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Euclidean Motion Group Representations and the Singular Value Decomposition of the Radon Transform

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The matrix elements of the unitary irreducible representations of the Euclidean motion group are closely related to the Bessel and Gegenbauer functions. These special functions also arise in the singular functions of the singular value decomposition (SVD) of the Radon transform. In this paper, our objective is to study the Radon transform using harmonic analysis over the Euclidean motion group and explain the origin of the special functions present in the SVD of the Radon transform from the perspective of group representation theory. Starting with a convolution representation of the Radon transform over the Euclidean motion group, we derive a method of inversion for the Radon transform using harmonic analysis over the Euclidean motion group. We show that this inversion formula leads to an alternative derivation of the SVD of the Radon transform. This derivation reveals the origin of the special functions present in the SVD of the Radon transform. The derivation of the SVD of the Radon transform is a special case of a general result developed in this paper. This result shows that an integral transform with a convolution kernel is decomposable if the matrix elements of the irreducible unitary representations of the underlying group is separable.

Key Words: Radon transform, singular value decomposition, special functions, Euclidean motion group, harmonic analysis.

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1 Introduction

Since the seminal work of Radon in 1917 [28], the Radon transform has become a major area of research in both pure and applied mathematics (see [6, 8, 11–17, 23, 25, 30] and references therein). The importance of the Radon transform for today's imaging technologies is another motivation for investigating the properties of the Radon transform [9, 10, 18, 19, 24, 27].

In this paper we present a new derivation of the singular value decomposition (SVD) of the Radon transform using harmonic analysis over the Euclidean motion group, $M(N)$. The SVD of the Radon transform has been studied in [5, 20, 21]. However, the origin of the specific special functions that arise in the SVD of the Radon transform was not discussed. Our derivation explains the presence of the special functions in the SVD of the Radon transform by using the relationship between the irreducible unitary representations of the Euclidean motion group and special functions.

It is known that the Euclidean motion group provides the natural symmetries of the Radon transform since the Radon transform is a mapping between the two homogeneous spaces of the Euclidean motion group [14, 17]. Starting with a convolution representation of the Radon transform over the Euclidean motion group, we diagonalize the Radon transform in the Euclidean motion group Fourier ($M(N)$-Fourier) domain and obtain the spherical harmonic decomposition of the projection slice theorem. After showing that an integral transform with a convolution kernel is separable if the matrix elements of the unitary representations of the underlying group are separable, we derive the singular functions of the inverse Radon transform, which leads to the SVD of the Radon transform. It is the separability of the matrix elements of the irreducible unitary representations of the Euclidean motion group that gives rise to the SVD of the Radon transform and the special functions present in this decomposition.

The rest of the paper is organized as follows: Section 2 provides a general overview of harmonic analysis over groups. Section 3 introduces harmonic analysis over the Euclidean motion group and related special functions. Section 4 introduces the Radon transform and spherical harmonic decomposition of the projection slice theorem by using Fourier analysis over $M(N)$. Section 5 presents the main results of the paper. Finally, Section 6 concludes our discussion. Appendices A and B provide the computation of the $M(N)$-Fourier coefficients of the projections and the convolution kernel of the Radon transform.

2 Harmonic Analysis over Groups

Harmonic analysis over groups is directly related to the decomposition of unitary representations of a group as a direct sum of its irreducible unitary representations. This decomposition enables spectral or Fourier decomposition of functions defined over the group. Fourier transform over groups has the important property of mapping convolution into multiplication in the Fourier transform domain.

Representation of a group G over a vector space V is a homomorphism $T : g \to T(g)$ of G into the linear transformations of V, $GL(V)$. It follows from this definition that,

$$T(g_1)T(g_2) = T(g_1 g_2), \quad T(g)^{-1} = T(g^{-1}),$$
$$T(e) = I, \quad g_1, g_2 \in G,$$

where $e \in G$ is the identity element and I is the identity transformation in $GL(V)$.

Given an orthonormal basis $\{\boldsymbol{e}_i\}$ of V, for each $T(g)$, we can associate the matrix $(t_{ij}(g))$, $1 \leq i, j \leq n$ given by

$$T(g)\boldsymbol{e}_j = \sum_{i=1}^{n} t_{ij}(g)\boldsymbol{e}_i,$$

where n is the dimension of the vector space V. The matrix $(t_{ij}(g))$ is called *matrix representation* for the representation $T(g)$ with respect to the basis $\{\boldsymbol{e}_i\}$. The n^2 continuous functionals $\{t_{ij}(g)\}$ also defines the $n-$dimensional representation of the group G. If V is an infinite-dimensional vector space, then $(t_{ij}(g))$ is an infinite matrix with elements given by

$$t_{ij}(g) = (\boldsymbol{e}_i, T(g)\boldsymbol{e}_j), \quad 1 \leq i, j < \infty.$$

By the homomorphism property of $T(g)$ the matrices $(t_{ij}(g))$, $g \in G$ satisfies the usual matrix multiplication rule, i.e.

$$t_{ij}(g_1 g_2) = \sum_k t_{ik}(g_1) t_{kj}(g_2).$$

If $T(g)$ and $Q(g)$ are two representations of a group G over the spaces V and W, respectively, then $T(g)$ and $Q(g)$ are said to be *equivalent* if there is an invertible linear mapping between V and W such that

$$Q(g) = AT(g)A^{-1}.$$

Equivalent representations have the same matrix representations, i.e.

$$Q(g)A\boldsymbol{e}_k = AT(g)\boldsymbol{e}_k = A\sum_j t_{jk}(g)\boldsymbol{e}_j = \sum_j t_{jk}(g)A\boldsymbol{e}_j$$

and form an equivalence class. Therefore, it is sufficient to find one representation $T(g)$ in each equivalence class.

A representation $T(g)$ of a group G in an inner-product space V is called *unitary* if it preserves the inner-product over V, i.e.

$$(T(g)\boldsymbol{x}, T(g)\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y}).$$

A measure $d(g)$ on the group G is called *left-invariant* if for any measurable subset H of G, and for any $g \in G$,

$$\int_H d(h) = \int_{gH} d(gh),$$

and *right-invariant* if

$$\int_H d(h) = \int_{Hg} d(hg).$$

A measure $d(g)$ on the group G is called *invariant* if it is both left- and right-invariant. A group having an invariant measure is called a *unimodular* group. Otherwise it is called a *non-unimodular* group. For any given representation $T(g)$ of the group G on the inner-product space V, one can define a new inner-product, $(.,.)_M$, on V

$$(\boldsymbol{x}, \boldsymbol{y})_M = \int_G (T(g)\boldsymbol{x}, T(g)\boldsymbol{y})\, d(g),$$

such that $T(g)$ becomes a unitary representation over V with respect to this new inner-product $(.,.)_M$, where, $d(g)$ being the invariant measure over G. Therefore, without loss of generality, given any representation, we may assume that it is unitary, and denote it by $U(g)$.

A subspace $V^{(0)}$ of V is called *invariant* if for any $\boldsymbol{x} \in V^{(0)}$, $U(g)\boldsymbol{x} \in V^{(0)}$, for all $g \in G$. The null-subspace and the whole space V are the *trivial* invariant subspaces. A representation is called *irreducible* if the only invariant subspaces are the trivial ones. Otherwise it is called *reducible*. Decomposing V as a direct sum of mutually orthogonal invariant subspaces $\{V^{(\lambda)}\}$ such that $U(g)$ is irreducible on each $V^{(\lambda)}$,

$$V = \sum_\lambda \oplus V^{(\lambda)},$$

enables decomposition of $U(g)$ as an orthogonal direct sum

$$U(g) = \sum_{\lambda} \oplus U^{(\lambda)}(g),$$

where $U^{(\lambda)}(g)$ is the restriction of $U(g)$ to the subspace $V^{(\lambda)}$. A representation is called *completely reducible* if it is the direct sum of irreducible representations.

Let G be a unimodular, separable locally compact group of Type I [7, 22]. Then, the *Fourier transform* of $f \in L^2(G)$ is defined as,

$$\mathcal{F}\{f\}(\lambda) = \widehat{f}(\lambda) = \int_G d(g) f(g) U^{(\lambda)}(g^{-1})$$

and the *inverse Fourier transform* is given by

$$\mathcal{F}^{-1}\{\widehat{f}\}(g) = f(g) = \int_{\widehat{G}} \mathrm{Tr}\left\{\widehat{f}(\lambda) U^{(\lambda)}(g)\right\} d(\lambda),$$

where $\widehat{G}$ is the collection of all $\{\lambda\}$ values and is called the dual of G [31]. Using the matrix representations, the matrix elements of the Fourier transform and the corresponding inverse Fourier transform are given by

$$\widehat{f}_{ij}(\lambda) = \int_G f(g) u_{ij}^{(\lambda)}(g^{-1}) d(g),$$

$$f(g) = \int_{\widehat{G}} \sum_{i,j} \widehat{f}_{ij}(\lambda) u_{ji}^{(\lambda)}(g) d(\lambda).$$

In the next section, we will look into the case where G is the group of motions of the $N-$dimensional Euclidean space.

3 Harmonic Analysis over the Euclidean Motion Group

In this section, we will provide a review of the irreducible representations and the Fourier transform of the Euclidean motion group and some related properties. A more detailed discussion on these topics can be found in [31,32].

3.1 *Euclidean Motion Group*

The isometries of $\mathbb{R}^N$ that preserve orientation form a group called the *Euclidean motion group* or motion group of N-dimensions, denoted by $M(N)$. The elements of $M(N)$ are formed by tuples $(R_\theta, \boldsymbol{r})$, where $\boldsymbol{r} \in \mathbb{R}^N$ is the translation component, and $R_\theta \in SO(N)$ is the rotation component parameterized by θ. For $N = 2$, θ can be parameterized as an element of the unit circle. For $N = 3$, θ can be parameterized by, for example, axis-angle description of the rotation, Euler angles, Caley-Klein parameters or any other parameterization for three dimensional rotation.

The group operation of $M(N)$ is given by

$$(R_\theta, \boldsymbol{r})(R_\phi, \boldsymbol{x}) = (R_\theta R_\phi, R_\theta \boldsymbol{x} + \boldsymbol{r}). \tag{1}$$

This defines $M(N)$ as the semi-direct product of the additive group $\mathbb{R}^N$ and the rotation group $SO(N)$. The columns of $R_\theta \in SO(N)$ forms an orthonormal basis for $\mathbb{R}^N$, meaning that each column is an element of $N-1$ dimensional sphere, S^{N-1}. Consequently, rows of R_θ also forms an orthonormal basis.

The identity element of the group is given by $(I, \boldsymbol{0})$, I being the identity rotation, and the inverse element of $g = (R_\theta, \boldsymbol{r})$ is given by $g^{-1} = (R_\theta^{-1}, -R_\theta^{-1}\boldsymbol{r})$.

Alternatively, the elements of $M(N)$, can be represented as $(N+1)\times(N+1)$ dimensional matrices of the form

$$A(g) = \begin{bmatrix} R_\theta \ \boldsymbol{r} \\ \mathbf{0}^T \ 1 \end{bmatrix}. \tag{2}$$

Then, the group operation becomes the usual matrix multiplication. The identity element is the identity matrix and the inverse of each element can be obtained by matrix inversion.

3.2 *Irreducible Unitary Representations of $M(N)$*

Irreducible, unitary representations $U^{(\lambda)}(g)$, $\lambda > 0$, of $M(N)$ on $L^2(S^{N-1})$ is given by

$$(U^{(\lambda)}(g)F)(\boldsymbol{s}) = \mathrm{e}^{-\mathrm{i}\lambda(\boldsymbol{r}\cdot\boldsymbol{s})}F(R_\theta^{-1}\boldsymbol{s}), \qquad F \in L^2(S^{N-1}), \tag{3}$$

where $g = (\theta, \boldsymbol{r})$ is an element of $M(N)$, $\boldsymbol{s}$ is a point on the unit sphere S^{N-1}, and $(\ \cdot\ )$ is the standard inner product over $\mathbb{R}^N$ [32].

Since the spherical harmonics form an orthonormal basis for $L^2(S^{N-1})$, matrix elements for the unitary representation $U^{(\lambda)}(g)$ of $M(N)$ is expressed by

$$u_{mn}^{(\lambda)}(g) = (S_m, U^{(\lambda)}(g)S_n) = \int_{S^{N-1}} \overline{S_m(\boldsymbol{s})}\mathrm{e}^{-\mathrm{i}\lambda(\boldsymbol{r}\cdot\boldsymbol{s})}S_n(R_\theta^{-1}\boldsymbol{s})d(\boldsymbol{s}), \tag{4}$$

where m and n are multi-indices for the spherical harmonics, with the first element denoting the degree of the spherical harmonics, and $d(\boldsymbol{s})$ is the invariant normalized measure on S^{N-1}.

The matrix elements of $U^{(\lambda)}(g)$ satisfy the following properties:

1 *Adjoint property:*

$$(u^{(\lambda)}_{mn}(g^{-1})) = (u^{(\lambda)}_{mn}(g))^{-1} = \overline{(u^{(\lambda)}_{nm}(g))} \tag{5}$$

2 *Homomorphism property:*

$$u^{(\lambda)}_{mn}(g_1 g_2) = \sum_k u^{(\lambda)}_{mk}(g_1) u^{(\lambda)}_{kn}(g_2) \tag{6}$$

3 *Orthogonality property:*

$$\int_G u^{(\lambda)}_{mk}(g) \overline{u^{(\mu)}_{ls}(g)} d(g) = \delta_{\mu\lambda}\delta_{ml}\delta_{ks}, \tag{7}$$

and $\{u^{(\lambda)}_{mn}(g)\}$ form a complete orthonormal system in $L^2(M(N), d(g))$.

3.3 *Associated Spherical and Zonal Spherical Functions*

Let $\mathfrak{S}^l$ denote the space of spherical harmonics of degree l over S^{N-1}. $\{\mathfrak{S}^l\}_{l=0}^{\infty}$ are orthogonal and span $L^2(S^{N-1})$. A basis for $\mathfrak{S}^l$ is given by

$$\Xi_m(\boldsymbol{s}) = A^l_m \prod_{j=0}^{N-3} r_{n-j}^{m_j - m_{j+1}} C_{m_j - m_{j+1}}^{\frac{N-j-2}{2} + m_{j+1}} \left(\frac{s_{N-j}}{r_{N-j}} \right) (s_1 + i s_2)^{m_{N-2}},$$

where $C^{\nu}_n(r)$ is the Gegenbauer polynomial with associated weight function given by $(1-r^2)^{\nu-1/2}$ for $\nu > 1/2$, $m = (m_0, m_1, \ldots, \pm m_{N-2})$ such that $l = m_0 \geq m_1 \geq \ldots \geq m_{N-2}$, $r_{N-j} = \left(\sum_{k=1}^{N-j} s_k^2\right)^{1/2}$, and A^l_m is the normalizing factor such that

$$\int_{S^{N-1}} |\Xi_m(\boldsymbol{s})|^2 \, d(\boldsymbol{s}) = 1.$$

If $l = 0$, $m = (0, \ldots, 0) = \mathbf{0}$. Then, $\Xi_{\mathbf{0}}(\boldsymbol{s})$ is equal to 1, and invariant under the rotations. For $h_1, h_2 \in SO(n-1) \times \mathbf{0}$ and $g = (R_\theta, \boldsymbol{r}) \in M(N)$, the functions $u^{(\lambda)}_{m\mathbf{0}}(g)$ are invariant under right rotations,

$$u^{(\lambda)}_{m\mathbf{0}}(gh_2) = u^{(\lambda)}_{m\mathbf{0}}(g) = u^{(\lambda)}_{m\mathbf{0}}((I, \boldsymbol{r})),$$

and the functions $u^{(\lambda)}_{\mathbf{00}}(g)$ are invariant under left and right rotations,

$$u^{(\lambda)}_{\mathbf{00}}(h_1 g h_2) = u^{(\lambda)}_{\mathbf{00}}(g) = u^{(\lambda)}_{\mathbf{00}}((I, |\boldsymbol{r}|\boldsymbol{e_n})).$$

The matrix elements $\{u^{(\lambda)}_{m\mathbf{0}}(g)\}_m$ depend only on the translation component of g and are called the *associated spherical functions* of the representation $U^{(\lambda)}(g)$. The matrix element $u^{(\lambda)}_{\mathbf{00}}(g)$ depends only on the radius of the translation component and is called the *zonal spherical function* of the representation $U^{(\lambda)}(g)$.

3.3.1 ***Zonal Spherical Functions.*** The zonal spherical functions are given by

$$u^{(\lambda)}_{\mathbf{00}}(g) = (\Xi_{\mathbf{0}}, U^{(\lambda)}(g)\Xi_{\mathbf{0}}) = \int_{S^{N-1}} \mathrm{e}^{-\mathrm{i}\lambda|\boldsymbol{r}|s_N} d(\boldsymbol{s}) \tag{8}$$

$$= \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi}\,\Gamma(\frac{N-1}{2})} \int_0^\pi \mathrm{e}^{-\mathrm{i}\lambda|\boldsymbol{r}|\cos(\varphi)} \sin^{N-2}(\varphi) d\varphi. \tag{9}$$

This integral representation can be rewritten in terms of Bessel functions by

$$u^{(\lambda)}_{\mathbf{00}}(g) = \Gamma\left(\frac{N}{2}\right) \frac{J_{\frac{N-2}{2}}(\lambda|\boldsymbol{r}|)}{\left(\frac{\lambda|\boldsymbol{r}|}{2}\right)^{\frac{N-2}{2}}}. \tag{10}$$

Equating (8) and (10), one obtains the identity

$$\int_{-1}^{1} \mathrm{e}^{\mathrm{i}xt}(1-t^2)^{\frac{n-1}{2}}dt = \frac{\sqrt{\pi}\ \Gamma(\frac{n+1}{2})J_{n/2}(x)}{(x/2)^{n/2}}. \tag{11}$$

We will use identity (11) in the derivation of the identity (17), which is a key property towards the derivation of the SVD of the Radon transform.

3.3.2 ***Associated Spherical Functions.*** The associated spherical functions are given by

$$u^{(\lambda)}_{m\mathbf{0}}(g) = (\Xi_m, U^{(\lambda)}(g)\Xi_{\mathbf{0}}) = \int_{S^{N-1}} \mathrm{e}^{-\mathrm{i}\lambda(\boldsymbol{r}\cdot\boldsymbol{s})}\overline{\Xi_m(\boldsymbol{s})}d(\boldsymbol{s}). \tag{12}$$

For $\boldsymbol{r} = r\boldsymbol{e_n}$, $r \in \mathbb{R}$, (12) is nonzero if and only if $m = (l, 0, \ldots, 0)$ and takes the form

$$u^{(\lambda)}_{m\mathbf{0}}(g) = \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi}\ \Gamma(\frac{N-1}{2})}\sqrt{\frac{l!\ \Gamma(N-2)(2l+N-2)}{\Gamma(N+l-2)(N-2)}} \times \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\lambda r x}C_l^{\frac{N-2}{2}}(x)(1-x^2)^{\frac{n-3}{2}}dx \tag{13}$$

$$= \frac{\Gamma(N/2)}{2^l\sqrt{\pi}\ \Gamma(l+\frac{N-1}{2})}\sqrt{\frac{\Gamma(N+l-2)(2l+N-2)}{l!\ \Gamma(N-1)}} \times(-\mathrm{i}\lambda r)^l \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\lambda r x}(1-x^2)^{l+\frac{n-3}{2}}dx. \tag{14}$$

Using the equality (11), (14) becomes

$$u^{(\lambda)}_{m\mathbf{0}}(g) = \mathrm{i}^{-l}\Gamma\left(\frac{N}{2}\right)\sqrt{\frac{\Gamma(N+l-2)(2l+N-2)}{l!\ \Gamma(N-1)}}\frac{J_{l+\frac{N-2}{2}}(\lambda r)}{\left(\frac{\lambda r}{2}\right)^{\frac{n-2}{2}}}. \tag{15}$$

Equating (13) and (15), one obtains the identity

$$\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\lambda r} C_l^{\frac{N-2}{2}}(r)(1-r^2)^{\frac{N-3}{2}} dr = \frac{\mathrm{i}^l \sqrt{\pi}\, \Gamma\left(\frac{N-1}{2}\right) \Gamma(N+l-2)}{l!\, \Gamma(N-2)} \frac{J_{l+\frac{N-2}{2}}(\lambda)}{(\lambda/2)^{\frac{N-2}{2}}}. \tag{16}$$

By the orthogonality relationship of the Gegenbauer polynomials,

$$\mathrm{e}^{\mathrm{i}\lambda r} = \Gamma\left(\frac{N-2}{2}\right) \sum_{l=0}^{\infty} \mathrm{i}^l \left(l + \frac{N-2}{2}\right) \frac{J_{l+\frac{N-2}{2}}(\lambda)}{(\lambda/2)^{\frac{N-2}{2}}} C_l^{\frac{N-2}{2}}(r).$$

More generally, it can be verified that (see 9.3.8 equation (8) in [34])

$$\mathrm{e}^{\mathrm{i}\lambda r} = \Gamma(\nu)\, (\lambda/2)^{-\nu} \sum_{l=0}^{\infty} \mathrm{i}^l\, (l+\nu)\, J_{l+\nu}(\lambda) C_l^{\nu}(r), \quad -1 < x < 1, \quad \nu > -1/2. \tag{17}$$

The associated spherical functions and the zonal spherical functions of the irreducible unitary representation of the group $M(N)$ are closely related to the Gegenbauer polynomials and Bessel functions. By the orthogonality of Gegenbauer polynomials and Bessel functions, (17) provides a decomposition of $\mathrm{e}^{\mathrm{i}\lambda r}$. This property plays an important role in the derivation of the SVD of the Radon transform.

Further properties involving Bessel function and Gegenbauer polynomials can be derived through the properties of the unitary representations, which is beyond the scope of our study. For an extensive study on the group representations and their relationship with the special functions, we refer the reader to [32–35].

3.4 *Fourier Transform over M(N)*

Let $f \in L^2(M(N))$, then its *Fourier transform* is defined as

$$\mathcal{F}(f)(\lambda) = \widehat{f}(\lambda) = \int_{M(N)} d(g)\, f(g) U^{(\lambda)}(g^{-1}), \qquad \lambda > 0 \tag{18}$$

and the *inverse Fourier transform* is given by

$$\mathcal{F}^{-1}(\widehat{f})(g) = f(g) = \int_0^\infty \mathrm{Tr}\left(\widehat{f}(\lambda) U^{(\lambda)}(g)\right) \lambda^{N-1} d\lambda. \tag{19}$$

Using the matrix elements of the unitary representation, the matrix elements of the Fourier and the inverse Fourier transforms over $M(N)$ can be expressed as

$$\widehat{f}_{mn}(\lambda) = \int_{M(N)} f(g) u_{mn}^{(\lambda)}(g^{-1})\, d(g), \tag{20}$$

$$f(g) = \int_0^\infty \sum_{m,n} \widehat{f}_{mn}(\lambda) u_{nm}^{(\lambda)}(g) \lambda^{N-1} d\lambda. \tag{21}$$

Properties of M(N)-Fourier transform

Fourier transform over $M(N)$ satisfies the following properties:

1 *Plancherel Equality:*

$$\int_{M(N)} f_1(g)\overline{f_2(g)}\ d(g) = \textstyle\int_0^\infty \mathrm{Tr}\left(\widehat{f_2}^{\dagger}(\lambda)\widehat{f_1}(\lambda)\right) \lambda^{N-1} d\lambda. \tag{22}$$

For $f_1 = f_2 = f$,

$$\int_{M(N)} |f(g)|^2\ d(g) = \int_0^\infty \left\|\widehat{f}(\lambda)\right\|_2^2 \lambda^{N-1} d\lambda, \tag{23}$$

This property is analogous to the Parsevals equality of the standard Fourier transform.

2 *Convolution Property:*

$$\mathcal{F}(f_1 * f_2)(\lambda) = \mathcal{F}(f_2)(\lambda)\mathcal{F}(f_1)(\lambda). \tag{24}$$

Equivalently,

$$\mathcal{F}(f_1 * f_2)_{mn}(\lambda) = \sum_q \mathcal{F}(f_2)_{mq}(\lambda)\mathcal{F}(f_1)_{qn}(\lambda) \ \ . \tag{25}$$

Note that since $M(N)$ is not commutative, $\mathcal{F}(f_1)(\lambda)\mathcal{F}(f_2)(\lambda)$ is not necessarily equal to $\mathcal{F}(f_2)(\lambda)\mathcal{F}(f_1)(\lambda)$.

3 *Adjoint Property:*

$$\mathcal{F}\left(f^*\right)(\lambda) = \left[\mathcal{F}(f)(\lambda)\right]^\dagger, \tag{26}$$

where $f^*(g) = \overline{f(g^{-1})}$ and $U^\dagger$ denotes the adjoint of the operator U. Equivalently,

$$\widehat{f^*}_{mn}(\lambda) = \overline{\widehat{f}_{nm}(\lambda)}. \tag{27}$$

3.5 *M(N)-Fourier Transform over the Homogeneous Space* $\mathbb{R}^N$

Any function $f \in L^2(\mathbb{R}^N)$ can be treated as a rotation invariant function over $M(N)$, by $f(g) = f(R_\theta, \boldsymbol{r}) = f(\boldsymbol{r})$. This extension is not only well-defined, but also treats f as an $L^2(M(N))$ function, since $SO(N)$ is a compact subgroup of $M(N)$, and the measure on $\mathbb{R}^N$ is invariant under the action of $M(N)$. The matrix coefficients of $M(N)$-Fourier transform of f is given by

$$\begin{aligned}\widehat{f}_{mn}(\lambda) &= \int_{M(N)} f(g) u^{(\lambda)}_{mn}(g^{-1}) d(g) \\ &= \delta_m \int_{\mathbb{R}^N} f(\boldsymbol{r}) \int_{S^{N-1}} \mathrm{e}^{\mathrm{i}\lambda \boldsymbol{r}\cdot\boldsymbol{s}} S_n(\boldsymbol{s}) d(\boldsymbol{s}) d\boldsymbol{r} \qquad (28) \\ &= \delta_m \widetilde{f}_n(-\lambda), \qquad (29)\end{aligned}$$

where $\widetilde{f}_n(\lambda)$ is the spherical harmonic decomposition of the standard Fourier transform of f. Observe from (28) that taking Fourier transform of f is equivalent to taking an $N-$dimensional standard Fourier transform followed by spherical harmonic decomposition. This decomposes f in terms of the associated spherical functions $\{u_{n0}(g)\}$ in $L^2(\mathbb{R}^N)$, i.e.

$$\int_{\mathbb{R}^N} f(\boldsymbol{r}) \int_{S^{N-1}} \mathrm{e}^{\mathrm{i}\lambda \boldsymbol{r}\cdot\boldsymbol{s}} S_n(\boldsymbol{s}) d(\boldsymbol{s}) d\boldsymbol{r} = (f, u^{(\lambda)}_{n0}(g)),$$

where $(f_1, f_2) = \int_{\mathbb{R}^N} f_1(\boldsymbol{r}) \overline{f_2(\boldsymbol{r})} d\boldsymbol{r}$ is the inner-product over $L^2(\mathbb{R}^N)$. The inverse $M(N)$-Fourier transform then becomes,

$$\begin{aligned}f(g) &= \int_0^\infty \sum_{m,n} \widehat{f}_{mn}(\lambda) u^{(\lambda)}_{nm}(g) \lambda^{N-1} d\lambda \\ &= \int_0^\infty \sum_{n} \widetilde{f}_n(-\lambda) u^{(\lambda)}_{n0}(g) \lambda^{N-1} d\lambda.\end{aligned}$$

It is straightforward to show that $u^{(\lambda)}_{m0}(g) = u^{(\lambda)}_{0m}(g)$, when $u^{(\lambda)}_{m0}(g)$ is calculated with respect to Ξ_m [32].

3.6 *Distributions and $M(N)$-Fourier Transform*

Let $\mathcal{S}(M(N))$ denote the space of rapidly decreasing functions on $M(N)$. The $M(N)$-Fourier transform can be extended to $\mathcal{S}(M(N))$, and is injective [31,32]. Let $\mathcal{S}'(M(N))$ denote the space of linear functionals over $\mathcal{S}(M(N))$. $\mathcal{S}'(M(N))$ is called the space of *tempered distributions* over $M(N)$. Let $u \in \mathcal{S}'(M(N))$ and $\varphi \in \mathcal{S}(M(N))$. The value $u(\varphi)$ is denoted by $\langle u, \varphi\rangle$ or $\int_{M(N)} u(g)\varphi(g)d(g)$.

Let $\varphi \in \mathcal{S}(M(N))$ and $u \in \mathcal{S}'(M(N))$. The $M(N)$-Fourier transform $\widehat{u}$ of u is defined by

$$\langle \widehat{u}, \widehat{\varphi}\rangle = \langle u, \varphi\rangle. \tag{30}$$

Let u and v be two distributions, at least one of which has compact support. Then the convolution of u and v is a distribution that can be computed using either of the following equations

$$\langle u *_{M(N)} v, \varphi\rangle = \langle u(h), \langle v(g), \varphi(hg)\rangle\rangle \tag{31}$$

$$= \langle v(g), \langle u(h), \varphi(hg)\rangle\rangle. \tag{32}$$

If either of u or v is a tempered distribution, and the other is compactly supported, then $u *_{M(N)} v$ is a tempered distribution. Without loss of generality, assume that u is compactly supported and $v \in \mathcal{S}'(M(N))$. Then $\widehat{u}$ can be computed using (30) by

$$\widehat{u} = \langle u(g), u^{(\lambda)}_{mn}(g^{-1})\rangle. \tag{33}$$

Using (31) and (33), the $M(N)$-Fourier transform of the convolution $u *_{M(N)} v$

is obtained to be

$$\mathcal{F}_{M(N)}(u *_{M(N)} v)_{mn}(\lambda) = \sum_k \widehat{v}_{mk}(\lambda)\widehat{u}_{kn}(\lambda). \tag{34}$$

For the rest of the paper we will use the integral representations for distributions.

4 Spherical Harmonic Decomposition of the Radon Transform and $M(N)$

4.1 *Radon Transform*

The Radon transform of a compactly supported function over $\mathbb{R}^N$ is defined as [17, 28]

$$\mathcal{R}f(\boldsymbol{\vartheta}, t) = \int_{\mathbb{R}^N} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\vartheta} - t)d\boldsymbol{x}, \tag{35}$$

where $\boldsymbol{\vartheta} \in S^{N-1}$, $t \in \mathbb{R}$ and δ is the generalized Dirac delta distribution. Without loss of generality, we assume that the support of f is within the unit ball.

Let $\widetilde{f}(\xi)$ be the $N-$dimensional standard Fourier transform of a real valued function $f \in L^2(\mathbb{R}^N)$,

$$\widetilde{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{x},$$

and $\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma)$ be the $1-$dimensional standard Fourier transform of $\mathcal{R}f(\boldsymbol{\vartheta}, t)$

with respect to t,

$$\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma) = \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, t)\mathrm{e}^{-\mathrm{i}t\sigma} dt.$$

The Fourier slice theorem states that $\widetilde{f}(\sigma\boldsymbol{\vartheta}) = \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma)$ for $\sigma \in \mathbb{R}$, $\boldsymbol{\vartheta} \in S^{N-1}$ [17]. Let $\widetilde{f}_l(\sigma)$ and $\widetilde{\mathcal{R}f}_l(\sigma)$ denote the spherical harmonic decomposition of the Fourier transforms of the function f and its projections $\mathcal{R}f$. Then, by the Fourier slice theorem, one can show that $\widetilde{f}_l(\sigma) = \widetilde{\mathcal{R}f}_l(\sigma)$. This relationship leads to the spherical harmonic decomposition of Radon transform in terms of f [2, 4]. In the following section, we will derive this relationship from a convolution type representation of the Radon transform over $M(N)$.

4.2 *Radon Transform as a Convolution over $M(N)$*

In [26], Radon transform over $\mathbb{R}^2$ was recognized as a cross-correlation over $M(2)$. The cross-correlation representation was restated and generalized as a convolution over $M(N)$ in Chapter 13 of [3]. The convolution representation of the Radon transform over $M(N)$ is an alternative representation of the Radon transform for the double fibration (see Section 2.2 of [17]) using distributions. We also adapt a convolution formulation similar to the one introduced in [3], and derive the SVD of the Radon transform from the convolution representation using the Fourier analysis over $M(N)$.

In order to express the Radon transform as a convolution over $M(N)$, let us revisit the convolution over $M(N)$. Let f_1 and f_2 be real valued functions over $M(N)$. Then, convolution of f_1 and f_2^* over $M(N)$ is given by

$$(f_1 *_{M(N)} f_2^*)(g) = \int_{M(N)} f_1(h) f_2(g^{-1}h)\, d(h)$$
$$= \int_{SO(N)} \int_{\mathbb{R}^N} f_1(R_\phi, \boldsymbol{x}) f_2(R_\phi R_\theta^{-1}, R_\theta^{-1}\boldsymbol{x} - R_\theta^{-1}\boldsymbol{r})\, d(\phi) d\boldsymbol{x}, \quad (36)$$

where $g = (R_\theta, \boldsymbol{r})$ and $h = (R_\phi, \boldsymbol{x})$.

Let $\delta(R_\phi)$ denote the distribution over $SO(N)$ defined as follows

$$\int_{SO(N)} \delta(R_\phi)\varphi(R_\phi) d(R_\phi) = \varphi(I), \quad (37)$$

where I is the identity rotation. Then, setting $f_\delta(R_\phi, \boldsymbol{x}) = f(\boldsymbol{x})\delta(R_\phi)$ and $\Lambda(R_\phi, \boldsymbol{x}) = \delta(\boldsymbol{x} \cdot \boldsymbol{e_1})$, we can write the Radon transform of f by

$$\mathcal{R}f(\boldsymbol{\vartheta}, t) = (f_\delta *_{M(N)} \Lambda^*)(g), \quad (38)$$

where $\boldsymbol{\vartheta} = R_\theta \boldsymbol{e_1} \in S^{N-1}$, the first column of R_θ, and $t = (R_\theta^{-1}\boldsymbol{r}) \cdot \boldsymbol{e_1}$.

Using an equivalent formulation,

$$(f_1 *_{M(N)} f_2^*)(g) = \int_{M(N)} f_1(R_\theta R_\phi, R_\theta \boldsymbol{x} + \boldsymbol{r}) f_2(\phi, \boldsymbol{x})\, d(h). \quad (39)$$

In this case, the Radon transform of f is given by,

$$\mathcal{R}f(\boldsymbol{\vartheta}, r_1) = (\Lambda *_{M(N)} f^*)(g), \quad (40)$$

where $\Lambda(R_\phi, \boldsymbol{x}) = \delta(\boldsymbol{x} \cdot \boldsymbol{e_1})$, $f(R_\phi, \boldsymbol{x}) = f(\boldsymbol{x})$, and $\boldsymbol{\vartheta} = -R_\theta^{-1}\boldsymbol{e_1}$.

The formulation in (40) leads to a simpler algorithm for the inversion because, unlike the formulation in (38), the projection data can be used without any preprocessing. The formulation in (40) and (38) carries the tomographic inversion problem over to $M(N)$. Now, we wish to determine f given $\mathcal{R}f$ and Λ over $M(N)$.

Equation (40) should be understood in the distribution sense where $\mathcal{R}f$ is a tempered distribution given by the convolution of the tempered distribution Λ and the compactly supported distribution f over $M(N)$.

4.3 *Spherical Harmonic Decomposition of the Radon Transform*

$M(N)$-Fourier transform maps the convolution into multiplication in the transform domain. Hence, taking $M(N)$-Fourier transform of both sides of (40) gives the relationship between the $M(N)$-Fourier coefficients of $\mathcal{R}f$ and f as

$$\widehat{\mathcal{R}f}(\lambda) = \widehat{f}(\lambda)^{\dagger}\widehat{\Lambda}(\lambda), \tag{41}$$

or in matrix form

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \sum_q \overline{\widehat{f}_{qm}(\lambda)}\widehat{\Lambda}_{qn}(\lambda). \tag{42}$$

Since $f \in L^2(\mathbb{R}^N)$, $M(N)$-Fourier coefficients of f are non-zero if and only if $q = 0$, Equation (42) becomes

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \overline{\widehat{f}_{0m}(\lambda)}\widehat{\Lambda}_{0n}(\lambda). \tag{43}$$

From (43), $\widehat{f}_{0m}(\lambda)$ is independent of the choice of n. Therefore, as long as $\widehat{\Lambda}_{0n}(\lambda)$ is not equal to zero, $M(N)$-Fourier coefficients of f are computed as

$$\overline{\widehat{f}_{0m}(\lambda)} = \frac{\widehat{\mathcal{R}f}_{mn}(\lambda)}{\widehat{\Lambda}_{0n}(\lambda)}. \tag{44}$$

The $M(N)$-Fourier coefficients, $\widehat{\mathcal{R}f}_{mn}(\lambda)$ and $\widehat{\Lambda}_{0n}(\lambda)$ are (see Sections A and B) given by

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta}), \tag{45}$$

$$\widehat{\Lambda}_{mn}(\lambda) = \delta_m C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}}. \tag{46}$$

Note that, whenever $\widehat{\Lambda}_{0n}(\lambda)$ is zero, so is $\widehat{\mathcal{R}f}_{mn}(\lambda)$. Substituting (45) and (46) in the right hand side of (44), the $M(N)$-Fourier coefficients of f becomes

$$\widehat{f}_{0m}(\lambda) = \int_{S^{N-1}} \overline{\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda)} S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}). \tag{47}$$

Using the $M(N)$-Fourier transform over $\mathbb{R}^N$, and the property of the standard Fourier transform under conjugation, one obtains

$$\widetilde{f}_m(-\lambda) = \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}).$$

This is nothing but the spherical harmonic decomposition of the Radon transform

$$\widetilde{f}_m(\lambda) = \widetilde{\mathcal{R}f}_m(\lambda). \tag{48}$$

Equation (48) shows that $M(N)$-Fourier analysis of the Radon transform implicitly combines the spherical harmonic decomposition and the Fourier slice theorem.

While the harmonic analysis over $M(N)$ provides a framework to analyze the Radon transform, the relationship between the representations of $M(N)$ and the special functions provides a natural approach in understanding the presence of the special functions in the SVD of the Radon transform.

5 Inversion of the Radon Transform

Equation (47) provides an inversion formula for the Radon transform based on the $M(N)$-Fourier transform. This inversion method facilitates the definition of the inverse Radon transform, $\mathcal{R}^{-1}$, as an integral operator over $M(N)$.

In the following sections we will first derive the sufficient conditions for the separability of the convolution kernels. This will be followed by the decomposition of the kernel of $\mathcal{R}^{-1}$, leading to the SVD of the Radon transform.

5.1 *Linear Integral Operators over Groups*

Let G be a unimodular, separable locally compact group of Type I, and A be a linear integral operator acting on the functions defined over a homogeneous space Z of G with kernel $\tilde{K}_A(g,h)$:

$$Af(h) = \int_Z \tilde{K}_A(g,h) f(g) d(g), \tag{49}$$

where h is an element of another homogeneous space H of G. If $\tilde{K}_A(g,h)$ can be decomposed as

$$\tilde{K}_A(g,h) = \sum_i c_i a_i(g) b_i(h), \tag{50}$$

where $\{c_i\}$ are constants and $\{a_i(g)\}$ and $\{b_i(h)\}$ are linearly independent normalized functions, then we shall call $\tilde{K}_A(g,h)$ a *separable* kernel. For linear integral operators with separable kernels, (49) becomes

$$Af(h) = \sum_i c_i b_i(h) \int_Z a_i(g) f(g) d(g). \tag{51}$$

If $\{a_i(g)\}$ and $\{b_i(h)\}$ are orthonormal and complete with respect to some weight functions $w_a(g)$ and $w_b(h)$,

$$\begin{aligned} Aa_j w_a(h) &= \sum_i c_i b_i(h) \int_Z a_i(g) a_j(g) w_a(g) d(g) \\ &= \sum_i c_i b_i(h) \delta_{ij} = c_j b_j(h), \end{aligned} \tag{52}$$

where

$$c_i = \int_{M(N)} Aa_j w_a(h) b_j(h) w_b(h) d(h),$$

then $\{a_i w_a, b_i, c_i\}$ becomes the *singular value decomposition* of A. Furthermore, if A is invertible, the SVD of A^{-1} is given by $\{b_i, a_i w_a, c_i^{-1}\}$.

Let $\tilde{K}_A(g,h) = K_A(g^{-1}h)$, and K_A has a Fourier transform over G. Then we can write K_A as

$$\begin{aligned} K_A(g^{-1}h) &= \int_0^\infty \mathrm{Tr}\left[\left[\int_H K_A(g^{-1}h_0)U^{(\lambda)}(h_0^{-1})d(h_0)\right] U^{(\lambda)}(h)\right] d(\lambda) \\ &= \int_0^\infty \mathrm{Tr}\left[\widehat{K_A}(\lambda)U^{(\lambda)}(g^{-1})U^{(\lambda)}(h)\right] d(\lambda) \\ &= \sum_{m,n,k} \int_0^\infty \widehat{K_A}_{mk}(\lambda)u_{kn}^{(\lambda)}(g^{-1}h)d(\lambda) \\ &= \sum_{m,n,k,l} \int_0^\infty \widehat{K_A}_{mk}(\lambda)\overline{u_{lk}^{(\lambda)}(g)}u_{ln}^{(\lambda)}(h)(d\lambda). \end{aligned} \tag{53}$$

(53) implies that if $u_{mn}^{(\lambda)}(g)$ is separable in λ and g, then K_A is separable. Let $u_{mn}^{(\lambda)}(g) = \sum_p u_{mnp}^{(1)}(\lambda)\ u_{mnp}^{(2)}(g)$. Then,

$$K_A(g^{-1}h) = \sum_{m,n,k,l,p,q} \left(\int_0^\infty \widehat{K_A}_{mk}(\lambda)\overline{u_{lkp}^{(1)}(\lambda)}u_{lnq}^{(1)}(\lambda)\right) \overline{u_{lkp}^{(2)}(g)}u_{lnq}^{(2)}(h). \tag{54}$$

Furthermore, if $u_{lkp}^{(2)}(g)$ and $u_{lnq}^{(2)}(h)$ are orthogonal and complete over Z and H, respectively, (54) leads to the SVD of A.

This is indeed the case for the unitary representations of $M(N)$, and can be seen by substituting the identity (17) into the matrix elements $u_{mn}^{(\lambda)}(g)$ of the unitary representations given in (4). In the next section, we will derive the singular functions arising in the SVD of the inverse Radon transform by using the identity (17).

5.2 *Inverse Radon Transform as a Linear Integral Operator*

By (47) the $M(N)$-Fourier coefficients of f are given by

$$\begin{aligned}\widehat{f}_{0m}(\lambda) &= \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}) \\ &= \int_{S^{N-1}} \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, r) \mathrm{e}^{\mathrm{i}r\lambda} dr S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}). \end{aligned} \tag{55}$$

Using the inverse $M(N)$-Fourier transform, f can be obtained as

$$f(h) = \int_0^\infty \int_{S^{N-1}} \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, r_1) \mathrm{e}^{\mathrm{i}r_1\lambda} dr S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}) u_{m0}^{(\lambda)}(h) \lambda^{N-1} d\lambda, \tag{56}$$

where $h = (R_\phi, \boldsymbol{x})$. Since $u_{m0}^{(\lambda)}(h)$ is equal to $u_{m0}^{(\lambda)}(h)$, f does not depend on the rotation component of h. Hence, Equation (56) is an exact inversion formula for the Radon transform.

Furthermore, Equation (56) defines the inverse Radon transform as an integral operator with the following kernel:

$$K_{\mathcal{R}^{-1}}(g, h) = \int_0^\infty \mathrm{e}^{\mathrm{i}r_1\lambda} S_m(\boldsymbol{\vartheta}) u_{m0}^{(\lambda)}(h) \lambda^{N-1} d\lambda, \tag{57}$$

where $g = (\boldsymbol{\vartheta}, r_1) \in S^{N-1} \times Z$. Now, our objective is to decompose the kernel $K_{\mathcal{R}^{-1}}(g, h)$ as in Equation (50). To do so, we first decompose $\mathrm{e}^{\mathrm{i}\lambda r_1}$ by using the equality (17), i.e.

$$\mathrm{e}^{\mathrm{i}\lambda r_1} = \Gamma(\nu) (\lambda/2)^{-\nu} \sum_{k=0}^{\infty} \mathrm{i}^k (k+\nu) J_{k+\nu}(\lambda) C_k^\nu(r_1), \quad -1 < x < 1, \;\; \nu > -1/2. \tag{58}$$

Secondly, we compute the associated spherical functions $u_{m0}^{(\lambda)}(h)$. $u_{m0}^{(\lambda)}(h)$ is given by

$$u_{m0}^{(\lambda)}(h) = \int_{S^{N-1}} S_m(\boldsymbol{\omega}) \mathrm{e}^{-\mathrm{i}\lambda(\boldsymbol{x}\cdot\boldsymbol{\omega})} d(\boldsymbol{\omega}) = \int_{S^{N-1}} S_m(\boldsymbol{\omega}) \mathrm{e}^{-\mathrm{i}\lambda|\boldsymbol{x}|(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{\omega})} d(\boldsymbol{\omega}), \quad (59)$$

showing that $u_{m0}^{(\lambda)}(h)$ is independent of the rotation component of h. Therefore, we will use h and $\boldsymbol{x}$ interchangeably. Furthermore, by Funk-Hecke theorem [29],

$$u_{m0}^{(\lambda)}(\boldsymbol{x}) = c_{N,l} \left[\int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\lambda|\boldsymbol{x}|t} C_l^{(N-2)/2}(t)(1-t^2)^{(N-3)/2} dt \right] \; S_m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \quad (60)$$

where l is the first component of the multi-index m representing the degree of the spherical harmonics. Using the identity (11), $u_{m0}^{(\lambda)}(\boldsymbol{x})$ becomes

$$u_{m0}^{(\lambda)}(\boldsymbol{x}) = c_{N,l} \frac{(-\mathrm{i})^l \pi \; \Gamma(N+l-2)}{2^{(N-4)/2} l! \; \Gamma(\frac{N-2}{2})} \frac{J_{l+(N-2)/2}(\lambda\,|\boldsymbol{x}|)}{(\lambda\,|\boldsymbol{x}|)^{(N-2)/2}} \; S_m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right). \quad (61)$$

Note that (15) is a special case for $u_{m0}^{(\lambda)}(g)$, where $S_m = \Xi_m$.

Substituting (61) and (58) in (57) and collecting all the constants under a single expression $C_1(N,k,l,\nu)$, and reorganizing the terms, $K_{\mathcal{R}^{-1}}(g,h)$ can be expressed as

$$K_{\mathcal{R}^{-1}}(g,\boldsymbol{x}) = \sum_{k,m} C_1(N,k,l,\nu) C_k^{\nu}(r_1) S_m(\boldsymbol{\vartheta}) \quad (62)$$

$$\times\, |\boldsymbol{x}|^{1-N/2} \left(\int_0^{\infty} \lambda^{-\nu} J_{k+\nu}(\lambda) \frac{J_{l+(N-2)/2}(\lambda\,|\boldsymbol{x}|)}{\lambda^{(N-2)/2}} \lambda^{N-1} d\lambda \right) S_m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right). \quad (63)$$

From (62), we obtain the first set of functions, $\{a_{km}\}$, of the kernel:

$$a_{km}(g) = C_k^{\nu}(r_1) S_m(\boldsymbol{\vartheta}). \quad (64)$$

The integral inside the brackets in (63) when constants are neglected, gives the second set of functions, $\{b_{km}\}$, of the kernel: (See equation (3.6) in [20])

$$b_{km}(\boldsymbol{x}) = (1-|\boldsymbol{x}|^2)^{\nu-N/2} Q_{k,l}^{\nu,N}(|\boldsymbol{x}|) S_m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \tag{65}$$

where

$$Q_{k,l}^{\nu,N}(|\boldsymbol{x}|) = |\boldsymbol{x}|^l P_{(k-l)/2}^{(\nu-N/2,l+N/2-1)}(2|\boldsymbol{x}|^2-1).$$

Using $\{a_{km}\}$ and $\{b_{km}\}$, $K_{\mathcal{R}^{-1}}(g,\boldsymbol{x})$ is rewritten as

$$K_{\mathcal{R}^{-1}}(g,h) = \sum_{k,m} c(N,k,l,\nu) a_{km}(g) b_{km}\left(|\boldsymbol{x}|, \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \tag{66}$$

where

$$c(N,k,l,\nu) = C_1(N,k,l,\nu) \times \left[\int_0^\infty |\boldsymbol{x}| \left(\int_0^\infty \lambda^{-\nu} J_{k+\nu}(\lambda) \frac{J_{l+(N-2)/2}(\lambda\,|\boldsymbol{x}|)}{\lambda^{(N-2)/2}} \lambda^{N-1} d\lambda\right)^2 d|\boldsymbol{x}|\right]^{1/2}. \tag{67}$$

An explicit computation of $c(N,k,l,\nu)$ is given in [20].

5.3 *Singular Value Decomposition of the Radon Transform*

By Equation (51),

$$f(\boldsymbol{x}) = \sum_{m,k} c(N,k,l,\nu) b_{km}(\boldsymbol{x}) \int_{S^{N-1}\times\mathbb{R}} a_{km}(g) \mathcal{R}f(g) d(g).$$

$\{a_{km}\}$ are orthogonal with respect to the weight function $w_{a,\nu}(r_1) = (1-r_1^2)^{\nu-1/2}$; and $\{b_{km}\}$ are orthogonal with respect to the weight function

$w_{b,\nu}(|\boldsymbol{x}|) = (1-|\boldsymbol{x}|^2)^{\nu-N/2}$; and both basis are complete. Both the orthogonality and completeness are direct consequence of the Gegenbauer polynomials, Jacobi polynomials and spherical harmonics (see [1, 20, 32]).

Therefore, the SVD of the inverse Radon transform is given by $\{a_{km}w_{a,\nu}, b_{km}, c(N,k,l,\nu)\}$. Hence, the SVD of the Radon transform is $\{b_{km}, a_{km}w_{a,\nu}, c(N,k,l,\nu)^{-1}\}$.

6 Conclusions

We presented a new derivation of the SVD of the Radon transform using harmonic analysis over the Euclidean motion group. We showed that the separability of the unitary representations of the underlying group leads to the decomposition of the integral transforms with convolution kernels. In our case, the decomposition of the matrix elements of the unitary representation of the Euclidean motion group is the key that leads to the SVD of the Radon transform. The separability of the unitary representations also gives rise to the special functions present in the SVD of the Radon transform. The decomposition method presented here can be generalized to the weighted Radon transform. This leads to the SVD of the weighted Radon transform [36].

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Appendix A: Computation of $\widehat{\mathcal{R}f}_{mn}(\lambda)$

Let $S_n(\boldsymbol{s})$ and $S_m(\boldsymbol{s})$ be any two n^{th} and m^{th} order spherical harmonics. Then, the $M(N)$ Fourier transform of the projections is computed as

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \int_{M(N)} \mathcal{R}f(\boldsymbol{\vartheta}, r_1) \int_{S^{N-1}} S_n(\boldsymbol{s}) e^{i\lambda(\boldsymbol{r}\cdot\boldsymbol{s})} \overline{S_m(R_\theta^{-1}\boldsymbol{s})} d(\boldsymbol{s}) d(g)$$

$$= \int_{SO(N)} \int_{\mathbb{R}^N} \mathcal{R}f(\boldsymbol{\vartheta}, r_1) \int_{S^{N-1}} S_n(\boldsymbol{s}) e^{i\lambda(\boldsymbol{r}\cdot\boldsymbol{s})} \overline{S_m(R_\theta^{-1}\boldsymbol{s})} d(\boldsymbol{s}) d\boldsymbol{r} d(\theta)$$

$$= (2\pi)^{N-1} \int_{SO(N)} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda s_1) \delta(\lambda \boldsymbol{s_2}) S_n(\boldsymbol{s}) \overline{S_m(R_\theta^{-1}\boldsymbol{s})} d(\boldsymbol{s}) d(\theta)$$

where $\boldsymbol{s_2} = (s_2, \ldots, s_N)$, and $\|\boldsymbol{s_2}\| = 1 - s_1^2 \Longrightarrow s_1 = \pm 1$ when $\boldsymbol{s_2} = \boldsymbol{0}$

$$= \frac{C_1}{\lambda^{N-1}} \left(\int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_n(\boldsymbol{e_1}) \overline{S_m(R_\theta^{-1}\boldsymbol{e_1})} d(\theta) + \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) S_n(-\boldsymbol{e_1}) \overline{S_m(-R_\theta^{-1}\boldsymbol{e_1})} d(\theta) \right)$$

where $C_1 = (2\pi^{N-1})/\left|S^{N-1}\right|$

$$= \frac{C_1}{\lambda^{N-1}} \left(\int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_n(\boldsymbol{e_1}) \overline{S_m(-\boldsymbol{\vartheta})} d(\theta) + \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) S_n(-\boldsymbol{e_1}) \overline{S_m(\boldsymbol{\vartheta})} d(\theta) \right)$$

$$= C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}} \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\theta)$$

$$= C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}} \int_{SO(N-1)} d(\theta) \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta})$$

$$= C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta})$$

Appendix B: Computation of $\widehat{\Lambda}_{0n}(\lambda)$

Similar to $\widehat{\mathcal{R}f}_{mn}(\lambda)$, we compute $\widehat{\Lambda}_{mn}(\lambda)$,

$$\begin{aligned}
\widehat{\Lambda}_{mn}(\lambda) &= \int_{M(N)} \delta(r_1) \int_{S^{N-1}} S_n(\boldsymbol{s}) \mathrm{e}^{\mathrm{i}\lambda(\boldsymbol{r}\cdot\boldsymbol{s})} \overline{S_m(R_\theta^{-1}\boldsymbol{s})} d(\boldsymbol{s}) d(g) \\
&= \int_{SO(N)} \int_{\mathbb{R}^N} \delta(r_1) \int_{S^{N-1}} S_n(\boldsymbol{s}) \mathrm{e}^{\mathrm{i}\lambda(\boldsymbol{r}\cdot\boldsymbol{s})} \overline{S_m(R_\theta^{-1}\boldsymbol{s})} d(\boldsymbol{s}) d\boldsymbol{r} d(\theta) \\
&= \delta_m (2\pi)^{N-1} \int_{S^{N-1}} \delta(\lambda \boldsymbol{s_2}) S_n(\boldsymbol{s}) d(\boldsymbol{s}) \\
&= \delta_m C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}}.
\end{aligned}$$

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