\ &= t^H \int_1^\infty h_H(\lambda) x\left(\frac{t}{\lambda}\right) d\ln\lambda, \qquad t > 0. \end{aligned} \tag{2.11}$$

Before we proceed, let us define a concept of stability for LSS systems in order to give a precise mathematical meaning to the integral representation of the input output relationships given in (2.8a) and (2.8b). For a given scale-invariant, finite energy signal, we shall say that an LSS system with parameter H is stable if the output is self-similar, finite energy with some parameter H. The following theorem characterizes the stability in terms of impulse response function.

Theorem 2.1: A linear self-similar system with pseudo impulse response function h_H is stable whenever

$$\int_0^\infty |h_H(t)|\, d\ln t < \infty. \tag{2.12}$$

□

Proof 2.1: See Appendix I.

The input output relationship of an LSS system can be simplified by means of the Mellin transform. Given a scale-invariant, finite energy signal, its Mellin transform is defined by

$$F(s) = \int_0^\infty f(t) t^{-s}\, d\ln t \tag{2.13a}$$

where $s = \sigma + j\omega$ takes values ensuring the convergence of the integral. For a purely imaginary s, the Mellin transform is an analog of the Fourier transform. It can be easily seen that if the input of an LSI system is, t^s, the output becomes

$$\begin{aligned} y(t) &= t^s \int_0^\infty h_0(\lambda) \lambda^{-s}\, d\ln\lambda \\ &= t^s H_0(s) \end{aligned} \tag{2.14}$$

showing that $\{t^s,\, s = \sigma + j\omega\}$ are the eigenfunctions of the system, whereas the corresponding eigenvalues are given by the Mellin transform of the impulse response function of the system. As a consequence, the inverse Mellin transform is given by the superposition of the eigenfunctions with the corresponding eigenvalues, i.e.,

$$f(t) = \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) t^s\, ds. \tag{2.13b}$$

As a result of the eigenfunction property, the time domain input–output relationship of an LSI system is conveniently mapped to a multiplication operation in the Mellin domain.

$$Y(s) = H_0(s) X(s). \tag{2.15}$$

Hence, H_0 is appropriately referred to as the transfer function of an LSI system [13]. Similar results can be obtained for LSS systems, via generalized Mellin transforms. The synthesis and analysis equations for the self-similar case are given by

$$\overline{F}(s) = \int_0^\infty f(t)t^{-s-H}\, d\ln t,$$
$$f(s) = \int_{\sigma-j\infty}^{\sigma+j\infty} \overline{F}(s)t^{s+H}\, ds, \qquad t > 0 \tag{2.16}$$

where f is any finite energy self-similar signal with parameter H. However, due to the nonorthogonality of generalized Mellin transforms, there are differences between the results of the scale-invariant and self-similar systems. For instance, the input output relationship of an LSS system in the generalized Mellin domain is given by

$$\overline{Y}(s) = H_H(s)X(s) \tag{2.17}$$

where $\overline{Y}$ is the generalized Mellin transform with parameter H of the output and H_H, and X are the ordinary Mellin transform of the pseudo impulse response function and the input, respectively.

In order to elucidate the physical meaning of the Mellin transform of a signal, we now introduce a new concept of periodicity.

Definition 2.1: Let f be a function satisfying the following relation:

$$f(t) = f(\lambda t), \qquad t > 0 \tag{2.18}$$

for some $\lambda > 1$. Then, we shall call the function f *periodic in scale* and the smallest $\lambda > 1$ the period in scale, or S-period for short. □

Informally speaking, the period λ defines a natural scale or a degree of self-similarity for the signal in the sense that the signal does not change if the observation axis is scaled with the integer exponents of the period λ, i.e., $f(t) = f(\lambda^n t)$. The prime example of the scale periodic signals are the eigenfunctions of an LSI system. It is easy to show that $t^{jn} = (e^{2\pi/n}t)^{jn}$. Thus, $e^{2\pi/n}$ is the S-period of the function t^{jn} in scale. The Mellin transform coefficient at $j\omega$ can be interpreted as the correlation of the signal with the scale periodic function $t^{j\omega}$. Therefore, the magnitude of the Mellin transform of a signal at ω can be viewed as a measure of scalability or self-similarity of the signal at S-period $e^{2\pi/\omega}$.

As a last but important item, we want to introduce a class of time-varying ordinary differential equations that can be utilized to represent the dynamics of an LSS system. It is straightforward to check that the following system is self-similar with parameter H:

$$a_N t^N \frac{d^N}{dt_N} t^{-H}y(t) + \cdots + a_1 t \frac{d}{dt} t^{-H}y(t) + a_0 t^{-H}y(t)$$
$$= b_M t^M \frac{d^M}{dt^M} x(t) + \cdots + b_1 t \frac{d}{dt} x(t) + b_0 x(t). \tag{2.19a}$$

Equation (2.19a) can be alternatively expressed as

$$a_N t^N \frac{d^N}{dt_N} y(t) + \cdots + \alpha_1 t \frac{d}{dt} y(t) + \alpha_0 y(t)$$
$$= \beta_M t^{M+H} \frac{d^M}{dt^M} x(t) + \cdots + \beta_1 t^{1+H} \frac{d}{dt} x(t)$$
$$+ \beta_0 t^H x(t). \tag{2.19b}$$

For $H = 0$, this system is known as the Euler–Cauchy system. It was used in the engineering literature to represent time-varying systems [16]. The transfer function of the system is given by

$$H_H(s) = \frac{\sum_{i=0}^{M} \beta_i s^i}{\sum_{i=0}^{N} \alpha_i s^i} \tag{2.20}$$

where s takes values in the appropriate region of convergence. The importance of the Euler–Cauchy system comes from the observation that given any stable LSS system, it can be always approximated by a sufficiently high-order Euler–Cauchy system. This is due to the fact that there is always a sequence of rational functions converging to a given finite energy, scale-invariant signal, namely, the transfer function of the stable LSS system. For a causal LSS system with $M = 0$, the pseudo impulse response function of the system is given by

$$h_H(t) = \sum_{j=0}^{n}\sum_{i=1}^{m_j} a_{ij}(\ln t^j)t^{-b_{ij}}, \qquad t \geq 1$$

and

$$N = \sum_{j=0}^{n} (m_j + 1) \tag{2.21}$$

where m_j, $j = 1, \cdots, n$, is the number of the repeated poles in the transfer function. Note that the system is stable whenever all $b_{ij} > 0$.

Finally, we want to note that there is a one-to-one correspondence between the linear self-similar systems and signals discussed in this section and the theory of linear shift invariant systems and signals. One can easily show that through the following logarithmic distortion of the time axis

$$t \to \ln t \tag{2.22}$$

finite energy, scale-invariant signals become finite energy shift invariant signals, ordinary Mellin transform becomes Laplace transform, and the Euler–Cauchy system becomes an ordinary constant coefficient differential equation system.

III. A Class of Second Order Self Similar Processes

In this section, we will introduce a class of stochastic processes that is a natural extension of the scale-invariant signals in the probabilistic setting. As indicated in the introduction, we have several objectives in our development. Our primary objective is to introduce models for $1/f$ processes that can be specified by finitely many parameters in order to develop efficient algorithms. That is, we would like to develop models, based on the notion of statistical self-similarity, analogous to those specified by finite order ordinary ARMA models. In addition, we would like to develop analysis tools, such as a spectral decomposition method, that could provide powerful algorithms for modeling, estimation, and other signal processing tasks. Finally, we want to demonstrate that the notion of statistical self-similarity provides a foundation for $1/f$ signal processing, which is similar in spirit to the foundations of classical time series analysis based on the concept of shift stationarity. We shall first introduce an important subclass of self-similar processes, which we refer to as scale stationary processes.

A. Scale Stationary Processes

A stochastic process $\{X(t), -\infty < t < \infty\}$ is called *statistically self-similar* with parameter H if it satisfies the following scaling condition:

$$X(t) \equiv a^{-H} X(at), \qquad -\infty < t < \infty, \quad \text{for any } a > 0 \tag{3.1}$$

where $\equiv$ denotes equality in finite-dimensional probability distributions [6]. Note that in Section II, self-similarity was defined with respect to the measure $d \ln t$, whereas statistical self-similarity is defined with respect to a scale-invariant probability measure. For the rest of the paper, unless otherwise stated, we shall use the term self-similarity in the statistical sense.

Brownian motion is a typical example of a statistically self-similar process. Its self-similarity parameter is $H = \frac{1}{2}$. fBm is a generalization of the Brownian motion with parameter $0 < H < 1$. Notice that for $H = 0$, self-similarity has an intuitively appealing interpretation. It indicates that the statistics of the process is absolutely independent of the time-scale chosen. This property is analogous to the independence of the shift stationary processes from the time origin. This subclass of self-similar processes exhibits many properties that can be utilized for practical applications and, therefore, deserves a special treatment. We will start our treatment by identifying this subclass with a new term.

Definition 3.1: We shall call a stochastic process $\{X(t), t > 0\}$ strictly scale stationary if it is self-similar with parameter 0, i.e.,

$$X(t) \equiv X(at), \qquad t > 0, \quad \text{for any } a > 0. \tag{3.2}$$

□

Note that since strictly scale stationary processes are non-stationary in the ordinary sense, the time origin becomes important while the time scale looses its significance.

Before proceeding further, we want to show that there is a natural isometry between strictly scale stationary processes and strictly shift stationary processes. Given any shift stationary process $Y(t) - \infty < t < \infty$, the process $X(\lambda), \lambda > 0$ obtained through the following exponential distortion of the time axis

$$X(\lambda) = Y(\ln \lambda), \qquad \lambda > 0 \tag{3.3}$$

is scale stationary. Note that strictly scale stationary processes are symmetric with respect to $\lambda = 1$ in the sense that $X(\lambda)$ and $X(1/\lambda)$, $\lambda > 0$ have the same finite probability distributions. This is due to the fact that the ordinary stationary processes are symmetric in distribution with respect to $t = 0$ and that the exponential distortion maps $t = 0$ to $\lambda = 1$. Therefore, we shall refer to $\lambda = 1$ as the *origin* of the strictly scale stationary processes.

B. Self-Similar Processes Generated by Scale Stationary Processes

Strictly scale stationary processes allow us to construct a class of self-similar processes with arbitrary self-similarity parameter H, $-\infty < H < \infty$. Such a construction is given by the following theorem, which is a straightforward corollary to Definition 3.1 and the isometry relationship described in (3.3).

Theorem 3.1: Define

$$X(t) = t^H \tilde{X}(t), \qquad \text{for all } t > 0, \quad \text{for some } -\infty < H < \infty. \tag{3.4}$$

Then, $\{X(t), t > 0\}$ is a statistically self-similar process with parameter H if and only if $\{\tilde{H}(t), t > 0\}$ is a strictly scale stationary process. □

Proof 3.1: See Appendix II.

We shall refer to the scale stationary process $\{\tilde{X}(t), t > 0\}$ as the *generating* scale stationary process and t^H as the *trend* term of the self-similar process $\{X(t), t > 0\}$. In order to distinguish a self-similar process defined only on the positive time axis from an arbitrary self-similar process, we shall refer to this class of self-similar process as *p-self-similar* processes. Obviously, the generating scale stationary process constitutes the random part, and the trend term constitutes the deterministic part of a p-self-similar process. Obviously, if the generating process is free of parameter H, the self-similarity parameter contributes only to the trend of the process. Although, p-self-similar processes form a subclass of self-similar processes, we shall demonstrate that for all practical purposes, p-self-similar processes provide a rich class of models, and a mathematical analysis framework for $1/f$ signals. This is due to the observation that given any self-similar process $\{X(t), -\infty < t < \infty\}$, one can find two scale stationary processes $\{\tilde{X}_1(t), \tilde{X}_2(t), t > 0\}$ such that the following holds in distribution:

$$X(t) = \begin{cases} t^H \tilde{X}_1(t) & t > 0 \\ t^H \tilde{X}_2(-t) & t < 0 \end{cases}. \tag{3.5}$$

C. Second-Order P-Self-Similar Processes

In practice, it is favorable to work with models based on first- and second-order statistics since probability density function estimation is a more involved task. In addition, second-order structures lead to a larger class of models, which includes probability density function-based models as a special case. In order to extend the concept of scale stationarity into second-order models, we now propose the following definition:

Definition 3.2: A random process $\{X(t), t > 0\}$ will be called wide sense scale stationary if it satisfies the following conditions:

i) $E[X(t)] = \text{constant}$, for all $t > 0$.
ii) $E[X^2(t)] < \infty$, for all $t > 0$.
iii) $E[X(t_1)X(t_2)] = E[X(\lambda t_1)X(\lambda t_2)]$ for all $t_1, t_2, \lambda > 0$. □

Obviously, ordinary wide sense stationary processes and wide sense scale stationary processes are also isometric through the relationship described in (3.3). Note that a strictly scale stationary process does not need to have a finite variance. The finite variance requirement in Definition 3.2 assures that wide sense scale stationary processes are physically realizable. From the last condition of Definition 3.2, it is immediate that

$$E[X(t)X(\lambda t)] = R(\lambda), \qquad \text{for any } t, \lambda > 0. \tag{3.6}$$

We shall refer to R as *scale autocorrelation* or, for short, the S-autocorrelation function. We shall show in the following sections that the concept of S-autocorrelation facilitates the development of practical analytical tools and estimation methods for wide sense scale stationary processes. Now, as a natural extension of wide sense scale stationarity, we propose the following definition:

Definition 3.3: A random process $\{X(t), t > 0\}$ will be called wide sense p-self-similar with parameter H if it satisfies the following conditions:

i) $E[X(t)] = \lambda^{-H} E[X(\lambda t)]$ for all $t, \lambda > 0$.
ii) $E[X^2(t)] < \infty$, for each $t > 0$.
iii) $E[X(t_1)X(t_2)] = \lambda^{-2H} E[X(\lambda t_1)X(\lambda t_2)]$ for all t_1, t_2, $\lambda > 0$. □

Obviously, Theorem 3.1 applies to the wide sense p-self-similar processes. Given any wide sense p-self-similar process $\{X(t), t > 0\}$, there is a generating wide sense scale stationary process such that

$$\begin{aligned} E[X(t)X(\lambda t)] &= \lambda^H t^{2H} E[\tilde{X}(t)\tilde{X}(\lambda t)] \\ &= t^{2H} \lambda^H R(\lambda), \qquad \text{for all } t, \lambda > 0 \end{aligned} \tag{3.7}$$

where R is the S-autocorrelation function of the generating process. Now, we want to introduce a notion of autocorrelation function for wide sense p-self-similar processes that will represent both the underlying scale stationary and the trend structure. Consider the following candidate:

$$\begin{aligned} E[X(\lambda)X(1)] &= \lambda^H R(\lambda) \\ &= \Gamma(\lambda), \qquad \lambda > 0. \end{aligned} \tag{3.8}$$

We shall refer to Γ as the *basic autocorrelation* function of the wide sense p-self-similar processes. It is a measure of correlation between the samples at $\lambda > 0$ and the origin, i.e., $\lambda = 1$. Since p-self-similar processes are trended wide sense scale stationary processes, it is sensible to choose an autocorrelation function that is also the trended version of the S-autocorrelation function. In fact, the concept of basic autocorrelation function will prove to be a very natural choice as we develop spectral analysis methods for the wide sense p-self-similar processes. The following properties of the basic autocorrelation function can be derived easily from the Definition 3.2 and (3.8). We state them here without proof.

i) $\Gamma(1) \geq 0$ for all $\lambda > 0$.
ii) $\Gamma(\lambda) \leq \Gamma(1)\lambda^H$.
iii) $\Gamma(1/\lambda) = \lambda^{-2H}\Gamma(\lambda)$ for all $\lambda > 0$. □

For the rest of this paper, we will focus on the wide sense p-self-similar and scale stationary processes and explore their practical utility for signal processing problems, such as signal modeling and estimation. Unless otherwise stated, for the rest of the paper, we shall use the terms p-self-similarity and scale stationarity in the second-order sense. Now, we give various examples of p-self-similar processes to illustrate their potential applications.

D. Examples

Example 3.1: Given a random variable φ uniform in the $(-\pi, \pi)$ and a constant f_0, we form the process

$$z(t) = at^H \cos\left(\frac{2\pi}{f_0} \ln t + \varphi\right), \qquad t > 0. \tag{3.9a}$$

By direct calculation, one can show that $\{z(t), t > 0\}$ is wide sense p-self-similar with zero mean and basic autocorrelation function

$$\Gamma(\lambda) = \frac{a^2}{2}\lambda^H \cos\left(\frac{2\pi}{f_0} \ln \lambda\right), \qquad \lambda > 0. \tag{3.9b}$$

The model can be utilized to analyze $1/f$ physical processes that exhibit scale periodicity as well as non-Gaussian behavior. Note that the S-autocorrelation function R of the generating process is scale periodic with S-period e^{f_0}, i.e.,

$$R(\lambda) = R(e^{f_0}\lambda), \qquad \lambda > 0 \tag{3.9c}$$

assuming that $f_0 > 0$. As a consequence of periodicity, it suffices to define the process on the interval $[1, e^{f_0})$. In addition, one can take advantage of the generative structure of the model in some problems, such as prediction, where data synthesis is needed. Fig. 1 shows the sample paths of $\{z(t), t > 0\}$ for various values of H and f_0.

Example 3.2: The second example is the well-known fractional Brownian motion. For $t \geq 0$, fBm is formally defined as follows:

$$\begin{aligned} B_H(0) &= 0, \qquad \text{with probability 1.} \\ B_H(t) &= \frac{1}{G(H+\frac{1}{2})}\left\{\int_{-\infty}^{t} (t-\tau)^{H-1/2}\, dB(\tau) \right. \\ &\qquad \left. - \int_{-\infty}^{0} (-\tau)^{H-1/2}\, dB(\tau)\right\}, \\ &\qquad \text{for } t > 0 \end{aligned} \tag{3.10a}$$

where H is a parameter between 0 and 1, G is the gamma function, and $\{B(\tau), -\infty < \tau < \infty\}$ is the Brownian motion. The fBm possesses numerous interesting properties. Among them, it is a generalization of the Brownian motion, in the sense that for $H = \frac{1}{2}$, it reduces to the Brownian motion. In addition, fBm uniquely models a certain class of self-similar processes. It can be shown that for $0 < H < 1$, fBm constitutes the only self-similar, zero mean, mean square continuous, finite variance Gaussian random process with stationary increments satisfying $x(0) = 0$.

The covariance function of fBm is given by

$$\begin{aligned} E[B_H(t)B_H(\lambda t] &= t^{2H}\frac{\sigma^2}{2}\{1 + \lambda^{2H} - |1-\lambda|^{2H}\}, \\ &\qquad t, \lambda > 0 \end{aligned} \tag{3.10b}$$

where

$$\sigma^2 = G(1-2H)\,\frac{\cos(\pi H)}{2\pi H}. \tag{3.10c}$$

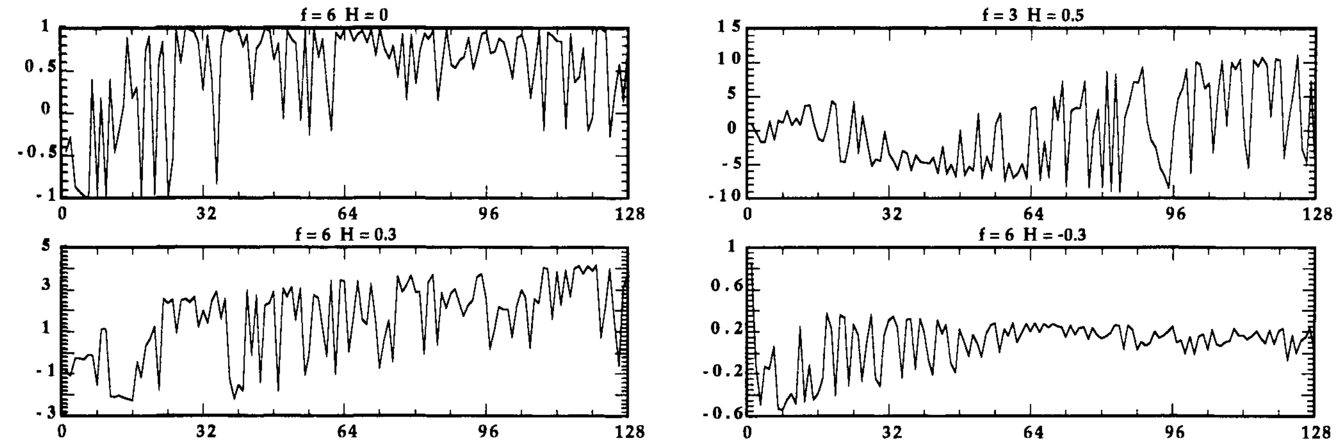

Fig. 1. Sample paths of the sinusoidal self-similar process.

Now, we shall derive three different p-self-similar processes using the fBm process:

a) In order to understand the analytical properties of the fBm, it may be useful to study the underlying generating process. Let $\{\tilde{B}_H(t), t > 0\}$ be the generating process of the fBm, and let $\{A_H(t), -\infty < t < \infty\}$ be the shift stationary process obtained by the exponential distortion of the time axis of $\{\tilde{B}_H(t), t > 0\}$, i.e.

$$A_H(t) = \tilde{B}_H(e^t), \qquad -\infty < t < \infty. \tag{3.11a}$$

From (3.10b) and (3.10c), we can easily obtain the S-autocorrelation function of the generating process

$$E[\tilde{B}_H(t)\tilde{B}_H(\lambda t)] = \sigma^2\{\cosh(H \ln \lambda) - |\sinh(\tfrac{1}{2}\ln \lambda)|^{2H}\}, \quad \lambda > 0. \tag{3.11b}$$

Similarly, the lag-based autocorrelation function of $\{A_H(t), -\infty < t < \infty\}$ is given by

$$E[A_H(t)A_H(t+\tau)] = \sigma^2\left\{\cosh(H\tau) - \left|\sinh\left(\frac{\tau}{2}\right)\right|^{2H}\right\}, \quad -\infty < \tau < \infty. \tag{3.11c}$$

Fig. 2 illustrates the sample paths of the generating process of the fBm for various values of H. While for H close to 1 the process exhibits long-term correlations, which are displayed by slow variations in long and short durations, it appears to exhibit short-term correlations for H close to 0, which are displayed by rapid short-term oscillations.

The sample paths of $\{A_H(t), -\infty < t < \infty\}$ shown in Fig. 3 have a relatively smooth variation. We could hypothesize that the strong long-term correlation structure of the fBm is inherited by $\{A_H(t), -\infty < t < \infty\}$ through the isometric mapping of the coordinate system.

b) Note that unlike Example 3.1, both the trend term and the generating process of the fBm are governed by the same parameter, namely, H. We can define a new class of p-self-similar models by assigning different parameters to the trend term and the generating process of the fBm. Let

$$X(t) = t^{\tilde{H}}\tilde{B}_H(t), \qquad t > 0. \tag{3.12a}$$

The basic autocorrelation function of the resulting p-self-similar process is

$$\Gamma(\lambda) = \sigma^2\{\lambda^{\tilde{H}-H} + \lambda^{\tilde{H}+H} - \lambda^{\tilde{H}-H}|1-\lambda|^{2H}\}, \quad \lambda > 0. \tag{3.12b}$$

Since the parameter of the trend term is positive by definition, fBm always exhibits a growing trend. In fBm, the growth rate is dictated by the parameter that controls the long-term correlations. This limits the applicability of the fBm, as the model has to have high growth rate to be able to exhibit strong correlations. In the new model, on the other hand, the trend and generating processes are controlled by independent parameters. The advantage of the later model becomes evident when we compare the sample paths of the two models generated by the same scale stationary process. In the top left of Fig. 5, the rapid changes in sample paths of the new model indicate short-term correlations with a growing trend, whereas in the top left of Fig. 4, the sample paths of fBm indicate is slow growth with short-term correlations. Similarly, despite the same generating process, in the bottom left of Fig. 4, the sample paths of the fBm grow much faster than the sample paths of the new model shown in the bottom left of Fig. 5.

c) Let us now consider the increment process of the fBm.

$$Z_H(t) = B_H(t+s) - B_H(t), \qquad -\infty < t < \infty, \quad s > 0. \tag{3.13a}$$

As we have mentioned above, $\{Z_H(t), -\infty < t < \infty\}$ is shift stationary, and its autocorrelation function is given by

$$E[Z_H(t)Z_H(t+\tau)] = \frac{\sigma^2}{2}\{|\tau+1|^{2H} + |\tau-1|^{2H} - 2|\tau|^{2H}\}, \quad \tau > 0. \tag{3.13b}$$

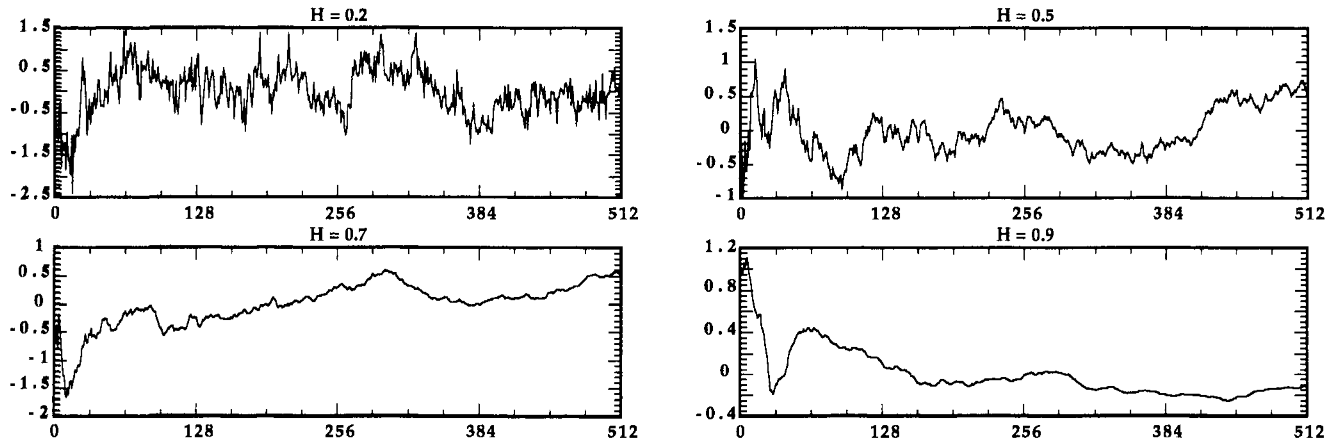

Fig. 2. Sample paths of the generating process of the fractional Brownian motion.

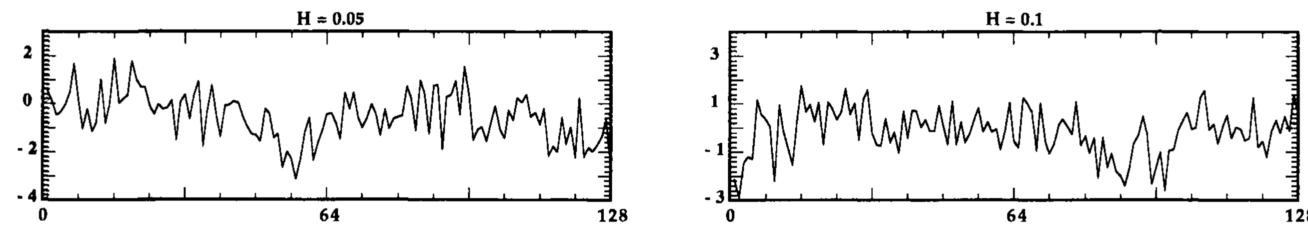

Fig. 3. Sample paths of the shift stationary process corresponding to the generating process of fBm.

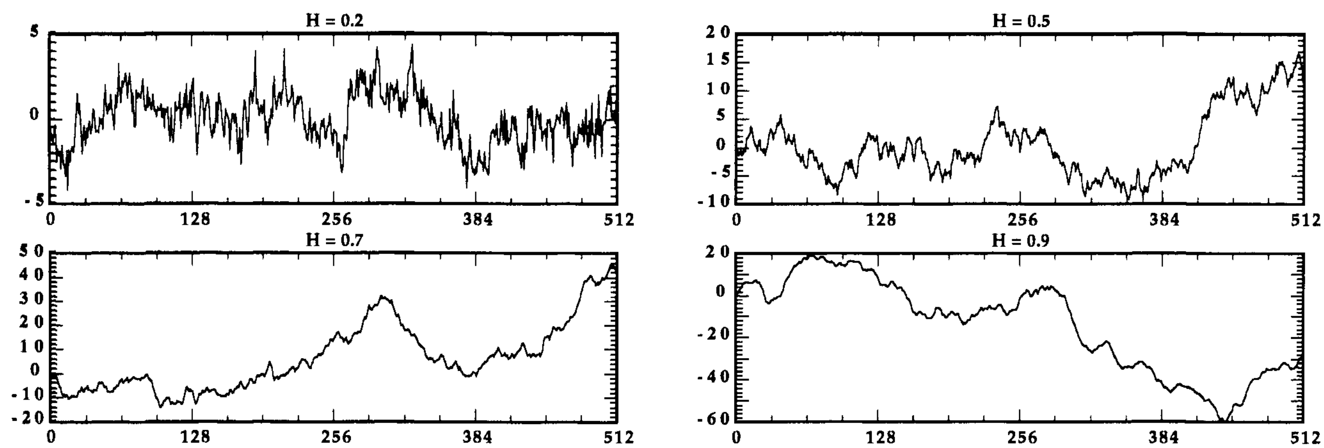

Fig. 4. Sample paths of the fractional Brownian motion.

For $\frac{1}{2} < H < 1$, the increment process is known to exhibit long-term correlations. Intuitively speaking, we expect this property to be strengthened and inherited by the scale stationary process obtained through the logarithmic coordinate distortion. Let

$$\tilde{Y}_H(t) = Z_H(\ln t),$$

and

$$Y_{\tilde{H}}(t) = t^{\tilde{H}} \tilde{Y}_H(t), \qquad t > 0. \tag{3.14a}$$

Then

$$E[Y_{\tilde{H}}(t) Y_{\tilde{H}}(t\lambda)] = \frac{\sigma^2 t^{2\tilde{H}}}{2} \lambda^{\tilde{H}} \{|\ln \lambda + 1|^{2H} + |\ln \lambda - 1|^{2H} - 2|\ln \lambda|^{2H}\}, \quad t, \lambda > 0. \tag{3.14b}$$

Fig. 6 illustrates sample paths of $\tilde{Y}_H$. As expected, it exhibits strong correlations in the long run.

Example 3.3: Let $P(t), -\infty < t < \infty$ be the Poisson process with parameter β. Consider the following process.

$$X(t) = t^H \{P(\ln t\alpha) - P(\ln t)\}, \qquad t, \alpha > 0. \tag{3.15a}$$

By direct calculation, we can show that $\{X(t), t > 0\}$ is p-self-similar with the basic autocorrelation function

$$\Gamma(\lambda) = \begin{cases} \beta \ln(\alpha) \lambda^H \{1 - |\ln \lambda|\} & \frac{1}{\alpha} < \lambda < \alpha \\ 0 & \text{else} \end{cases}. \tag{3.15b}$$

Unlike the first two examples $\{X(t), t > 0\}$ is p-self-similar but does not exhibit long term correlations.

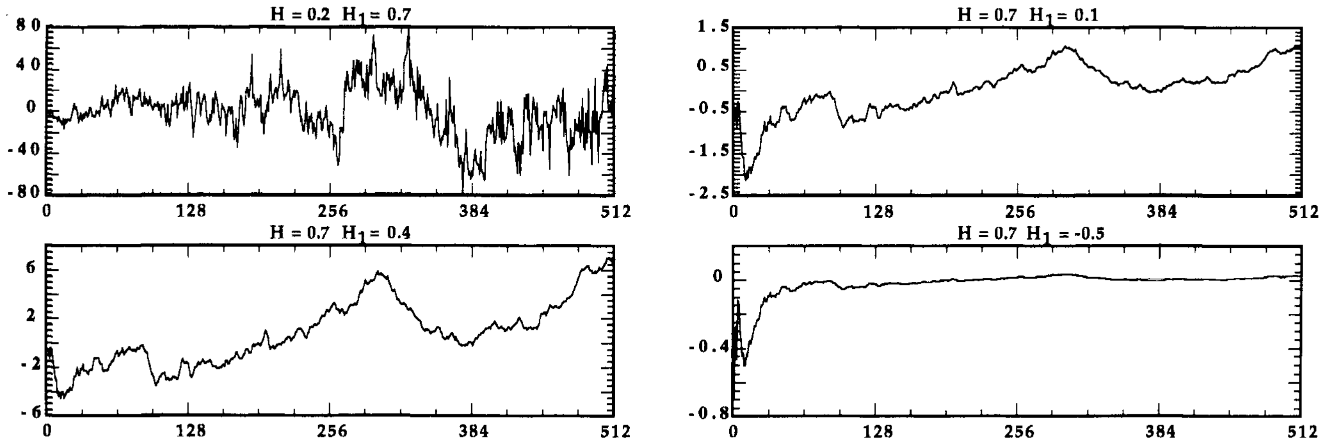

Fig. 5. Sample paths of the self-similar process induced from the generating process of fBm. H is the parameter of the generating process, and H_1 is the trend or the self-similarity parameter.

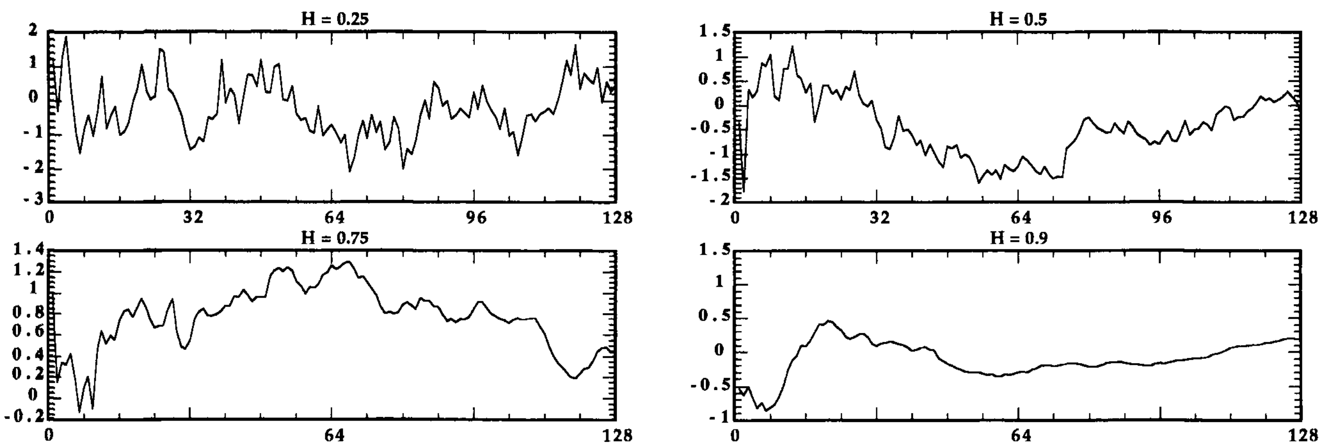

Fig. 6. Sample paths of the scale stationary process corresponding to the fractional Gaussian noise.

E. Spectral Representation of P-Self-Similar Processes

Among all available tools in statistical signal processing, spectral analysis is of special importance. This is due to the concepts of frequency and spectral density function by which the representation of a broad range of physical phenomena is simplified. In this subsection, we want to introduce similar concepts for wide sense p-self-similar processes. To motivate our development, let us first recall the spectral decomposition of the shift stationary processes. We know that given any ordinary wide sense stationary process $\{X(t), -\infty < t < \infty\}$ with square summable autocorrelation function, it can be represented in the following form [17]:

$$X(t) = \int_{-\infty}^{\infty} e^{-i\omega t} S(\omega)\, dB(\omega) \qquad (3.16)$$

where B is the Brownian motion, S is the spectral density function of the process, and the integration is defined in the mean square sense. Intuitively speaking, we can view $X(t)$ as the sum of the modulated statistically independent process $\{S(\omega)\, dB(\omega), -\infty < \omega < \infty\}$ with variance $S(\omega)$, where the modulating functions $\{e^{-i\omega t}, -\infty < \omega < \infty\}$ are periodic in shift. For p-self-similar processes, we want to develop a similar representation in which the modulating functions are periodic in scale. Such a representation is given by the following theorem, which is a corollary of the isometry relationship (3.3) and the spectral representation theorem of shift stationary processes.

Theorem 3.2: A function $\Gamma(\lambda)$, $\lambda > 0$ is the basic autocorrelation function of a wide sense p-self-similar process with parameter H if and only if there exists a nonnegative, symmetric measure F on $(-\infty, \infty)$ such that

$$\Gamma(\lambda) = \int_{-\infty}^{\infty} \lambda^{j\omega + H}\, dF(\omega), \qquad \lambda > 0. \qquad (3.17)$$

Proof 3.2: See Appendix II. □

The spectral representation theorem stated above leads to a new set of useful spectral domain tools to analyze and process $1/f$ phenomena. In particular, it leads to a whitening filter for p-self-similar processes via generalized Mellin transform by which a new concept of spectral density function can be derived.

Corollary 3.1: Any wide sense p-self-similar process with parameter H has the following spectral representation:

$$X(t) = \int_{-\infty}^{\infty} t^{j\omega + H}\, dB(\omega), \qquad t > 0 \qquad (3.18a)$$

where the integral is defined in the mean square sense, and $\{B(\omega), -\infty < \omega < \infty\}$ is the orthogonal increment process satisfying

$$\begin{aligned} &E[|B(\omega)|^2] \\ &\quad = F(\omega), \\ &E\{B(\omega)[\overline{B(\omega+\Delta)-B(\omega)})]\} \\ &\quad = 0, \qquad \Delta > 0 \text{ and } -\infty < \omega < \infty. \end{aligned} \tag{3.18b}$$

Moreover, if F is absolutely continuous, we have

$$\begin{aligned} f(\omega) &= \frac{dF}{d\omega}(\omega) \\ &= \frac{1}{2\pi}\int_0^\infty \lambda^{-j\omega-H-1}\Gamma(\lambda)\,d\lambda. \end{aligned} \tag{3.18c}$$

Proof 3.1: See Appendix II. □

Informally speaking, $f(\omega), -\infty < \omega < \infty$ defined in (3.18c) is the variance of the statistically independent process $dB(\omega), -\infty < \omega < \infty$, which can be regarded as unit constituents of the stochastic process. In addition, $f(\omega)$ can be viewed as a measure of correlation between the p-self-similar process $X(t)$ and the scale periodic sinusoid $t^{j\omega}$. Therefore, we shall refer to f as the *S-spectral density function,* or *S-spectrum, S* standing for scale. Note that the S-spectral density function quantifies only the random part, i.e., the generating process of a p-self-similar process. As a result, any two p-self-similar processes generated by the same process have the same S-spectral density functions. In addition, ω is directly associated with the S-period of the modulating functions, which is given by $e^{2\pi/\omega}$.

To elaborate on the physical meaning of S-spectral density function, let us examine the spectral behavior of the process introduced in Example 3.1.

$$z(t) = at^H \cos\left(\frac{2\pi}{f_0}\ln t + \varphi\right), \qquad t > 0. \tag{3.19a}$$

It is straightforward to show that the S-spectrum of the process is

$$f(\omega) = \frac{a^2}{4}\left\{\delta\left(\omega - \frac{2\pi}{f_0}\right) + \delta\left(\omega + \frac{2\pi}{f_0}\right)\right\} \tag{3.19b}$$

where $\delta(\cdot)$ is the delta function. Obviously, the energy of the process is concentrated at the S-period e^{f_0}. This implies that the process can be realized in the mean square sense by the modulated white noise process with variance $a^2/4$ in which the modulating function is $t^{j2\pi/f_0}$. As a result, the S-period e^{f_0} dominates the second-order behavior of the process. This observation is also verified by the S-autocorrelation function of the generating process given in (3.9c). Hence, we can conclude that given an empirical S-spectral density, the S-periods at which the spectral peaks occur can be interpreted as the dominant or the natural scales of the physical process.

F. Filtered P-Self-Similar Processes

For many practical problems, it may be of considerable interest to develop finite parameter, white noise driven models for p-self-similar processes since finite parameter models facilitate the development of efficient estimation and system identification methods. In this subsection, we discuss some system theoretic issues that will guide our development of finite parameter models in the subsequent section.

It is well known that any linear shift-invariant system, in particular, any casual, stable system, yields a wide sense shift stationary output when it is driven by a wide sense shift stationary input. The following theorem states the counterpart of this result for linear self-similar systems and p-self-similar processes.

Theorem 3.3: Let $\{X(t), t > 0\}$ be a p-self-similar process with parameter H_1 and h_{H_2} be the pseudo impulse response function of a linear self-similar system with parameter H_2 satisfying the following condition:

$$\int_0^\infty |h_{H_2}(t)|t^{-H_1}\,d\ln t < \infty. \tag{3.20a}$$

Then, the output process

$$Y(t) = t^{H_2}\int_0^\infty h_{H_2}\left(\frac{t}{\lambda}\right)X(\lambda)\,d\ln\lambda, \qquad t > 0 \tag{3.20b}$$

is p-self-similar with parameter $H_1 + H_2$.

Proof: See Appendix II. □

Note that for a scale stationary input, the condition (3.20a) is the sufficiency condition for the stability of a linear scale-invariant system. Now, we want to examine the output of linear scale-invariant systems driven by a special type of scale stationary process, which we refer to as scale stationary white noise.

The continuous time ordinary white noise process is interpreted as the derivative of the Brownian motion. In our development, we first introduce a process analogous to the Brownian motion and next interpret its increments as scale stationary white noise process.

Definition 3.4: We shall call $\{Z(\omega), \omega > 0\}$ a *scale orthogonal process* if it satisfies the following conditions:

$$\begin{aligned} E\{Z(\omega)[\overline{Z(\alpha\omega) - Z(\omega)}]\} &= 0, \\ E[|Z(\omega)|^2] &= \ln\omega, \qquad \alpha > 1 \\ &\text{and } \omega > 0. \end{aligned} \tag{3.21}$$

□

Note that one can rigorously derive such a process using Hilbert space methods [18]. Now, let us define an increment process:

$$dZ(\omega) = Z(\omega\,d\omega) - Z(\omega), \qquad d\omega > 1 \quad \omega > 0. \tag{3.22a}$$

By Definition 3.4, $\{dZ(\omega), \omega > 0\}$ are statistically independent. Similar to the interpretation of the continuous time white noise process, we can interpret the following process:

$$W_d(\omega) = \frac{Z(\omega\,d\omega) - Z(\omega)}{\ln d\omega} \tag{3.22b}$$

as scale stationary white noise process as the scale factor $d\omega \to 1$. The process $\{dZ(\omega), \omega > 0\}$ can be viewed as a stochastic counterpart of the unit driving force introduced in Section II (see (2.9)).

We shall adapt the following shorthand notation:

$$E[|dZ(\omega)|^2] = d\ln\omega, \qquad \omega > 0. \tag{3.22c}$$

for the variance of the process. Now, let $h(\cdot)$ be the pseudo impulse response function of a linear scale-invariant system with parameter H. Consider the following output process:

$$X(t) = t^H \int_0^\infty h\left(\frac{t}{\lambda}\right) dZ(\lambda), \qquad t > 0. \tag{3.23a}$$

Using the properties of the scale stationary white noise process, we can easily show that $\{X(t), t > 0\}$ is wide sense p-self-similar with the basic autocorrelation function

$$\Gamma(\lambda) = \lambda^H \int_0^\infty h\left(\frac{\lambda}{t}\right) h(t) \frac{dt}{t}, \qquad \lambda > 0. \tag{3.23b}$$

We will use the results of this subsection as a basis to develop a class of finite parameter p-self-similar processes in the following section.

IV. Self-Similar Autoregressive Models

In this section, we introduce a special class of p-self-similar processes, which we refer to as self-similar autoregressive models and investigate their practical value in modeling $1/f$ signals. Proposed models have two distinct advantages. First, we show that any wide sense p-self-similar processes can be approximated by a self-similar autoregressive model with a finite number of parameters. Therefore, they are rich enough to be useful in modeling broad range of $1/f$ physical phenomena. Second, they are linear in the sense that they are generated by white noise driven linear scale-invariant systems. As a result of this linear structure, it may be possible to develop efficient parameter estimation methods analogous to the methods developed for ordinary ARMA models.

As is well known, ordinary continuous-time autoregressive processes are generated by the ordinary white noise driven linear time-invariant systems whose dynamics can be represented by a linear constant coefficient differential equations. Motivated by this observation, we introduce a special class of wide sense p-self-similar processes generated by the white noise driven Euler–Cauchy system introduced in Section II. Recall that the generalized Euler–Cauchy system is given by

$$\begin{aligned} &\alpha_N t^N \frac{d^N}{dt^N} y(t) + \cdots + \alpha_1 t \frac{d}{dt} y(t) + \alpha_0 y(t) \\ &= \beta_M t^{M+H} \frac{d^M}{dt^M} x(t) + \cdots + \beta_1 t^{1+H} \\ &\quad \cdot \frac{d}{dt} x(t) + \beta_0 t^H x(t). \end{aligned} \tag{4.1}$$

For $M = 0$, the pseudo impulse response function of the system corresponding to the generalized Euler–Cauchy equation is equal to

$$h_H(t) = \sum_{j=0}^{n} \sum_{i=1}^{m_j} a_{ij} (\ln t^j) t^{-b_{ij}}, \qquad t \geq 1$$

and

$$N = \sum_{j=0}^{n} (m_j + 1) \tag{4.2}$$

where m_j, $j = 1, \cdots, n$ is the number of the repeated poles in the transfer function of the system. Assume that the system is stable, i.e., $b_{ij} > 0$, and consider the following output process:

$$y(t) = t^H \int_0^t h_H\left(\frac{t}{\lambda}\right) dZ(\lambda), \qquad t > 0 \tag{4.3}$$

where $\{Z(\lambda), \lambda > 0\}$ is the scale orthogonal increment process introduced in Definition 3.4. Then, the output of the Euler–Cauchy system is p-self-similar with the basic autocorrelation function

$$\Gamma(\lambda) = \begin{cases} \sum_{j=0}^{n} \sum_{i=0}^{m_j} a_{ij}^2 (\ln \lambda^{-j}) \lambda^{H+b_{ij}} & 0 < \lambda \leq 1 \\ \sum_{j=0}^{n} \sum_{i=0}^{m_j} a_{ij}^2 (\ln \lambda^{j}) \lambda^{H-b_{ij}} & \lambda > 1 \end{cases}. \tag{4.4a}$$

We shall refer to the process defined in (4.3) as the Nth-order *self-similar autoregressive* (SS-AR) process with parameter H. Since the filter is causal, SS-AR processes are Markov. For the first-order SS-AR process, the basic autocorrelation function is given by

$$\Gamma(\lambda) = \begin{cases} \sigma^2 \lambda^{H+v} & 0 < \lambda \leq 1 \\ \sigma^2 \lambda^{H-v} & \lambda > 1 \end{cases}.$$

Equation (4.1) is a symbolic representation of SS-AR processes and does not lend itself to a data synthesis method. Therefore, SS-AR processes basically serve as covariance models. To synthesize an SS-AR process, we utilize the S-autocorrelation function and Gaussian random generator. The sampling interval for the realizations are taken to be 1. Figs. 7 and 8 shows sample paths of first– and second-order SS-AR processes with various parameter values. As expected, the sample paths get smoother as the order of the model increases. Note that in a similar fashion, one can define self-similar autoregressive moving average models.

For any real valued p-self-similar process $\{X(t), t > 0\}$ with continuous, square summable S-spectral density f, it is possible to find a pth-order SS-AR process whose spectral density function is arbitrarily close to f. This suggests that $\{X(t), t > 0\}$ can be approximated in some sense by an SS-AR model. This result is rigorously stated by the following theorem.

Theorem 4.1: Let f be a continuous, square summable S-spectral density of a wide sense p-self-similar process with parameter H and $\varepsilon > 0$; then, there exists a pth-order SS-AR process with parameter H such that

$$|f_{AR}(\omega) - f(\omega)| < \varepsilon, \quad \text{for all } -\infty < \omega < \infty \tag{4.5}$$

where f_{AR} is the S-spectral density of the SS-AR process.

Proof: See Appendix III. □

In many cases, it may be sufficient to consider the properties of the SS-AR processes because Theorem 4.1 may provide the means to prove similar results for a wide range of wide sense p-self-similar processes through appropriate approximations.

As we proposed in the introduction, the p-self-similar processes may provide a framework for the analysis of $1/f$ physical signals. Now, we shall focus on SS-AR processes and justify our claim heuristically. Since $1/f$ processes are characterized by their empirical Fourier spectrum, we shall

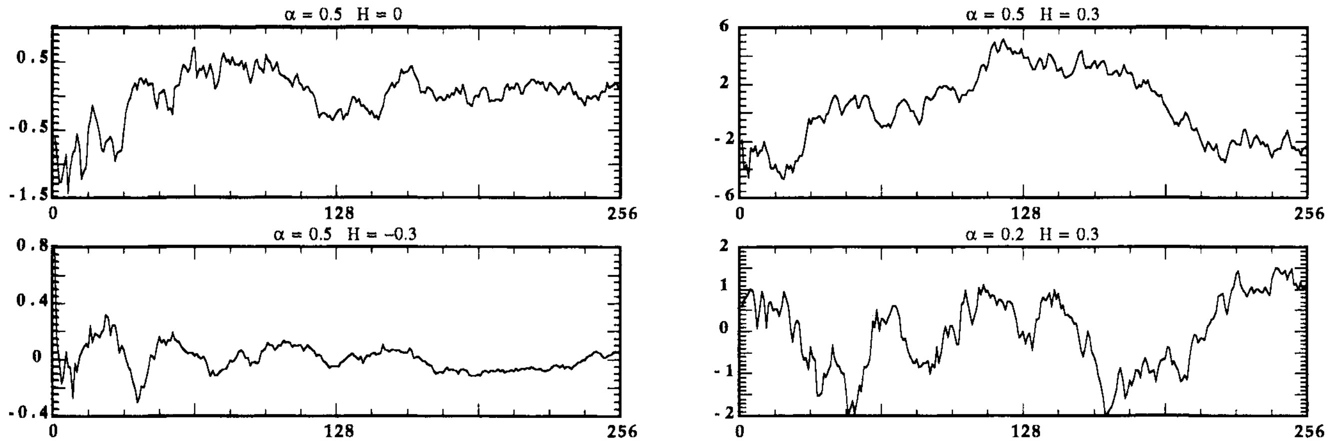

Fig. 7. Sample paths of the first order self-similar autoregressive process.

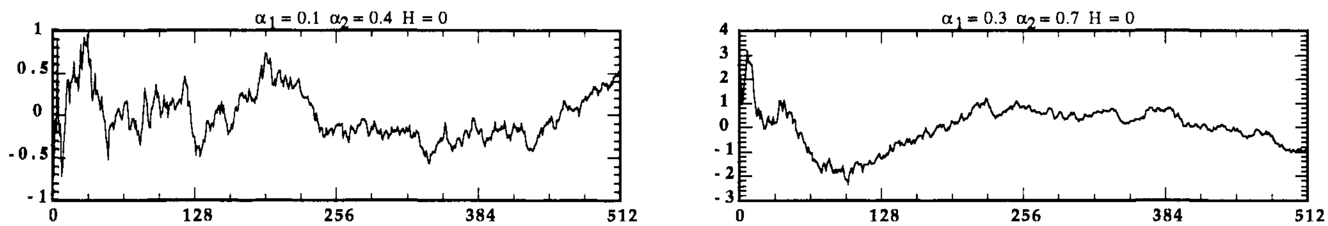

Fig. 8. Sample paths of the second order SS-AR process.

study the first-order SS-AR processes in the shift stationarity framework. Consider the covariance function of the first-order SS-AR process represented in terms of time lag.

$$E[X(t)X(t+\tau)] = \sigma^2\Big(1+\frac{\tau}{t}\Big)^{-\nu}, \quad \tau,\, t>0 \text{ and } \nu>0. \tag{4.6}$$

Consider the Fourier cosine transform of the covariance function in (4.6).

$$\hat{S}(f) = \int_0^\infty \sigma^2\Big(1+\frac{\tau}{t}\Big)^{-\nu} e^{j2\pi f\tau}\, d\tau \tag{4.7a}$$

$$= f^{\nu-1}t^{\nu}e^{-j2\pi ft}C(\nu), \quad \text{for } 0<\nu<1 \tag{4.7b}$$

where C is a complex-valued function of ν [20, pp. 1151–1152]. Equation (4.7b), together with Theorem 4.1, justify heuristically that the Fourier spectrum of a p-self-similar process can be approximated by a linear combination of $1/f$ spectrums.

V. Long-Term Correlations

As we have shown in Example 3.3, not all wide sense p-self-similar processes are long-term correlated. In statistics literature, long-term dependence is characterized in two ways: i) by the empirical correlation function decaying hyperbolically $\tau^{-\nu}, \nu > 0$ as the lag $\tau \to \infty$ and ii) by the sum of the lag based correlations increasing without limit as the lag increases. In order to identify those with long term correlations, we reinterpret the lag based criteria in terms of the S-autocorrelation function. Let $\{X(t),\, t>0\}$ be a wide sense p-self-similar process with parameter H, and let Γ be its basic autocorrelation function, i.e.,

$$E[X(t)X(t+\tau)] = t^{2H}\Big(1+\frac{\tau}{t}\Big)^{2H} R\Big(1+\frac{\tau}{t}\Big), \quad \tau > -t \tag{5.1}$$

where R is the S-autocorrelation function of the underlying generating process. To ensure the slow decay of the lag-based correlation function, we require (5.1) to be infinite for each fixed $t>0$. This requirement is equivalent to the following condition:

$$\int_0^\infty \Gamma(\lambda)\, d\lambda \to \infty. \tag{5.2}$$

This condition limits the range of the self-similarity parameter and the range of the parameters governing the generating process. In addition to the above condition, one can impose further restrictions on the parameters by imposing a hyperbolic decay on the lag-based correlations. For the sinusoidal model introduced in Example 3.1, the basic autocorrelation function is not summable for any H. However, hyperbolic decay requirement restricts the range of H to the negative real axis. For the first-order SS-AR process, the long-term correlation criteria is translated into $\nu - 1 < H < \nu, \nu > 0$, where H is the self-similarity parameter, and ν is the parameter of the generating process. In Fig. 9, we illustrate the validity of the long-term correlation criterion chosen by the empirical Fourier spectra of various p-self-similar processes. Note that for $H = 0$, the Fourier spectrum of the first-order SS-AR

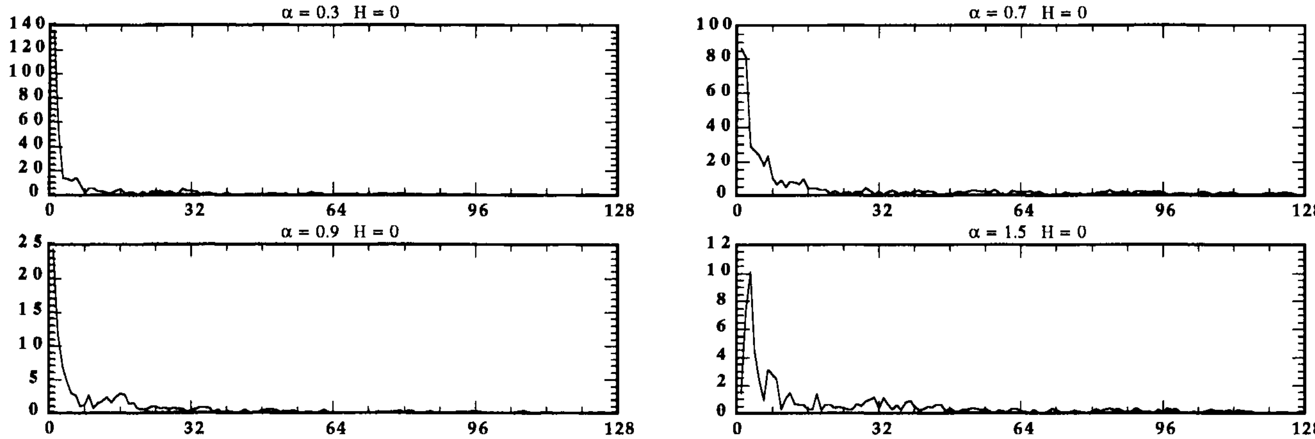

Fig. 9. Empiricial Fourier spectrum of the first order SS-AR process for various parameters.

process contains high energy at low frequencies for ν close to 0. For ν close to 1, the spectral energy of the process spreads over the entire frequency range.

VI. Conclusion

In this paper, we have proposed a class of statistically self-similar processes for the modeling and analysis of $1/f$ phenomena. In addition, we have developed a mathematical analysis framework whose foundation is based on the extensions of the basic concepts of classical time series analysis. We introduced new concepts, and practical analysis tools, such as S-periodicity, S-autocorrelation, and S-spectral density, which allow us to understand the structure of the proposed class of self-similar processes and guide us to devise practical signal processing schemes. We have established a relationship between the theory of linear scale-invariant systems and the analysis framework of the proposed class, which leads to a concrete physical understanding of the proposed class. As we discussed and illustrated by several examples, the proposed class of models are suitable for a variety of $1/f$ processes including Gaussian, non-Gaussian, generative, and Markovian processes. In particular, we introduced self-similar autoregressive models by which an arbitrary p-self-similar process can be approximated.

There are several promising directions for further research building on our formalism. In particular, essential topics for investigation are efficient parameter estimation methods for self-similar autoregressive models and discrete approximations. In addition, we expect that these models should be of value for segmentation of signals and, in two dimensions, for the identification of textures based on their self-similarity parameter. Work on these areas, as well as several applications of our formalism, is proceeding and will be reported in the future.

Appendix I

Proof of Theorem 2.1: Suppose h_H satisfies $\int_0^\infty |h_H(t)|\, d\ln t < \infty$.

Let

$$y(t) = t^H \int_0^\infty h_H(\lambda) x\left(\frac{t}{\lambda}\right) d\ln\lambda. \tag{I.1}$$

Then

$$\begin{aligned}\int_0^\infty |y(t)|^2 \frac{1}{t^{2H}}\, d\ln t &= \int_0^\infty \left\{ \int_0^\infty h_H(\lambda_1) x\left(\frac{t}{\lambda_1}\right) d\ln\lambda_1 \right.\\ &\quad \left. \cdot \overline{\int_0^\infty h_H(\lambda_2) x\left(\frac{t}{\lambda_2}\right) d\ln\lambda_2} \right\} d\ln t. \end{aligned} \tag{I.2a}$$

By Fubini's theorem

$$\begin{aligned}&= \int_0^\infty x\left(\frac{t}{\lambda_1}\right) \overline{x\left(\frac{t}{\lambda_2}\right)}\, d\ln t \int_0^\infty \int_0^\infty \\ &\quad \cdot h_H(\lambda_1)\overline{h_H(\lambda_2)}\, d\ln\lambda_1\, d\ln\lambda_2. \end{aligned} \tag{I.2b}$$

However, by the Cauchy–Schwartz inequality

$$\int_0^\infty x\left(\frac{t}{\lambda_1}\right)\overline{x\left(\frac{t}{\lambda_2}\right)}\, d\ln t \le \int_0^\infty |x(t)|^2\, d\ln t. \tag{I.3a}$$

In addition

$$\begin{aligned}&\int_0^\infty \int_0^\infty h_H(\lambda_1)\overline{h_H(\lambda_2)}\, d\ln\lambda_1\, d\ln\lambda_2 \\ &\quad \le \left(\int_0^\infty |h_H(\lambda)|\, d\ln\lambda\right)^2. \end{aligned} \tag{I.3b}$$

Hence

$$\int_0^\infty |y(t)|^2 \frac{1}{t^{2H}}\, d\ln t < \infty. \qquad \square$$

Appendix II

Proof of Theorem 3.1: Let

$$X(t) = t^H \tilde{X}(t), \quad t > 0 \text{ and some } -\infty < H < \infty. \tag{II.1}$$

Assume $\{\tilde{X}(t),\, t > 0\}$ is a strictly scale stationary process.

$$\begin{aligned}&\Pr[X(t_1) \le x_1, \cdots, X(t_N) \le x_N] \\ &\quad = \Pr[\tilde{X}(t_1) \le t_1^{-H} x_1, \cdots, \tilde{X}(t_N) \le t_N^{-H} x_N] \\ &\quad = \Pr[\tilde{X}(\lambda t_1) \le t_1^{-H} x_1, \cdots, \tilde{X}(\lambda t_N) \le t_N^{-H} x_N] \\ &\quad = \Pr[\tilde{X}(\lambda t_1)(\lambda t_1)^{-H} \le t_1^{-H} x_1, \cdots, \\ &\qquad X(\lambda t_N)(\lambda t_N)^{-H} \le t_N^{-H} x_N]. \end{aligned} \tag{II.2}$$

Now, assume $\{X(t), t > 0\}$ is a self-similar process with parameter H.

$$\begin{aligned}\Pr[\tilde{X}(t_1) \le x_1, \cdots, \tilde{X}(t_N) \le x_N] \\ = \Pr[X(t_1) \le t_1^H x_1, \cdots, \tilde{X}(t_N) \le t_N^H x_N] \\ = \Pr[X(\lambda t_1)\lambda^{-H} \le t_1^H x_1, \cdots, X(\lambda t_N)\lambda^{-H} \le t_N^H x_N] \\ = \Pr[\tilde{X}(\lambda t_1) \le x_1, \cdots, \tilde{X}(\lambda t_N) \le x_N]. \qquad \text{(II.3)}\end{aligned}$$

□

Proof of Theorem 3.2: This is a straightforward corollary of the classical theorem of Bochner and the isometry relationship between the wide sense p-self-similar and the wide sense shift stationary processes. Classical theorem of Bochner states that R is the autocorrelation function of a shift stationary process if and only if there is a symmetric, non-negative distribution F on $(-\infty, \infty)$ such that

$$R(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau}\, dF(\omega). \qquad \text{(II.4)}$$

Since for any wide sense p-self-similar process $\{X(t), t > 0\}$, there is a shift stationary process $\{Y(t), -\infty < t < \infty\}$ satisfying

$$Y(t) = t^{-H} X(e^t). \qquad \text{(II.5)}$$

The autocorrelation function R of Y is related to the basic autocorrelation function Γ of X by

$$R(\tau) = e^{-H\tau}\Gamma(e^\tau). \qquad \text{(II.6)}$$

Hence, by suitable change of variables, (II.4) becomes

$$\Gamma(\lambda) = \int_{-\infty}^{\infty} \lambda^{j\omega + H}\, dF(\omega), \qquad \lambda > 0. \qquad \text{(II.7)}$$

□

Proof of Corollary 3.1: As a consequence of the classical theorem of Bochner, any wide sense shift stationary process can be represented as

$$Y(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, dB(\omega), \qquad -\infty < t < \infty \qquad \text{(II.8)}$$

where the integral is defined in the mean square sense, and $\{B(\omega), -\infty < \omega < \infty\}$ is the orthogonal increment process satisfying

$$E[|B(\omega)|^2] = F(\omega), \qquad \text{(II.9a)}$$

$$E\{B(\omega)\overline{[B(\omega+\Delta) - B(\omega)]}\} = 0, \qquad \Delta > 0 \text{ and } -\infty < \omega < \infty. \qquad \text{(II.9b)}$$

Moreover, if F is absolutely continuous, we have

$$\begin{aligned} f(\omega) &= \frac{dF}{d\omega}(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\tilde{R}(\tau)\, d\tau. \qquad \text{(II.10)}\end{aligned}$$

where $\tilde{R}$ is the shift autocorrelation function. The corollary follows by the isometry relation (3.3) and (II.10). □

Proof of Theorem 3.3: It suffices to show that $E[Y(t)Y(1)] = t^{H_1+H_2}R(t)$, where R is the S-autocorrelation of some scale stationary process. By Fubini's theorem

$$\begin{aligned} E[Y(t)Y(1)] \\ = t^{H_2}\int_0^\infty\int_0^\infty h_{H_2}(\lambda_1)h_{H_2}(\lambda_2) \\ \cdot E\left[X\left(\frac{1}{\lambda_1}\right)X\left(\frac{1}{\lambda_2}\right)\right] d\ln\lambda_1\, d\ln\lambda_2. \qquad \text{(II.12)}\end{aligned}$$

However

$$E\left[X\left(\frac{1}{\lambda_1}\right)X\left(\frac{1}{\lambda_2}\right)\right] = \left(\frac{t}{\lambda_1}\right)^{H_1}\left(\frac{1}{\lambda_2}\right)^{H_1} R\left(t\frac{\lambda_2}{\lambda_1}\right) \qquad \text{(II.13)}$$

for some S-autocorrelation function R. Then, (II.12) becomes

$$\begin{aligned} E[Y(t)Y(1)] \\ = t^{H_2+H_1}\int_0^\infty\int_0^\infty R\left(t\frac{\lambda_2}{\lambda_1}\right)h_{H_2}(\lambda_1) \\ \cdot h_{H_2}(\lambda_2)\lambda_1^{-H_1}\lambda_2^{-H_2}\, d\ln\lambda_1\, d\ln\lambda_2. \qquad \text{(II.14a)} \\ \le t^{H_2+H_1}R(1)\left[\int_0^\infty |h_{H_2}(\lambda)|\lambda^{-H_1}\, d\ln\lambda\right]^2. \qquad \text{(II.14b)}\end{aligned}$$

□

IX. Appendix III

Proof of Theorem 4.1: It is sufficient to show that for a given $\varepsilon > 0$, there is a pth-order polynomial P with roots on the left half plane satisfying

$$\left|\frac{1}{P(\omega)} - f(\omega)\right| < \varepsilon \qquad \text{for } -\infty < \omega < \infty. \qquad \text{(III.1)}$$

The proof depends the following two results:

a) Given any continuous spectral density $\tilde{f}(\nu)$, $-\pi \le \nu \le \pi$ [19], there is a $K > 0$, and a pth-order polynomial A such that

$$\begin{aligned} A(e^{j\nu}) &= \prod_{k=1}^{p}(1 - \eta_k^{-1}e^{j\nu}) \\ &= 1 + a_1 e^{j\nu} + \cdots + a_p e^{j\nu p} \qquad \text{(III.2a)}\end{aligned}$$

with $|\eta_k| < 1$, and a_k are real valued for $k = 1, \cdots, p$ satisfying

$$\left|\frac{K}{|A(e^{j\nu})|^2} - \tilde{f}(\nu)\right| < \varepsilon \qquad \text{for a given } \varepsilon > 0. \qquad \text{(III.2b)}$$

b) The basic device to extend the result stated in part i) is the following mapping of the closed unit disc onto the left half plane, which can be achieved by

$$s = \frac{z-1}{z+1}. \qquad \text{(III.3)}$$

This maps the boundary $z = e^{j\nu}$ onto the boundary $s = j\omega$, and we have $\omega = \tan\nu/2$. For a given continuous spectral density $f(\omega)$, $-\infty < \omega < \infty$, define

$$\tilde{f}(\nu) = f\left(\frac{\tan\nu}{2}\right), \qquad -\pi \le \nu \le \pi. \qquad \text{(III.4)}$$

It is easy to check that $\tilde{f}$ is continuous, symmetric, positive, and square summable. Therefore, by i), it is immediate that there is a pth-order polynomial with real coefficients and roots on the left-hand plane satisfying

$$\left|\frac{K}{|A(j\omega)|^2} - f(\omega)\right| < \varepsilon \qquad \text{for all } -\infty < \omega < \infty. \quad \square \tag{III.5}$$

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