

Effect of discretization error and adaptive mesh generation in diffuse optical absorption imaging: I

Murat Guven¹, Birsen Yazici¹, Kiwoon Kwon¹, Eldar Giladi²
and Xavier Intes³

¹ Electrical, Computer, and Systems Engineering Department, Rensselaer Polytechnic Institute, Troy, NY, USA

² Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY, USA

³ Department of Biomedical Engineering, Rensselaer Polytechnic Institute, Troy, NY, USA

E-mail: yazici@ecse.rpi.edu

Received 27 April 2006, in final form 31 January 2007

Published 15 May 2007

Online at stacks.iop.org/IP/23/1115

Abstract

In diffuse optical tomography (DOT), the discretization error in the numerical solutions of the forward and inverse problems results in error in the reconstructed optical images. In this first part of our work, we analyse the error in the reconstructed optical absorption images, resulting from the discretization of the forward and inverse problems. Our analysis identifies several factors which influence the extent to which the discretization impacts on the accuracy of the reconstructed images. For example, the mutual dependence of the forward and inverse problems, the number of sources and detectors, their configuration and their orientation with respect to optical absorptive heterogeneities, and the formulation of the inverse problem. As a result, our error analysis shows that the discretization of one problem cannot be considered independent of the other problem. While our analysis focuses specifically on the discretization error in DOT, the approach can be extended to quantify other error sources in DOT and other inverse parameter estimation problems.

1. Introduction

Imaging in diffuse optical tomography (DOT) comprises two interdependent stages which seek solutions to the forward and inverse problems. The forward problem is associated with describing the near-infrared (NIR) light propagation, while the objective of the inverse problem is to estimate the unknown optical parameters from boundary measurements [2].

There are a variety of factors that affect the accuracy of the DOT imaging, such as model mismatch (due to the light propagation model and/or linearization of the inverse problem), measurement noise, discretization, numerical algorithm efficiency and inverse problem formulation. In this two-part study, we focus on the effect of discretization of the

forward and inverse problems. In the first part of our work, we present an error analysis to show the effect of discretization on the accuracy of the reconstructed optical absorption images. We identify the factors specific to the imaging problem, which determine the extent to which the discretization impacts on the accuracy of the reconstructed optical absorption images. In the following, first, we use the error analysis to develop novel adaptive discretization algorithms for the forward and inverse problems to reduce the error in the reconstructed optical images resulting from discretization. Next, we present numerical experiments that support the main results of part I and demonstrate the effectiveness of the developed adaptive mesh generation algorithms.

There has been extensive research on the estimation of discretization error in the solutions of partial differential equations (PDEs) [1, 5–7, 21, 22]. In contrast, relatively little has been published in the area of parameter estimation problems governed by PDEs. See for example [8] for an *a posteriori* error estimate for the Lagrangian in the inverse scattering problem for the time-dependent acoustic wave equation and [19] for *a posteriori* error estimates for distributed elliptic optimal control problems. In the area of DOT, it was numerically shown that the approximation errors resulting from the discretization of the forward problem can lead to significant errors in the reconstructed optical images [3]. However, an analysis regarding the error in the reconstructed optical images resulting from discretization has not been reported so far.

In this work, we model the forward problem by the frequency-domain diffusion equation. For the inverse problem, we focus on the estimation of the absorption coefficient. We consider the linear integral equation resulting from the iterative linearization of the inverse problem based on Born approximation and use zeroth-order Tikhonov regularization to address the illposedness of the resulting integral equation. We use finite elements with first-order Lagrange basis functions to discretize the forward and inverse problems and analyse the effect of the discretization of each problem on the reconstructed optical absorption image. Our analysis describes the dependence of the image quality on the optical image properties, the configuration of the source and detectors, the orientation of the source and detectors with respect to absorptive heterogeneities, and on the regularization parameter in addition to the discretization error in the solution of each problem. In our analysis, we first consider the impact of the inverse problem discretization when there is no discretization error in the solution of the forward problem, and provide a bound for the resulting error in the reconstructed optical image. Next, we analyse the effect of the forward problem discretization on the accuracy of the reconstructed image without discretizing the inverse problem, and obtain another bound for the resulting error in the reconstructed optical image. We see that each error bound comprises the discretization error in the corresponding problem solution, scaled spatially by the solutions of both problems. This is a direct consequence of the fact that the inverse problem solution depends on the model defined by the forward problem. As a result, the error analysis yields specific error estimates which are different than the conventional discretization error estimates (see equations (3.8)–(3.9) and (4.14)) which only take into account the smoothness and support of the function of interest, and the finite-dimensional space of approximating functions [9]. We further discuss the use of other basis functions and methods in the discretization of the forward and inverse problems and explain how the error bounds can be modified accordingly. Finally, we extend our analysis to show the effect of noise on the accuracy of the reconstructed optical images. Our analysis shows that the presence of noise results in error terms in addition to the error in the reconstructed optical images induced by the discretization of the forward and inverse problems.

This work not only provides an insight into the error in reconstructed optical absorption images resulting from discretization, but also motivates the development of novel adaptive

mesh generation algorithms to address this error [14]. In addition, the analysis presented in this work provides a means to identify and analyse the error in the reconstructed optical images resulting from the linearization of the Lippmann–Schwinger-type equations [10] using Born approximation [15]. Furthermore, the error analysis introduced in this paper is not limited to DOT, and can easily be extended for use in similar inverse parameter estimation problems such as electrical impedance tomography, bioluminescence tomography, optical fluorescence tomography, microwave imaging etc, in all of which the inverse problem can be interpreted in terms of a linear integral equation, whose kernel is the solution of a PDE that models the forward problem.

The outline of this paper is as follows: section 2 defines the forward and inverse problems. In section 3, we discuss the discretization of the forward and inverse problems. In section 4, we present two theorems that summarize the impact of discretization on the accuracy of the reconstructed optical images, which is followed by section 5. The appendices include results regarding the boundedness and compactness of the linear integral operator used to define the inverse problem, and the proof for the convergence of the inverse problem discretization.

2. Forward and inverse problems

In this section, we describe the model for NIR light propagation and define the forward and inverse DOT problems. Table 1 provides a list of the notation and table 2 provides the definition of function spaces and norms used throughout the paper. We note that we use calligraphic letters to denote the operators, e.g. $\mathcal{A}_a, \mathcal{I}, \mathcal{K}$ etc.

2.1. Forward problem

We use the following boundary value problem to model the NIR light propagation in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ [2, 9]:

$$-\nabla \cdot D(\mathbf{x})\nabla g_j(\mathbf{x}) + \left(\mu_a(\mathbf{x}) + \frac{i\omega}{c}\right) g_j(\mathbf{x}) = Q_j(\mathbf{x}) \quad \mathbf{x} \in \Omega, \quad (2.1)$$

$$g_j(\mathbf{x}) + 2aD(\mathbf{x})\frac{\partial g_j}{\partial n}(\mathbf{x}) = 0 \quad \mathbf{x} \in \partial\Omega, \quad (2.2)$$

where $g_j(\mathbf{x})$ is the photon density at \mathbf{x} , Q_j is the point source located at \mathbf{x}_s^j , $j = 1, \dots, N_s$, where N_s is the number of sources, $D(\mathbf{x})$ is the diffusion coefficient and $\mu_a(\mathbf{x})$ is the absorption coefficient at \mathbf{x} , $i = \sqrt{-1}$, ω is the modulation frequency of the source, c is the speed of the light, $a = (1 + R)/(1 - R)$ where R is a parameter governing the internal reflection at the boundary $\partial\Omega$, and $\partial \cdot / \partial n$ denotes the directional derivative along the unit normal vector on the boundary. Note that we assume the diffusion coefficient is isotropic. For the general anisotropic material, see [17].

The adjoint problem [2] associated with (2.1)–(2.2) is given by the following boundary value problem:

$$-\nabla \cdot D(\mathbf{x})\nabla g_i^*(\mathbf{x}) + \left(\mu_a(\mathbf{x}) - \frac{i\omega}{c}\right) g_i^*(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \quad (2.3)$$

$$g_i^*(\mathbf{x}) + 2aD(\mathbf{x})\frac{\partial g_i^*}{\partial n}(\mathbf{x}) = Q_i^*(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega, \quad (2.4)$$

where Q_i^* is the adjoint source located at \mathbf{x}_d^i , $i = 1, \dots, N_d$, where N_d is the number of detectors. We note that we approximate the point source Q_j in (2.1) and the adjoint source

Table 1. Definition of variables functions, and operators.

Notation	Explanation
Ω	Bounded domain in \mathbb{R}^3 with Lipschitz boundary
$\partial\Omega$	Lipschitz boundary of Ω
\mathbf{x}	Position vector in $\Omega \cup \partial\Omega$
$g_j(\mathbf{x})$	Solution of the diffusion equation at \mathbf{x} for the j th point source located at \mathbf{x}_s^j
$g_i^*(\mathbf{x})$	Solution of the adjoint problem at \mathbf{x} for the i th adjoint source located at \mathbf{x}_d^i
$G_j(\mathbf{x})$	Finite-element approximation of g_j at \mathbf{x}
$G_i^*(\mathbf{x})$	Finite-element approximation of g_i^* at \mathbf{x}
$e_j(\mathbf{x})$	The discretization error at \mathbf{x} in the finite-element approximation of g_j
$e_i^*(\mathbf{x})$	The discretization error at \mathbf{x} in the finite-element approximation of g_i^*
$\alpha(\mathbf{x})$	Small perturbation over the background μ_a at \mathbf{x}
$\Gamma_{i,j}$	Differential measurement at the i th detector due to the j th source
\mathcal{A}_a	The matrix-valued operator mapping $\alpha \in L^\infty(\Omega)$ to $\mathbb{C}^{N_d \times N_s}$
\mathcal{A}_a^*	The adjoint of \mathcal{A}_a mapping from $\mathbb{C}^{N_d \times N_s}$ to $L^1(\Omega)$
$H_{i,j}(\mathbf{x})$	The kernel in \mathcal{A}_a at \mathbf{x} for the i th detector and the j th source
$H_{i,j}^*(\mathbf{x})$	The kernel in \mathcal{A}_a^* at \mathbf{x} for the i th detector and the j th source
$\gamma(\mathbf{x})$	$\mathcal{A}_a^* \Gamma$ at \mathbf{x}
λ	The regularization parameter
$\alpha^\lambda(\mathbf{x})$	Solution of the regularized inverse problem at \mathbf{x}
$\alpha_n^\lambda(\mathbf{x})$	Solution of the discretized regularized inverse problem with exact kernel at \mathbf{x}
$\tilde{\alpha}^\lambda(\mathbf{x})$	Solution of the regularized inverse problem with degenerate kernel at \mathbf{x}
$\tilde{\alpha}_n^\lambda(\mathbf{x})$	Solution of the discretized regularized inverse problem with degenerate kernel at \mathbf{x}

Table 2. Definition of function spaces and norms.

Notation	Explanation
\bar{f}	The complex conjugate of the function f
$C(\Omega)$	Space of continuous complex-valued functions on Ω
$C^k(\Omega)$	Space of complex-valued k -times continuously differentiable functions on Ω
$L^\infty(\Omega)$	$L^\infty(\Omega) = \{f \text{ess sup}_\Omega f(\mathbf{x}) < \infty\}$
$L^p(\Omega)$	$L^p(\Omega) = \{f (\int_\Omega f(\mathbf{x}) ^p \, d\mathbf{x})^{1/p} < \infty\}, p \in [1, \infty)$
$D_w^z f$	z th weak derivative of f
$H^p(\Omega)$	$H^p(\Omega) = \{f (\sum_{ z \leq p} \ D_w^z f\ _0^2)^{1/2} < \infty\}, p \in [1, \infty)$
$\ f\ _0$	The $L^2(\Omega)$ norm of f
$\ f\ _p$	The $H^p(\Omega)$ norm of f
$\ f\ _\infty$	The $L^\infty(\Omega)$ norm of f
$\ f\ _{L^p(\Omega)}$	The $L^p(\Omega)$ norm of f
$\ f\ _{0,m}$	The L^2 norm of f over the m th finite element Ω_m
$\ f\ _{p,m}$	The H^p norm of f over the m th finite element Ω_m

Q_i^* in (2.4) by Gaussian functions with sufficiently low variance, whose centres are located at \mathbf{x}_s^j and \mathbf{x}_d^i , respectively.

In this work, we consider the finite-element approximations of the solutions of the forward problem. Hence, before we discretize the forward problem (see section 3.2), we consider the variational formulations of (2.1)–(2.2) and (2.3)–(2.4) by multiplying (2.1) by a test function $\bar{\phi} \in H^1(\Omega)$ and integrating over Ω [9]:

$$\int_{\Omega} \left[\nabla \bar{\phi} \cdot D \nabla g_j + \bar{\phi} \left(\mu_a + \frac{i\omega}{c} \right) g_j - \bar{\phi} Q_j \right] dx + \frac{1}{2a} \int_{\partial\Omega} \bar{\phi} g_j dl = 0, \quad (2.5)$$

where the boundary integral term results from the boundary condition (2.2).

Equivalently, we can express (2.5) by defining the sesquilinear form $b(\phi, g_j)$ [16]:

$$b(\phi, g_j) := A(\phi, g_j) + \left\langle \phi, \frac{1}{2a} g_j \right\rangle = (\phi, Q_j), \quad (2.6)$$

where

$$\begin{aligned} A(\phi, g_j) &:= \int_{\Omega} \left[\nabla \bar{\phi} \cdot D \nabla g_j + \left(\mu_a + \frac{i\omega}{c} \right) \bar{\phi} g_j \right] dx, \\ (\phi, Q_j) &:= \int_{\Omega} \bar{\phi} Q_j dx, \\ \left\langle \phi, \frac{1}{2a} g_j \right\rangle &:= \frac{1}{2a} \int_{\partial\Omega} \bar{\phi} g_j dl. \end{aligned}$$

Similarly, the variational problem for (2.3)–(2.4) can be formulated by defining the sesquilinear form $b^*(\phi, g_i^*)$:

$$b^*(\phi, g_i^*) := A(\phi, g_i^*) + \left\langle \phi, \frac{1}{2a} g_i^* \right\rangle = \left\langle \phi, \frac{1}{2a} Q_i^* \right\rangle, \quad (2.7)$$

where in $A(\phi, g_i^*)$, ω is replaced by $-\omega$.

The sesquilinear forms $b(\phi, g_j)$, $b^*(\phi, g_i^*)$ are continuous and positive definite for bounded D and μ_a [16]. As a result, the variational problems (2.6) and (2.7) have unique solutions, which follows from the Lax–Milgram lemma [9]. The solutions g_j and g_i^* of the variational problems (2.6) and (2.7) belong to $H^1(\Omega)$, which results from the H^1 -boundedness of the Gaussian function that approximates the point source Q_j and the adjoint source Q_i^* [16]. Assuming $D, \mu_a \in C^1(\Omega)$ and noting that $Q_j, Q_i^* \in H^1(\Omega)$; the solutions g_j, g_i^* satisfy $g_j, g_i^* \in H_{loc}^2(\Omega)$ (in [12, chapter 6.3, theorem 2]). This last condition implies (in [12, chapter 5.6, theorem 6])

$$g_j, g_i^* \in C(\Omega). \quad (2.8)$$

2.2. Inverse problem

In this work, we focus on the estimation of the absorption coefficient; therefore, we assume $D(\mathbf{x})$ is known for all $\mathbf{x} \in \Omega \cup \partial\Omega$. To address the nonlinear nature of the inverse DOT problem, we consider an iterative algorithm based on repetitive linearization of the inverse problem using first-order Born approximation [2]. As a result, at each linearization step, the following linear integral equation relates the differential optical measurements to a small perturbation α on the absorption coefficient μ_a :

$$\Gamma_{i,j} = - \int_{\Omega} \overline{g_i^*(\mathbf{x})} g_j(\mathbf{x}) \alpha(\mathbf{x}) dx \quad (2.9)$$

$$\begin{aligned} &:= \int_{\Omega} H_{i,j}(\mathbf{x}) \alpha(\mathbf{x}) dx \\ &:= (\mathcal{A}_a \alpha)_{i,j}, \end{aligned} \quad (2.10)$$

where $H_{i,j} = -\overline{g_i^* g_j}$ is the kernel in the (i, j) th entry of the matrix-valued operator $\mathcal{A}_a : L^\infty(\Omega) \rightarrow \mathbb{C}^{N_d \times N_s}$, g_j is the solution of (2.6), g_i^* is the solution of (2.7), and $\Gamma_{i,j}$ is the (i, j) th entry in the vector $\Gamma \in \mathbb{C}^{N_d \times N_s}$, which represents the differential measurement at the i th detector due to the j th source. Note that approximating Q_i^* in (2.4) by a Gaussian function centred at \mathbf{x}_d^i implies that $\Gamma_{i,j}$ corresponds to the scattered optical field evaluated at \mathbf{x}_d^i , after filtering it by that Gaussian function. Thus, the Gaussian approximation of the adjoint source models the finite size of the detectors. Similarly, approximating Q_j in (2.1) by a Gaussian function models the finite beam of the source.

The linear operator $\mathcal{A}_a : L^\infty(\Omega) \rightarrow \mathbb{C}^{N_d \times N_s}$ defined by (2.9) is compact and bounded by (see appendices A and B)

$$\|\mathcal{A}_a\|_{L^\infty(\Omega) \rightarrow l^1} \leq N_d N_s \max_i \|g_i^*\|_0 \max_j \|g_j\|_0. \quad (2.11)$$

For the given solution space $L^\infty(\Omega)$ for α , the compactness of the linear operator \mathcal{A}_a implies the illposedness of (2.9). Hence, we regularize (2.9) with a zeroth-order Tikhonov regularization. This yields the following equation which defines our inverse problem at each linearization step:

$$\gamma := \mathcal{A}_a^* \Gamma = (\mathcal{A}_a^* \mathcal{A}_a + \lambda \mathcal{I}) \alpha^\lambda \quad (2.12)$$

$$:= \mathcal{K} \alpha^\lambda, \quad (2.13)$$

where $\lambda > 0$ and α^λ is an approximation to α . In this representation, \mathcal{I} is the identity operator and $\mathcal{A}_a^* : \mathbb{C}^{N_d \times N_s} \rightarrow L^1(\Omega)$ is the adjoint of \mathcal{A}_a , defined by

$$(\mathcal{A}_a^* \beta)(\mathbf{x}) = \sum_{i,j}^{N_d, N_s} H_{i,j}^*(\mathbf{x}) \beta_{i,j} = \sum_{i,j}^{N_d, N_s} -g_i^*(\mathbf{x}) \overline{g_j(\mathbf{x})} \beta_{i,j}, \quad (2.14)$$

for all $\beta \in \mathbb{C}^{N_d \times N_s}$, where $H_{i,j}^* := -\overline{g_i^* g_j}$ is the (i, j) th kernel in the adjoint operator \mathcal{A}_a^* . Let $\mathcal{A} := \mathcal{A}_a^* \mathcal{A}_a$, then $\mathcal{A} : L^\infty(\Omega) \rightarrow L^1(\Omega)$ is defined as follows:

$$\begin{aligned} (\mathcal{A} \alpha)(\mathbf{x}) &= \sum_{i,j}^{N_d, N_s} H_{i,j}^*(\mathbf{x}) \int_{\Omega} H_{i,j}(\hat{\mathbf{x}}) \alpha(\hat{\mathbf{x}}) d\hat{\mathbf{x}} \\ &:= \int_{\Omega} \kappa(\mathbf{x}, \hat{\mathbf{x}}) \alpha(\hat{\mathbf{x}}) d\hat{\mathbf{x}}, \end{aligned} \quad (2.15)$$

where $\kappa(\mathbf{x}, \hat{\mathbf{x}})$ stands for the kernel of the integral operator \mathcal{A} and is given by

$$\kappa(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{i,j}^{N_d, N_s} H_{i,j}^*(\mathbf{x}) H_{i,j}(\hat{\mathbf{x}}). \quad (2.16)$$

Having defined the adjoint operator \mathcal{A}_a^* , we note that the operator $\mathcal{A} : L^\infty(\Omega) \rightarrow L^1(\Omega)$ is compact and that the operator $\mathcal{K} : L^\infty(\Omega) \rightarrow L^1(\Omega)$ is bounded by $\|\mathcal{K}\| \leq \|\mathcal{A}_a\|^2 + \lambda$. We assume that the solution $\alpha^\lambda \in L^\infty(\Omega)$ also satisfies $\alpha^\lambda \in H^1(\Omega)$. For the rest of the paper, we will denote $L^\infty(\Omega)$ and $L^1(\Omega)$ by X and Y , respectively.

3. Discretization of the inverse and forward problems

In this section, we outline the discretization of the inverse and forward problems.

3.1. Inverse problem discretization

In practice, we seek a finite-dimensional approximation to the solution of the inverse problem (2.13) at each linearization step. Therefore, we discretize (2.13) by projecting it onto a finite-dimensional subspace.

Let $X_n \subset X$ and $Y_n \subset Y$ denote a sequence of finite-dimensional subspaces of dimension $n = 1, 2, \dots$, spanned by first-order Lagrange basis functions $\{L_1, \dots, L_n\}$, and $\{\mathbf{x}_p\}$, $p = 1, \dots, n$, be the set of collocation points on Ω . Then, the collocation method approximates the solution of (2.13) by an element $\alpha_n^\lambda \in X_n$ which satisfies

$$(\mathcal{K}\alpha_n^\lambda)(\mathbf{x}_p) = \gamma(\mathbf{x}_p), \quad p = 1, \dots, n, \quad (3.1)$$

where we express α_n^λ on a set $\{\Omega_m\}$ of finite elements, $m = 1, \dots, N_\Delta$ such that $\bigcup_m^{N_\Delta} \Omega_m = \Omega$ as follows:

$$\alpha_n^\lambda(\mathbf{x}) = \sum_{k=1}^n a_k L_k(\mathbf{x}). \quad (3.2)$$

Note that in (3.2), $a_p = \alpha_n^\lambda(\mathbf{x}_p)$, $p = 1, \dots, n$. Then, (3.1) can explicitly be written as

$$\lambda a_p + \sum_{k=1}^n a_k \int_{\Omega} \kappa(\mathbf{x}_p, \hat{\mathbf{x}}) L_k(\hat{\mathbf{x}}) d\hat{\mathbf{x}} = \gamma(\mathbf{x}_p), \quad p = 1, \dots, n. \quad (3.3)$$

Equivalently, the collocation method can be interpreted as a projection with the interpolation operator $\mathcal{P}_n : Y \rightarrow Y_n$ defined by [18]

$$\mathcal{P}_n f(\mathbf{x}) := \sum_{p=1}^n f(\mathbf{x}_p) L_p(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (3.4)$$

for all $f \in Y$. Then, (3.1) is equivalent to

$$\mathcal{P}_n \mathcal{K} \alpha_n^\lambda = \mathcal{P}_n \gamma. \quad (3.5)$$

3.2. Forward problem discretization

In this section, we consider the finite-element discretizations of (2.6) and (2.7), and use their solutions to approximate $H_{i,j}$ and $H_{i,j}^*$. As a result, we obtain finite-dimensional approximations to \mathcal{K} and γ .

Let L_k denote the k th first-order Lagrange basis function. Replacing ϕ and g_j in (2.6) with their finite-dimensional counterparts $\Phi(\mathbf{x}) = \sum_{k=1}^{N_j} p_k L_k(\mathbf{x})$, $G_j(\mathbf{x}) = \sum_{k=1}^{N_j} c_k L_k(\mathbf{x})$, and replacing ϕ and g_i^* in (2.7) with $\Phi(\mathbf{x}) = \sum_{k=1}^{N_i} p_k L_k(\mathbf{x})$, $G_i^*(\mathbf{x}) = \sum_{k=1}^{N_i} d_k L_k(\mathbf{x})$ yields the matrix equations:

$$\mathbf{S} \mathbf{c}_j = \mathbf{q}_j, \quad (3.6)$$

$$\mathbf{S}^* \mathbf{d}_i = \mathbf{q}_i^*, \quad (3.7)$$

for $\mathbf{c}_j := [c_1, c_2, \dots, c_{N_j}]^T$ and $\mathbf{d}_i := [d_1, d_2, \dots, d_{N_i}]^T$. Here \mathbf{S} and \mathbf{S}^* are the finite-element matrices and \mathbf{q}_j and \mathbf{q}_i^* are the load vectors resulting from the finite-element discretization of (2.6) and (2.7). Note that for each source (detector), the dimension of the finite-element solution G_j (G_i^*) can be different; therefore, N_j (N_i) may vary.

The $H^1(\Omega)$ boundedness of the solutions g_j and g_i^* implies that the discretization errors e_j and e_i^* in G_j and G_i^* are bounded. Let $\{\Omega_m^j\}$ denote the set of linear elements used to

discretize (2.6) for $m = 1, \dots, N_{\Delta}^j$, such that $\bigcup_m^{N_{\Delta}^j} \Omega_m^j = \Omega$ for all $j = 1, \dots, N_s$. Similarly, let $\{\Omega_n^i\}$ denote the set of linear elements used to discretize (2.7) for $n = 1, \dots, N_{\Delta}^{*i}$, such that $\bigcup_n^{N_{\Delta}^{*i}} \Omega_n^i = \Omega$ for all $i = 1, \dots, N_d$. Then, a bound for e_j and e_i^* on each finite element can be found by using the discretization error estimates (in [9, theorem 4.4.4]):

$$\|e_j\|_{0,m^j} \leq C \|g_j\|_{1,m^j} h_m^j, \tag{3.8}$$

$$\|e_i^*\|_{0,n^i} \leq C \|g_i^*\|_{1,n^i} h_n^i, \tag{3.9}$$

where C is a positive constant, $\|\cdot\|_{0,m^j}$ ($\|\cdot\|_{0,n^i}$) and $\|\cdot\|_{1,m^j}$ ($\|\cdot\|_{1,n^i}$) are respectively the L^2 and H^1 norms on Ω_m^j (Ω_n^i), and h_m^j (h_n^i) is the diameter of the smallest ball containing the finite element Ω_m^j (Ω_n^i) in the solution G_j (G_i^*).

3.3. Discretization of the inverse problem with operator approximations

Substituting the finite-element approximations G_j and G_i^* in (2.15) and (2.14), and using the resulting finite-dimensional operator approximations in (3.5), we obtain the following linear system in terms of $\tilde{\alpha}_n^\lambda$ which approximates α^λ :

$$\mathcal{P}_n \tilde{\mathcal{K}} \tilde{\alpha}_n^\lambda = \mathcal{P}_n \tilde{\gamma}. \tag{3.10}$$

In (3.10), the operator $\tilde{\mathcal{K}} : X \rightarrow Y$ is the finite-dimensional approximation of \mathcal{K} in (2.13) and $\mathcal{P}_n \tilde{\mathcal{K}} : X_n \rightarrow Y_n$. Similarly,

$$\tilde{\gamma} := \tilde{\mathcal{A}}_a^* \Gamma, \tag{3.11}$$

where $\tilde{\mathcal{A}}_a^*$ is the approximation to the adjoint operator \mathcal{A}_a^* , obtained by substituting G_j and G_i^* in (2.14).

4. Discretization-based error analysis

As a result of the discretization of the forward and inverse problems, the reconstructed image $\tilde{\alpha}_n^\lambda$ in (3.10) is an approximation to the actual image α^λ . Thus, the accuracy of the reconstructed image depends on the error incurred by the discretization of the forward and inverse problems.

In this section, we analyse the effect of the discretization of the forward and inverse problems on the accuracy of DOT imaging. The analysis is carried out based on the inverse problem at each linearization defined by (2.13) and the associated kernel $\kappa(\mathbf{x}, \hat{\mathbf{x}})$.

In this work, we follow an approach which allows us to separately analyse the effect of the discretization of each problem on the accuracy of the reconstructed optical image. In this respect, we first consider the impact of projection (i.e. inverse problem discretization) by the collocation method when the associated kernel $\kappa(\mathbf{x}, \hat{\mathbf{x}})$ in (2.13) is exact. Next, we explore the case in which the kernel is replaced by its finite-dimensional approximation (i.e. degenerate kernel) and analyse the effect of the forward problem discretization on the accuracy of the reconstructed image without projecting (2.13).

Our analysis reveals that even if the kernel is exact, the accuracy of the solution approximation α_n^λ in (3.5) resulting from the inverse problem discretization depends on the kernel $\kappa(\mathbf{x}, \hat{\mathbf{x}})$ of the integral operator. Likewise, the error in the reconstructed optical image due to the discretization of the forward problem is a function of the inverse problem solution. These results suggest that the discretization of the inverse and forward problems cannot be considered independent of each other.

4.1. Case 1. The kernel $\kappa(\mathbf{x}, \hat{\mathbf{x}})$ is exact

In this section, we show the effect of projection on the optical imaging accuracy. In the analysis, we assume that the kernel $\kappa(\mathbf{x}, \hat{\mathbf{x}})$ is exact. We first prove the convergence of the projection method for the operator \mathcal{K} , and then analyse the effect of projection on the imaging accuracy.

Clearly, the inverse operator $\mathcal{K}^{-1} : Y \rightarrow X$ exists since \mathcal{K} is positive definite for $\lambda > 0$. Furthermore, by the compactness of \mathcal{A} and Riesz theorem, the inverse operator \mathcal{K}^{-1} is bounded by

$$\|\mathcal{K}^{-1}\|_{Y \rightarrow X} \leq \frac{1}{\lambda}. \quad (4.1)$$

Lemma. *Projection by the collocation method for the operator $\mathcal{K} : X \rightarrow Y$ converges. Specifically, the sequence of finite-dimensional operators $\mathcal{P}_n \mathcal{K} : X_n \rightarrow Y_n$ is invertible for sufficiently large n , and $(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K} \alpha^\lambda \rightarrow \alpha^\lambda, n \rightarrow \infty$. Furthermore,*

$$\|(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}\|_{X \rightarrow X_n} \leq C_M \frac{\|\mathcal{K}\|_{X \rightarrow Y}}{\lambda} \quad (4.2)$$

for some $C_M > 0$ independent of n .

Proof. See appendix C. □

Based on the lemma, the following theorem provides an upper bound for the $L^1(\Omega)$ norm of the error between the solution α^λ of (2.13) and the solution α_n^λ of (3.5).

Theorem 1. *Let $\{\Omega_m\}$ denote a set of linear finite elements used in the discretization of the inverse problem (2.13) for $m = 1, \dots, N_\Delta$, such that $\bigcup_m^{N_\Delta} \Omega_m = \Omega$, and h_m be the diameter of the smallest ball that contains the m th element. Then,*

$$\begin{aligned} \|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)} &\leq C \sqrt{V_\Omega} \|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \sum_{m=1}^{N_\Delta} \|\alpha^\lambda\|_{1,m} h_m \\ &+ \frac{C}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \max_{i,j} \|g_i^* g_j\|_{L^1(\Omega)} \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} \|g_i^* g_j\|_{0,m} \|\alpha^\lambda\|_{1,m} h_m, \end{aligned} \quad (4.3)$$

where C is a positive constant, V_Ω is the volume of Ω and $\mathcal{T}_n : Y \rightarrow X_n$ is a uniformly bounded operator given by $\mathcal{T}_n = (\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A})^{-1} \mathcal{P}_n$.

Proof.

$$\begin{aligned} \alpha^\lambda - \alpha_n^\lambda &= [\mathcal{I} - (\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}] \alpha^\lambda \\ &= [\mathcal{I} - (\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}] (\alpha^\lambda - \psi) \end{aligned} \quad (4.4)$$

since $[\mathcal{I} - (\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}] \psi = 0$, where $\psi \in X_n$ is the interpolant of α^λ [9]. Using (C.2),

$$\begin{aligned} [\mathcal{I} - (\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}] &= \mathcal{I} - \left(\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A} \right)^{-1} \frac{1}{\lambda} \mathcal{P}_n \mathcal{K} \\ &= \mathcal{I} - \mathcal{T}_n \frac{1}{\lambda} \mathcal{K}, \end{aligned} \quad (4.5)$$

where $\mathcal{T}_n := (\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A})^{-1} \mathcal{P}_n$ is a uniformly bounded operator (see appendix C). We use \mathcal{K} defined by (2.13) and (4.5) in (4.4) to obtain

$$\alpha^\lambda - \alpha_n^\lambda = (\mathcal{I} - \mathcal{T}_n) (\alpha^\lambda - \psi) - \frac{\mathcal{T}_n}{\lambda} \mathcal{A} (\alpha^\lambda - \psi). \quad (4.6)$$

Then, we use the definition of \mathcal{A} in (4.6) and find

$$\alpha^\lambda - \alpha_n^\lambda = (\mathcal{I} - \mathcal{T}_n)(\alpha^\lambda - \psi) - \frac{\mathcal{T}_n}{\lambda} \int_{\Omega} \kappa(\cdot, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}}) \, d\hat{\mathbf{x}}. \tag{4.7}$$

This leads to

$$\begin{aligned} \|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)} &\leq \|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \|\alpha^\lambda - \psi\|_{L^1(\Omega)} \\ &\quad + \frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \left\| \int_{\Omega} \kappa(\cdot, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}}) \, d\hat{\mathbf{x}} \right\|_{L^1(\Omega)} \\ &\leq \|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \|\alpha^\lambda - \psi\|_{L^1(\Omega)} \\ &\quad + \frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \int_{\Omega} d\mathbf{x} \int_{\Omega} |\kappa(\mathbf{x}, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}}, \end{aligned} \tag{4.8}$$

The second term in (4.8) can be rewritten as

$$\begin{aligned} &\frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \int_{\Omega} d\mathbf{x} \int_{\Omega} |\kappa(\mathbf{x}, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \\ &= \frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \int_{\Omega} d\mathbf{x} \left(\sum_{m=1}^{N_\Delta} \int_{\Omega_m} |\kappa(\mathbf{x}, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \right). \end{aligned} \tag{4.9}$$

Let e_α be the interpolation error:

$$e_\alpha := \alpha^\lambda - \psi. \tag{4.10}$$

Then, using (2.16),

$$\sum_{m=1}^{N_\Delta} \int_{\Omega_m} |\kappa(\mathbf{x}, \hat{\mathbf{x}})e_\alpha(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} = \sum_{m=1}^{N_\Delta} \int_{\Omega_m} \left| \sum_{i,j}^{N_d, N_s} g_i^*(\mathbf{x}) \overline{g_j(\mathbf{x})} g_i^*(\hat{\mathbf{x}}) g_j(\hat{\mathbf{x}}) e_\alpha(\hat{\mathbf{x}}) \right| \, d\hat{\mathbf{x}} \tag{4.11}$$

$$\begin{aligned} &\leq \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} |g_i^*(\mathbf{x}) g_j(\mathbf{x})| \int_{\Omega_m} |g_i^*(\hat{\mathbf{x}}) g_j(\hat{\mathbf{x}})| |e_\alpha(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \\ &\leq \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} |g_i^*(\mathbf{x}) g_j(\mathbf{x})| \|g_i^* g_j\|_{0,m} \|e_\alpha\|_{0,m}, \end{aligned} \tag{4.12}$$

where (4.12) follows from the Schwarz' inequality. Note that $g_i^* g_j \in L^2(\Omega)$ by considering (2.8) holds up to the boundary $\partial\Omega$ (see [11, theorem 2.1]).

We now use (4.9) and (4.12) to obtain

$$\begin{aligned} &\frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \int_{\Omega} d\mathbf{x} \left(\int_{\Omega} |\kappa(\mathbf{x}, \hat{\mathbf{x}})(\alpha^\lambda - \psi)(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \right) \\ &\leq \frac{1}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \int_{\Omega} d\mathbf{x} \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} |g_i^*(\mathbf{x}) g_j(\mathbf{x})| \|g_i^* g_j\|_{0,m} \|e_\alpha\|_{0,m}. \end{aligned} \tag{4.13}$$

Using the bound (4.13) in (4.8) and substituting the interpolation error bound [9]

$$\|e_\alpha\|_{0,m} \leq C \|\alpha^\lambda\|_{1,m} h_m, \tag{4.14}$$

and noting $\|e_\alpha\|_{L^1(\Omega)} \leq \sqrt{V_\Omega} \sum_{m=1}^{N_\Delta} \|e_\alpha\|_{0,m}$, we obtain

$$\begin{aligned} \|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)} &\leq C\sqrt{V_\Omega} \|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \sum_{m=1}^{N_\Delta} \|\alpha^\lambda\|_{1,m} h_m \\ &\quad + \frac{C}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} \|g_i^* g_j\|_{L^1(\Omega)} \|g_i^* g_j\|_{0,m} \|\alpha^\lambda\|_{1,m} h_m. \\ &\leq C\sqrt{V_\Omega} \|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \sum_{m=1}^{N_\Delta} \|\alpha^\lambda\|_{1,m} h_m \\ &\quad + \frac{C}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \max_{i,j} \|g_i^* g_j\|_{L^1(\Omega)} \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} \|g_i^* g_j\|_{0,m} \|\alpha^\lambda\|_{1,m} h_m. \end{aligned} \quad (4.15) \quad \square$$

Remark 1.

- (i) Theorem 1 shows the spatial dependence of the inverse problem discretization on the forward problem solution.
- (ii) The first term in (4.15) suggests that the mesh of the inverse problem be refined where $\|\alpha^\lambda\|_1$ is large.
- (iii) The second term in (4.15) shows that the term $\|\alpha^\lambda\|_{1,m}$ is scaled spatially by $\|g_i^* g_j\|_{0,m}$. Thus, the effect of the interpolation error e_α (see equation (4.10)) in the inverse problem solution is scaled spatially by the solution of the forward problem. As a result, the orientation of the sources and detectors with respect to the support of the optical heterogeneity determines the extent of the bound on $\|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)}$.
- (iv) The regularization parameter affects the bound on $\|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)}$.
- (v) Increasing the number of sources and detectors increases the bound on $\|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)}$.

Remark 2.

- (i) Note that the conventional interpolation error estimate given in (4.14) depends only on the smoothness and support of α^λ , and the finite-dimensional space of approximating functions [9]. On the other hand, the error estimate (4.3) in theorem 1 shows that the accuracy of the reconstructed image α_n^λ depends on the orientation of the absorptive heterogeneity with respect to the sources and detectors, as well as on the bound (4.14) on the interpolation error.
- (ii) An error bound similar to (4.3) follows if one uses the Galerkin method [18] instead of the collocation method for projection.
- (iii) The interpolation error bound (4.14) can be modified based on the choice of the basis function in (3.2) and the smoothness of the solution α^λ (theorem 4.4.4. in [9]). For instance, if $\alpha^\lambda \in H^2(\Omega)$ and quadratic Lagrange basis functions are used, then (4.14) can be replaced by

$$\|e_\alpha\|_{0,m} \leq C \|\alpha^\lambda\|_{2,m} h_m^2,$$

for some $C > 0$.

- (iv) An error bound similar to (4.3) can be derived for the error that occurs as a result of the discretization of the inverse problem in electrical impedance tomography, optical fluorescence tomography, bioluminescence tomography and microwave imaging. Note that in all these imaging modalities, the forward problem is modelled by a PDE and the inverse problem can be interpreted in terms of a linear integral equation, whose kernel is related to the solution of this PDE.

(v) Let γ^δ be the perturbed right-hand side γ of (3.5) due to the presence of noise, such that $\|\gamma^\delta - \gamma\|_{L^1(\Omega)} \leq \delta$. Then, an additional term is introduced to the error bound in (4.3) due to this perturbation:

$$\begin{aligned} \|\alpha^\lambda - \alpha_n^\lambda\|_{L^1(\Omega)} &\leq C\sqrt{V_\Omega}\|\mathcal{I} - \mathcal{T}_n\|_{Y \rightarrow X_n} \sum_{m=1}^{N_\Delta} \|\alpha^\lambda\|_{1,m} h_m \\ &\quad + \frac{C}{\lambda} \|\mathcal{T}_n\|_{Y \rightarrow X_n} \max_{i,j} \|g_i^* g_j\|_{L^1(\Omega)} \sum_{m=1}^{N_\Delta} \sum_{i,j}^{N_d, N_s} \|g_i^* g_j\|_{0,m} \|\alpha^\lambda\|_{1,m} h_m \\ &\quad + \frac{C_M}{\lambda} \delta, \end{aligned} \tag{4.16}$$

where $C_M > 0$ is the constant in (4.2) with the use of first-order Lagrange basis functions (see appendix C). The additional term $C_M \delta / \lambda$ indicates that the choice of basis functions may be critical in the presence of noise.

4.2. Case 2. The kernel is degenerate

In this section, we first derive approximate upper bounds for the approximation errors $\|\tilde{\mathcal{K}} - \mathcal{K}\|$ and $\|\tilde{\gamma} - \gamma\|$, which result from the discretization of the forward problem. Then, we show the effect of these approximation errors on the accuracy of the reconstructed optical image. For notational convenience, we will drop the subscripts on the norms $\|\cdot\|$ where necessary.

The operator $\mathcal{K} : X \rightarrow Y$ is bounded with a bounded inverse $\mathcal{K}^{-1} : Y \rightarrow X$. By the finite-element approximation of the associated kernel, the sequence of bounded linear finite-dimensional operators $\tilde{\mathcal{K}}$ is norm convergent $\|\tilde{\mathcal{K}} - \mathcal{K}\| \rightarrow 0$; $N_j, N_i \rightarrow \infty$, for $j = 1, \dots, N_s$ and $i = 1, \dots, N_d$, and

$$\|\tilde{\mathcal{K}}^{-1}\|_{Y \rightarrow X} < 1/\lambda, \tag{4.17}$$

which can be obtained analogous to (4.1).

In the following, we derive an explicit approximation to the error $\|\tilde{\mathcal{K}} - \mathcal{K}\|$ in terms of the associated kernel and the discretization error in the kernel approximation. The result is then used to compute the error in the reconstructed optical image due to $\|\tilde{\mathcal{K}} - \mathcal{K}\|$.

By definition,

$$\|(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\alpha\|_{l^1} = \sum_{i,j}^{N_d, N_s} \left| \int_\Omega (\overline{g_i^*(\mathbf{x})} g_j(\mathbf{x}) - \overline{G_i^*(\mathbf{x})} G_j(\mathbf{x})) \alpha(\mathbf{x}) \, d\mathbf{x} \right|, \tag{4.18}$$

where G_i^*, G_j are the finite-element approximations to g_i^* and g_j , respectively. We can expand $\overline{g_i^*} g_j - \overline{G_i^*} G_j$ as

$$\overline{g_i^*} g_j - \overline{G_i^*} G_j = \overline{e_i^*} e_j + G_j \overline{e_i^*} + \overline{G_i^*} e_j, \tag{4.19}$$

where $e_i^* := g_i^* - G_i^*$ and $e_j := g_j - G_j$. Replacing G_i^* and G_j respectively with $g_i^* - e_i^*$ and $g_j - e_j$, we get

$$\begin{aligned} \overline{g_i^*} g_j - \overline{G_i^*} G_j &= g_j \overline{e_i^*} + \overline{g_i^*} e_j - \overline{e_i^*} e_j \\ &\approx g_j \overline{e_i^*} + \overline{g_i^*} e_j, \end{aligned} \tag{4.20}$$

where we neglect the term $\overline{e_i^*} e_j$.

We can express $\mathcal{K} - \tilde{\mathcal{K}}$ as

$$\mathcal{K} - \tilde{\mathcal{K}} = \mathcal{A}_a^* \mathcal{A}_a - \tilde{\mathcal{A}}_a^* \tilde{\mathcal{A}}_a. \tag{4.21}$$

Following a similar approach as above,

$$\mathcal{A}_a^* \mathcal{A}_a - \tilde{\mathcal{A}}_a^* \tilde{\mathcal{A}}_a = (\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)(\mathcal{A}_a - \tilde{\mathcal{A}}_a) + \tilde{\mathcal{A}}_a^*(\mathcal{A}_a - \tilde{\mathcal{A}}_a) + (\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)\tilde{\mathcal{A}}_a. \quad (4.22)$$

As a result, the following condition holds

$$\|\tilde{\mathcal{K}} - \mathcal{K}\| \leq \|(\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\| + \|\tilde{\mathcal{A}}_a^*(\mathcal{A}_a - \tilde{\mathcal{A}}_a) + (\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)\tilde{\mathcal{A}}_a\|. \quad (4.23)$$

Since $\tilde{\mathcal{A}}_a = -(\mathcal{A}_a - \tilde{\mathcal{A}}_a) + \mathcal{A}_a$, (4.23) can be rewritten as

$$\begin{aligned} \|\tilde{\mathcal{K}} - \mathcal{K}\| &= \|\mathcal{A}_a^* \mathcal{A}_a - \tilde{\mathcal{A}}_a^* \tilde{\mathcal{A}}_a\| \\ &\leq \|(\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\| + 2\|\mathcal{A}_a^*(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\| \\ &\approx 2\|\mathcal{A}_a^*(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\|, \end{aligned} \quad (4.24)$$

where we neglect the term $\|(\mathcal{A}_a^* - \tilde{\mathcal{A}}_a^*)(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\|$.

Similarly, $\|\tilde{\gamma} - \gamma\|$ can be interpreted as

$$\begin{aligned} \|\tilde{\gamma} - \gamma\|_{L^1(\Omega)} &= \int_{\Omega} \left| \sum_{i,j}^{N_d, N_s} (g_i^*(\mathbf{x}) \overline{g_j(\mathbf{x})} - G_i^*(\mathbf{x}) \overline{G_j(\mathbf{x})}) \Gamma_{i,j} \right| \mathrm{d}\mathbf{x} \\ &\approx \int_{\Omega} \left| \sum_{i,j}^{N_d, N_s} (e_i^*(\mathbf{x}) \overline{g_j(\mathbf{x})} + g_i^*(\mathbf{x}) \overline{e_j(\mathbf{x})}) \Gamma_{i,j} \right| \mathrm{d}\mathbf{x}, \end{aligned} \quad (4.25)$$

where the error in $\Gamma_{i,j}$ due to discretization is neglected and the last approximation is derived similar to (4.20).

We now analyse the effect of the forward problem discretization on the accuracy of the reconstructed optical image. Let $\tilde{\alpha}^\lambda$ be the solution of

$$\tilde{\mathcal{K}} \tilde{\alpha}^\lambda = \tilde{\gamma}, \quad (4.26)$$

where $\tilde{\mathcal{K}}$ and $\tilde{\gamma}$ are the finite-dimensional approximations to \mathcal{K} and γ , respectively. Then, by theorem 10.1 in [18], the error in the solution $\tilde{\alpha}^\lambda$ with respect to the actual solution α^λ is bounded by

$$\|\alpha^\lambda - \tilde{\alpha}^\lambda\| \leq \frac{1}{\lambda} \{ \|(\tilde{\mathcal{K}} - \mathcal{K})\alpha^\lambda\| + \|\tilde{\gamma} - \gamma\| \}. \quad (4.27)$$

In the next theorem, we will expand the terms in (4.27) to show explicitly the effect of the forward problem discretization on the accuracy of the inverse problem solution.

Theorem 2. Let $\{\Omega_m^j\}$ denote the set of linear elements used to discretize (2.6) for $m = 1, \dots, N_\Delta^j$, such that $\bigcup_m^{N_\Delta^j} \Omega_m^j = \Omega$ and h_m^j be the diameter of the smallest ball that contains the element Ω_m^j in the solution G_j , for all $j = 1, \dots, N_s$. Similarly, let $\{\Omega_n^i\}$ denote the set of linear elements used to discretize (2.7) for $n = 1, \dots, N_\Delta^i$, such that $\bigcup_n^{N_\Delta^i} \Omega_n^i = \Omega$ and h_n^i be the diameter of the smallest ball that contains the element Ω_n^i in the solution G_i^* , for all $i = 1, \dots, N_d$. Then, a bound for the error between the solution α^λ of (2.13) and the solution $\tilde{\alpha}^\lambda$ of (4.26) due to the approximations $\tilde{\mathcal{K}}$ and $\tilde{\gamma}$ is given by

$$\begin{aligned} \|\alpha^\lambda - \tilde{\alpha}^\lambda\|_{L^1(\Omega)} &\leq \frac{C}{\lambda} \max_{i,j} \|g_i^* g_j\|_{L^1(\Omega)} \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^i, N_s} (2\|g_j \alpha^\lambda\|_{0,ni} + \|\alpha\|_\infty \|g_j\|_{0,ni}) \|g_i^*\|_{1,ni} h_n^i \right. \\ &\quad \left. + \sum_{j=1}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} (2\|g_i^* \alpha^\lambda\|_{0,mj} + \|\alpha\|_\infty \|g_i^*\|_{0,mj}) \|g_j\|_{1,mj} h_m^j \right), \end{aligned} \quad (4.28)$$

where C is a positive constant.

Proof. Using (4.24), (4.18) and (4.20), we can write

$$\begin{aligned} \|(\tilde{\mathcal{K}} - \mathcal{K})\alpha^\lambda\|_{L^1(\Omega)} &\approx 2\|\mathcal{A}_a^*(\mathcal{A}_a - \tilde{\mathcal{A}}_a)\alpha^\lambda\|_{L^1(\Omega)} \\ &\approx 2\left\|\sum_{i,j}^{N_d, N_s} g_i^*(\cdot)\overline{g_j(\cdot)} \int_{\Omega} (g_j(\mathbf{x})\overline{e_i^*(\mathbf{x})} + \overline{g_i^*(\mathbf{x})}e_j(\mathbf{x}))\alpha^\lambda(\mathbf{x}) \, d\mathbf{x}\right\|_{L^1(\Omega)} \\ &\leq 2\max_{i,j} \|g_i^*g_j\|_{L^1(\Omega)} \sum_{i,j}^{N_d, N_s} \int_{\Omega} |(g_j(\mathbf{x})\overline{e_i^*(\mathbf{x})} + \overline{g_i^*(\mathbf{x})}e_j(\mathbf{x}))\alpha^\lambda(\mathbf{x})| \, d\mathbf{x}. \end{aligned} \tag{4.29}$$

An upper bound for the integral in (4.29) can be obtained as follows:

$$\begin{aligned} &\int_{\Omega} |(g_j(\mathbf{x})\overline{e_i^*(\mathbf{x})} + \overline{g_i^*(\mathbf{x})}e_j(\mathbf{x}))\alpha^\lambda(\mathbf{x})| \, d\mathbf{x} \\ &\leq \sum_{n=1}^{N_\Delta^{ni}} \|e_i^*\|_{0,n^i} \|g_j\alpha^\lambda\|_{0,n^i} + \sum_{m=1}^{N_\Delta^j} \|e_j\|_{0,m^j} \|g_i^*\alpha^\lambda\|_{0,m^j}. \end{aligned} \tag{4.30}$$

Note that $g_j\alpha^\lambda \in L^2(\Omega)$ since $|g_j\alpha^\lambda| \leq |g_j|\|\alpha^\lambda\|_\infty$. Similarly, $g_i^*\alpha^\lambda \in L^2(\Omega)$ since $|g_i^*\alpha^\lambda| \leq |g_i^*|\|\alpha^\lambda\|_\infty$. Using (4.30) in (4.29),

$$\begin{aligned} \|(\tilde{\mathcal{K}} - \mathcal{K})\alpha^\lambda\|_{L^1(\Omega)} &\leq 2\max_{i,j} \|g_i^*g_j\|_{L^1(\Omega)} \\ &\quad \times \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^{ni}, N_s} \|e_i^*\|_{0,n^i} \|g_j\alpha^\lambda\|_{0,n^i} + \sum_{j=1}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} \|e_j\|_{0,m^j} \|g_i^*\alpha^\lambda\|_{0,m^j} \right). \end{aligned} \tag{4.31}$$

To compute an upper bound for $\|\tilde{\gamma} - \gamma\|$ using (4.25), we first write

$$\begin{aligned} &\int_{\Omega} \left| \sum_{i,j}^{N_d, N_s} (e_i^*(\mathbf{x})\overline{g_j(\mathbf{x})} + \overline{g_i^*(\mathbf{x})}e_j(\mathbf{x}))\Gamma_{i,j} \right| \, d\mathbf{x} \\ &\leq \max_{i,j} |\Gamma_{i,j}| \int_{\Omega} \sum_{i,j}^{N_d, N_s} |e_i^*(\mathbf{x})\overline{g_j(\mathbf{x})} + \overline{g_i^*(\mathbf{x})}e_j(\mathbf{x})| \, d\mathbf{x} \\ &\leq \max_{i,j} |\Gamma_{i,j}| \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^{ni}, N_s} \|e_i^*\|_{0,n^i} \|g_j\|_{0,n^i} + \sum_{i=j}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} \|g_i^*\|_{0,m^j} \|e_j\|_{0,m^j} \right). \end{aligned} \tag{4.32}$$

Noting (2.9),

$$\max_{i,j} |\Gamma_{i,j}| \leq \max_{i,j} \|g_i^*g_j\|_{L^1(\Omega)} \|\alpha\|_\infty, \tag{4.33}$$

which leads to

$$\begin{aligned} &\max_{i,j} |\Gamma_{i,j}| \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^{ni}, N_s} \|e_i^*\|_{0,n^i} \|g_j\|_{0,n^i} + \sum_{i=j}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} \|g_i^*\|_{0,m^j} \|e_j\|_{0,m^j} \right) \\ &\leq \max_{i,j} \|g_i^*g_j\|_{L^1(\Omega)} \|\alpha\|_\infty \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^{ni}, N_s} \|e_i^*\|_{0,n^i} \|g_j\|_{0,n^i} + \sum_{i=j}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} \|g_i^*\|_{0,m^j} \|e_j\|_{0,m^j} \right). \end{aligned} \tag{4.34}$$

We now use (4.31), (4.34), the corresponding discretization error estimates (3.8)–(3.9), and (4.27) to obtain (4.28). \square

Remark 3.

- (i) Theorem 2 suggests the use of meshes designed individually for the solutions G_j , $j = 1, \dots, N_s$ and G_i^* , $i = 1, \dots, N_d$.
- (ii) Theorem 2 states explicitly the effect of the forward problem discretization on the accuracy of the inverse problem solution. In this context, theorem 2 suggests a discretization scheme for the forward problem, where the discretization criterion is based on the inverse problem solution accuracy, rather than the accuracy of the forward problem solution.
- (iii) For each source, when solving for G_j , h_m^j has to be kept small where $(2\|g_i^* \alpha^\lambda\|_{0,m^i} + \|\alpha\|_\infty \|g_i^*\|_{0,m^i}) \|g_j\|_{1,m^i}$ is large. Note that $\|g_j\|_{1,m^i}$ will be large on the elements close to the j th source.
- (iv) For each detector, when solving for G_i^* , h_n^i has to be kept small where $(2\|g_j \alpha^\lambda\|_{0,n^i} + \|\alpha\|_\infty \|g_j\|_{0,n^i}) \|g_i^*\|_{1,n^i}$ is large. Note that $\|g_i^*\|_{1,n^i}$ will be large on the elements close to the i th detector.
- (v) $|g_j|$ and $|g_i^*|$ are higher close to the sources and detectors, respectively. Therefore, h_m^j has to be small around the j th source and around all detectors, where α^λ is nonzero. Likewise, h_n^i has to be small around the i th detector and around all sources, where α^λ is nonzero.
- (vi) If α^λ is nonzero on the whole domain Ω , then the error may become higher depending on the magnitude of $|g_j|$ and $|g_i^*|$.
- (vii) The regularization parameter affects the bound on $\|\alpha^\lambda - \tilde{\alpha}^\lambda\|_{L^1(\Omega)}$.
- (viii) Increasing the number of sources and detectors increases the bound on $\|\alpha^\lambda - \tilde{\alpha}^\lambda\|_{L^1(\Omega)}$.

Remark 4.

- (i) Note that the finite-element discretization error estimates (3.8)–(3.9) depend on only the smoothness and support of g_j and g_i^* , and the finite dimensional space of approximating functions [9]. However, the error estimate (4.28) in theorem 2 shows that the accuracy of the reconstructed image $\tilde{\alpha}^\lambda$ depends on the orientation of the absorptive heterogeneity with respect to the sources and detectors, as well as on the finite-element discretization error estimates (3.8)–(3.9). In this respect, the estimate (4.28) in theorem 2 shows that reducing the discretization error in the solutions G_j and G_i^* of the forward problem may not ensure the accuracy of the reconstructed absorption image (see [14]).
- (ii) In case a different discretization approach such as finite difference [20] or finite volume [13] is used to solve the forward problem, theorem 2 can be modified in a straightforward manner by replacing the discretization error estimates (3.8) and (3.9) with the corresponding error estimates specific to the method of choice [13, 20].
- (iii) Let $\tilde{\gamma}^\delta$ be the perturbed right-hand side $\tilde{\gamma}$ of (4.26) due to the presence of noise, such that $\|\tilde{\gamma}^\delta - \tilde{\gamma}\|_{L^1(\Omega)} \leq \delta$. Then, an additional term is introduced to the bound in (4.28) due to this perturbation:

$$\begin{aligned} \|\alpha^\lambda - \tilde{\alpha}^\lambda\|_{L^1(\Omega)} &\leq \frac{C}{\lambda} \max_{i,j} \|g_i^* g_j\|_{L^1(\Omega)} \\ &\times \left(\sum_{i=1}^{N_d} \sum_{n,j}^{N_\Delta^i, N_s} (2\|g_j \alpha^\lambda\|_{0,n^i} + \|\alpha\|_\infty \|g_j\|_{0,n^i}) \|g_i^*\|_{1,n^i} h_n^i \right. \\ &\left. + \sum_{j=1}^{N_s} \sum_{m,i}^{N_\Delta^j, N_d} (2\|g_i^* \alpha^\lambda\|_{0,m^j} + \|\alpha\|_\infty \|g_i^*\|_{0,m^j}) \|g_j\|_{1,m^j} h_m^j \right) + \frac{\delta}{\lambda}. \end{aligned} \quad (4.35)$$

Clearly, the additional term $\tilde{\delta}/\lambda$ due to the presence of noise in (4.35) is independent of the discretization of the forward problem.

- (iv) Theorem 2 provides a general framework to analyse the error in reconstructed optical images resulting from the perturbations in the kernel of the linear integral equation (2.16). In general, a perturbation in the kernel of the linear integral equation (2.16) can occur due to errors resulting from the numerical integration of (2.6)–(2.7), the approximation of the boundary $\partial\Omega$, the inaccurate approximation of the source Q_j and/or the background optical properties. Furthermore, the analysis framework in theorem 2 can be used to analyse the effect of linearization of the Lippmann–Schwinger-type equations [10] using Born approximation on the accuracy of the reconstructed optical images [15].
- (v) A bound similar to (4.28) can be derived for the error that occurs as a result of the discretization of the forward problem in electrical impedance tomography, optical fluorescence tomography, bioluminescence tomography and microwave imaging.

4.3. Iterative Born approximation

In this section, we explore the error in the inverse problem solution within an iterative linearization approach.

The error analysis presented in this paper covers the error which results from the discretization of the forward and inverse problems. If α is sufficiently low, then one iteration suffices to solve the inverse problem and the error analysis discussed above applies. When iterative linearization is considered to address the nonlinearity of the inverse problem, we can make use of the error analysis at each linearized step as follows: let $\alpha_{(t)}^\lambda$ and $\tilde{\alpha}_{n(t)}^\lambda$ be the actual solution of the regularized inverse problem (2.13) and the solution of (3.10) at the t th linearization step, respectively. At the end of the $(r - 1)$ th linearization step, the absorption coefficient estimate at \mathbf{x} is given by $\hat{\mu}_a^{(r-1)}(\mathbf{x}) = \mu_a^{(0)}(\mathbf{x}) + \sum_{t=1}^{r-1} \tilde{\alpha}_{n(t)}^\lambda(\mathbf{x})$, where $\tilde{\alpha}_{n(t)}^\lambda$ has an error due to discretization with respect to the actual solution $\alpha_{(t)}^\lambda$, and $\mu_a^{(0)}$ is the initial guess for the background absorption coefficient. In the next linearization, an error on the new solution update $\hat{\mu}_a^{(r)}$ will be introduced due to

- (i) projection (inverse problem discretization),
- (ii) the error $(\tilde{\mathcal{K}} - \mathcal{K})^{(r-1)}$ in the operator $(\tilde{\mathcal{K}})^{(r-1)}$ and the error $(\tilde{\gamma} - \gamma)^{(r-1)}$ in $(\tilde{\gamma})^{(r-1)}$ resulting from the forward problem discretization, and
- (iii) the error in the $(r - 1)$ th update $\hat{\mu}_a^{(r-1)}$, resulting from the discretization of the forward and inverse problems. Note that $\hat{\mu}_a^{(r-1)}$ appears as a coefficient in the boundary value problems (2.1)–(2.2) and (2.3)–(2.4). An error in this coefficient implies perturbation in the solutions of (2.1)–(2.2) and (2.3)–(2.4). As a result, G_j and G_i^* will have error terms in addition to the discretization error.

As a result, the error in $\hat{\mu}_a^{(r)}$ at the r th iteration is bounded by

$$\|\mu_a - \hat{\mu}_a^{(r)}\| = \left\| \sum_{t=1}^r \alpha_{(t)}^\lambda - \tilde{\alpha}_{n(t)}^\lambda \right\| \leq \sum_{t=1}^r \|\alpha_{(t)}^\lambda - \tilde{\alpha}_{n(t)}^\lambda\|, \quad (4.36)$$

assuming that the initial guess $\mu_a^{(0)}$ for the background absorption is approximated accurately while solving the boundary value problems (2.1)–(2.2) and (2.3)–(2.4) at the first iteration, that is $\mu_a^{(0)}(\mathbf{x}) - \sum_{k=1}^n \mu_a^{(0)}(\mathbf{x}_k) L_k(\mathbf{x}) \rightarrow 0$, for all $\mathbf{x} \in \Omega$.

5. Conclusion

In this work, we presented an error analysis to show the relationship between the error in the reconstructed optical absorption images and the discretization of the forward and inverse problems. We summarized the implications of the error analysis in two theorems which provide an insight into the impact of forward and inverse problem discretizations on the accuracy of the reconstructed optical absorption images. These theorems show that the error in the reconstructed optical image due to the discretization of each problem is bounded by roughly the multiplication of the discretization error in the corresponding solution and the solution of the other problem. In particular, theorem 2 shows that solving the diffusion equation and the associated adjoint problem accurately may not ensure small values for $\|\tilde{\mathcal{K}} - \mathcal{K}\|$ and $\|\gamma - \tilde{\gamma}\|$, which may lead to large errors in the reconstructed optical images, depending on the value of the regularization parameter. Similarly, relatively large discretization error in the solution of the forward problem may have relatively low impact on the accuracy of the reconstructed optical images, depending on the source–detector configuration, and orientation with respect to the optical heterogeneities. We have also shown that the error estimates can be extended to include the effect of noise on the overall error in the reconstructed images.

The error analysis presented in this work motivates the development of novel adaptive discretization schemes based on the error estimates in theorems 1 and 2. In the sequel of this work, we propose two novel adaptive discretization algorithms for the forward and inverse problems [14], and justify the validity of theorems 1 and 2.

The error analysis can be extended to show the effect of the discretization error on the accuracy of the simultaneous reconstruction of scattering and absorption coefficients, which will be the focus of our future work. Finally, we note that the error analysis introduced in this paper is not limited to DOT, and can easily be adapted for similar inverse parameter estimation problems such as electrical impedance tomography, bioluminescence tomography, optical fluorescence tomography, microwave imaging etc.

Acknowledgments

This material is based upon work supported by NSF-BES-0353160, ONR-N00014-04-1-0694, and US Army Medical Research-W81XWH-04-1-0559. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. Various portions of this research were supported by the Center for Subsurface Sensing and Imaging Systems, under the Engineering Research Centers Program of the National Science Foundation (Award Number EEC-9986821) and Rensselaer Polytechnic Institute.

Appendix A. Boundedness of \mathcal{A}_a

$$\|\mathcal{A}_a \alpha\|_{l^1} = \sum_{i,j}^{N_d, N_s} \left| \int_{\Omega} H_{i,j}(\mathbf{x}) \alpha(\mathbf{x}) \, d\mathbf{x} \right|. \quad (\text{A.1})$$

We can write the following inequality:

$$\|\mathcal{A}_a \alpha\|_{l^1} \leq \sum_{i,j}^{N_d, N_s} \int_{\Omega} |H_{i,j}(\mathbf{x}) \alpha(\mathbf{x})| \, d\mathbf{x} \leq \left(\sum_{i,j}^{N_d, N_s} \int_{\Omega} |H_{i,j}(\mathbf{x})| \, d\mathbf{x} \right) \|\alpha\|_{\infty}. \quad (\text{A.2})$$

Using Schwarz' inequality, we can write an upper bound for the summation as follows:

$$\begin{aligned} \sum_{i,j}^{N_d, N_s} \int_{\Omega} |H_{i,j}(\mathbf{x})| \, d\mathbf{x} &= \sum_{i,j}^{N_d, N_s} \|g_i^* g_j\|_{L^1(\Omega)} \\ &\leq \sum_{i,j}^{N_d, N_s} \|g_i^*\|_0 \|g_j\|_0 \\ &\leq N_d N_s \max_i \|g_i^*\|_0 \max_j \|g_j\|_0, \end{aligned} \quad (\text{A.3})$$

which leads to

$$\|\mathcal{A}_a \alpha\|_{l^1} \leq N_d N_s \max_i \|g_i^*\|_0 \max_j \|g_j\|_0 \|\alpha\|_{\infty}.$$

Therefore an upper bound for the norm of \mathcal{A}_a is given by

$$\|\mathcal{A}_a\|_{L^{\infty}(\Omega) \rightarrow l^1} \leq N_d N_s \max_i \|g_i^*\|_0 \max_j \|g_j\|_0. \quad (\text{A.4})$$

The boundedness of g_j and g_i^* imply that \mathcal{A}_a is bounded.

Appendix B. Compactness of \mathcal{A}_a

\mathcal{A}_a is bounded by (A.4). Furthermore \mathcal{A}_a maps the infinite-dimensional subspace $L^{\infty}(\Omega)$ to a finite-dimensional subspace $\mathbb{C}^{N_d \times N_s}$, that is the range $R(\mathcal{A}_a)$ of \mathcal{A}_a satisfies $R(\mathcal{A}_a) \in \mathbb{C}^{N_d \times N_s}$ due to the finite number of sources and detectors. As a result, \mathcal{A}_a is compact [18]. The inverse problem is illposed as a consequence of compactness [18].

Appendix C. Proof of the lemma

The identity operator \mathcal{I} is a bounded operator with bounded inverse and $(\mathcal{P}_n \mathcal{I})^{-1} = \mathcal{I} : X_n \rightarrow X_n$. Furthermore, $\|\mathcal{P}_n\|_{X \rightarrow X_n}$ is bounded for first-order Lagrange basis functions [4, 18]. Thus, projection by collocation converges for the identity operator. \mathcal{A} is bounded and compact, and $\mathcal{K} = \lambda \mathcal{I} + \mathcal{A}$ is injective, with bounded inverse given by (4.1). As a result, by theorem 13.7 in [18], the projection method also converges for $\mathcal{K} = \lambda \mathcal{I} + \mathcal{A}$. Convergence of projection for \mathcal{K} implies $(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K} \alpha^{\lambda} \rightarrow \alpha^{\lambda}$, $n \rightarrow \infty$ for $(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K} : X \rightarrow X_n$ [18].

It follows from the proof of theorem 13.7 in [18] that $(\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A})^{-1} : Y_n \rightarrow X_n$ exists and is uniformly bounded for all sufficiently large n . Then from $\mathcal{P}_n \mathcal{K} = \lambda \mathcal{P}_n (\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A}) = \lambda (\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A})$, it follows that $\mathcal{P}_n \mathcal{K} : X_n \rightarrow Y_n$ is invertible for all sufficiently large n with the inverse given by

$$(\mathcal{P}_n \mathcal{K})^{-1} = \left(\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A} \right)^{-1} \frac{1}{\lambda}. \quad (\text{C.1})$$

As a result we can write $(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}$ as follows:

$$(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K} = \left(\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A} \right)^{-1} \frac{1}{\lambda} \mathcal{P}_n \mathcal{K}. \quad (\text{C.2})$$

Thus,

$$\|(\mathcal{P}_n \mathcal{K})^{-1} \mathcal{P}_n \mathcal{K}\|_{X \rightarrow X_n} \leq C_M \frac{\|\mathcal{K}\|_{X \rightarrow Y}}{\lambda} \quad (\text{C.3})$$

where $C_M > 0$ is independent of n , using the facts that projection by collocation method converges for the identity operator and $(\mathcal{I} + \frac{1}{\lambda} \mathcal{P}_n \mathcal{A})^{-1}$ is uniformly bounded.

References

- [1] Ainsworth M and Oden J T 1993 A unified approach to *a posteriori* error estimation using elemental residual methods *Numer. Math.* **65** 23–50
- [2] Arridge S R 1999 Optical tomography in medical imaging *Inverse Problems* **15** R41–93
- [3] Arridge S R, Kaipio J P, Kolehmainen V, Schwejger M, Somersalo E, Tarvainen T and Vauhkonen M 2006 Approximation errors and model reduction with an application in optical diffusion tomography *Inverse Problems* **22** 175–95
- [4] Atkinson K and Weimin H 2001 *Theoretical Numerical Analysis: A Functional Analysis Framework* (Berlin: Springer)
- [5] Babuška I and Rheinboldt W C 1978 Error estimates for adaptive finite element computations *SIAM J. Numer. Anal.* **15** 736–54
- [6] Babuška I, Zienkiewicz O C, Gago J and Oliveira E R de A 1986 *Accuracy Estimates and Adaptive Refinements in Finite Element Computations* (New York: Wiley)
- [7] Bank R E and Weiser A 1985 Some *a posteriori* error estimators for elliptic partial differential equations *Math. Comput.* **44** 283–301
- [8] Beilina L 2002 Adaptive hybrid FEM/FDM methods for inverse scattering problems *PhD Thesis* Chalmers University of Technology
- [9] Brenner S C and Scott L R 2002 *The Mathematical Theory of Finite Element Methods* (Berlin: Springer)
- [10] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Theory* (Berlin: Springer)
- [11] Daners D 2000 Robin boundary value problems on arbitrary domains *Trans. Am. Math. Soc.* **352** 4207–36
- [12] Evans L C 1998 *Partial Differential Equations* (Providence, RI: American Mathematical Society)
- [13] Eymard R, Gallouët T and Herbin R 2000 The finite volume method *Handbook of Numerical Analysis* vol 7 ed P G Ciarlet and J L Lions (Amsterdam: Elsevier) pp 715–1022
- [14] Guven M, Yazici B, Kwon K, Giladi E and Intes X 2007 Effect of discretization error and adaptive mesh generation in diffuse optical absorption imaging: II *Inverse Problems* **23** 1135–60
- [15] Guven M, Yazici B, Kwon K, Giladi E and Intes X Error in optical absorption images due to Born approximation in diffuse optical tomography, in preparation
- [16] Ihlenburg F 1998 *Finite Element Analysis of Acoustic Scattering* (Berlin: Springer)
- [17] Kaipio J and Somersalo E 2005 Statistical and computational inverse problems *Applied Mathematical Sciences* vol 160 (New York: Springer)
- [18] Kress R 1999 Linear integral equations *Applied Mathematical Sciences* vol 82 2nd edn (Berlin: Springer)
- [19] Li R, Liu W, Ma H and Tang T 2002 Adaptive finite element approximation for distributed elliptic optimal control problems *SIAM J. Control Optim.* **41** 1321–49
- [20] Smith G D 1985 *Numerical Solution of Partial Differential Equations: Finite Difference Methods (Oxford Applied Math. Comput. Sci. Series)* 3rd edn (Oxford: Clarendon)
- [21] Strouboulis T and Hague K A 1992 Recent experiences with error estimation and adaptivity: Part I. Review of error estimators for scalar elliptic problems *Comput. Methods Appl. Mech. Eng.* **97** 399–436
- [22] Verfurth R 1996 *A Review of A Posteriori Error Estimation and Adaptive Mesh Refinement Techniques* (New York: Teubner-Wiley)