ESTIMATION OF RADAR TARGET REFLECTIVITY IN ULTRAWIDEBAND REGIME – A GROUP THEORETIC APPROACH

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ABSTRACT

This paper presents a fundamentally new mathematical framework based on the *group representation theory* for the modeling and processing of signals in ultrawideband (UWB) regime. A group theoretical approach is motivated by the fact that in UWB regime, the underlying mathematical structure of the inverse scattering is governed by the affine group. In particular, received echo can be viewed as the affine Fourier transform of the range-Doppler image evaluated at the transmitted waveform. Based on group representation theory and a novel Wiener filtering method over the affine group, we derived a regularized closed form analytical estimate of the range-Doppler target reflectivity density in the presence of nonstationary noise and clutter in UWB regime. This estimate also leads to a method of data fusion to form synthetic UWB high resolution images from multiple radars with narrowband transmission.

1. INTRODUCTION

In radar/sonar imaging, the transmitter emits an electromagnetic signal. The signal is reflected off a target and detected by the transmitter/receiver as the echo signal. Assuming negligible acceleration of the reflector, the echo model from a point reflector is given as the delayed and scaled replica of the transmitted pulse [1],-[4]:

$$e(t)=\sqrt{s}f(st+\tau), \tag{1.1}$$

where f is the transmitted pulse, τ is the time delay, and s is the time scale or Doppler stretch. The term s is given as $s=(c-\upsilon)/(c+\upsilon)$, where c is the speed of the transmitted signal and υ is the radial velocity of the reflector. This echo model is also referred as the "wideband" echo model. When the bandwidth of the transmitted signal is narrow as compared to main frequency component, the echo model becomes the delayed and the Doppler shifted version of the transmitted signal, i.e., $e(t)=f(t-\tau)e^{j\omega t}$, known as the narrowband echo model. The echo model in (1.1) is more general and approximates the narrowband model in the limit [5]. In general, the echo model described above is valid for ultrawideband (UWB) signals. According to the definition introduced by DARPA in 1990, an UWB signal is the one with large relative bandwidth, $\eta=(f_H-f_L)/(f_H+f_L)$ in the range $0.25\le\eta<1$ where f_H and f_L are the highest and lowest frequencies of interest. The term "wideband" is typically used for signals with a relative bandwidth of 5- 10%. See also Sholtz et al [6]-[9] for information theoretic modeling of UWB channels.

It is often desirable to image a "dense" group of reflectors. This means that the target environment is composed of several objects, or a physically large object with continuum of reflectors and that the reflectors are very close in range-Doppler space. This dense group of reflectors is described by *a reflectivity density function* in the range-Doppler space. The received signal is modeled as a weighted average [1], [10]-[11] echo signal of the continuum of scatters. For UWB signals, the echo model is given as

$$e_W(t)=\int_{-\infty}^{\infty}\int_{0}^{\infty}T_W(s,\tau)\frac{1}{\sqrt{s}}f\left(\frac{t-\tau}{s}\right)\frac{ds}{s^2}d\tau\,, \tag{1.2}$$

where $T_W(s,\tau)$ is the wideband reflectivity density function associated with each time delayed and time scaled version of the transmitted signal. Note that the UWB echo model in is in fact the affine group Fourier transform of the target reflectivity function T_W evaluated at the transmitted pulse f. The narrowband echo model is given by

$$e_N(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T_N(\omega,\tau)f(t-\tau)e^{j\omega t}d\tau d\omega\,, \tag{1.3}$$

where $T_N(\omega,\tau)$ is the narrowband reflectivity density function associated with each time delayed and frequency shifted version of the transmitted signal.

The goal in range-Doppler imaging is to estimate $T_W(s,\tau)$ and $T_N(\omega,\tau)$ given the transmitted and the received signals. Typically, the received echo in a radar or sonar system is very weak due to clutter and system noise. Therefore, the detection at the receiver side is performed by matched filtering (estimator/correlator), which amounts to correlating the received echo with the transmitted pulse. Estimator/Correlator is an optimal filter maximizing the output signal to noise ratio (SNR) for a given waveform, and received signal embedded in white Gaussian noise [12]. When the wideband echo model described in Equation (1.2) inserted into the wideband correlation receivers, the resulting output is expressed as an affine group convolution integral:

$$C_W(s,\tau)=\int_{-\infty}^{\infty}\int_{0}^{\infty}T_W(a,b)\mathrm{K}_W\left(\frac{s}{a},\frac{\tau-b}{a}\right)\frac{da}{a^2}db \quad (1.4)$$

where K_W is the wideband auto-ambiguity function and C_W is the wideband cross-ambiguity function or the estimator/correlator output. Thus, at the output of the estimator/correlator, the wideband echo model is expressed as an *affine group* convolution. $C_W = T_W *_A \mathrm{K}_W$ where $*_A$ denotes the affine group convolution. On the other hand, for narrowband processing, the echo model is expressed as a *Heisenberg group* convolution.

The fundamental problem that we will address can be stated as follows: How can we recover the UWB target reflectivity function T_W in range-Doppler space from the measurements $y(t)$, $t\in\mathbb{R}$, embedded in nonstationary noise/clutter given a priori target and clutter information?

2. MATHEMATICAL PRELIMINERIES

2.1. Fourier Analysis on the Affine Group

Affine group has exactly two nonequivalent, infinite dimensional, irreducible, unitary representations. We denote them by $\pi_\pm$. Let $\pi_\pm$ act on the Hilbert space $H_\pm$ that consist of functions $\phi_\pm$ whose Fourier transform are supported on the right and left half-line, respectively. Then, the affine Fourier transform is given as follows:

$$\pi_\pm(a,b)\phi(x)=\frac{1}{\sqrt{a}}\phi_\pm\left(\frac{x-b}{a}\right) \quad (2.1)$$

$$\mathcal{F}_{U_\pm}(f)=\int_{-\infty}^{\infty}\int_{0}^{\infty}a^{-2}dadbf(a,b)\pi_\pm(a,b) \quad (2.2)$$

$$f(a,b)=\sum_{\pm}tr\left(\pi_\pm^\dagger(a,b)\mathcal{F}_\pm(f)\delta\right) \quad (2.3)$$

Note that the linear operator δ corresponds to $\delta\phi_\pm(t)=-\frac{i}{2\pi}\frac{d\phi_\pm}{dt}(t)$, [13]. The extra operator δ in the Affine group Fourier inversion formula is due to the non-unimodular nature of the affine group. The operator valued Fourier transform maps any function f in $L^2(G,dg)$ to a family $\{\hat{f}(\lambda)\}$ of bounded operators. The collection of Fourier transforms $\{\hat{f}(\lambda)\}$ for all $\lambda\in\hat{G}$ is called the spectrum of the function f. Note that an important property of Fourier transform is that the group convolution is mapped to operator multiplication in the Fourier domain, i.e.,

$$\mathcal{F}(f_1 *_G f_2)=\mathcal{F}(f_2)\mathcal{F}(f_1), \quad (2.4)$$

2.2. Wiener Filtering over the Affine Group

Theorem : Let G be a separable locally compact group of Type-I, and $x(g)$ and $n(g)$, $g\in G$, be two zero mean left group stationary processes, referred to as signal and noise, respectively. Assume that the measurements obey the following convolution integral and noise model:

$$y(g)=\int_G x(h)f(h^{-1}\circ g)dh+n(g) \text{ and } E\left[x(g)\overline{n(g)}\right]=0 \quad (2.5)$$

where the filter f is known and belongs to $L^2(G,dg)$. Then, the optimum linear least squares deconvolution filter W_{opt}, minimizing the least squares error variance

$$J(\varepsilon_w)=\int_G E\left[\left|\varepsilon_w(g)\right|^2\right]dg$$

where

$$\varepsilon_w(g)=\int_G W(g,h)y(h)dh-x(g), \quad (2.6)$$

is left group invariant and the estimate of the signal is given by the following convolution integral

$$\hat{x}(g)=\int_G y(h)W_{opt}(h^{-1}\circ g)dh. \quad (2.7)$$

The Fourier transform of the optimal filter W_{opt} is given by

$$\hat{W}_{opt}(\lambda)=S_x(\lambda)\hat{f}^\dagger(\lambda)\left[\hat{f}(\lambda)S_x(\lambda)\hat{f}^\dagger(\lambda)+S_n(\lambda)\right]^{-1},\ \lambda\in\hat{G} \quad (2.8)$$

Here, $\hat{f}$ is the Fourier transform of the convolution filter f, and, $\hat{f}^\dagger$ denotes the adjoint of the operator $\hat{f}$. S_x and S_n are operator valued spectral density functions of the signal and noise, respectively. The spectral density function of the least square error between the signal and its filtered estimate is given as

$$S_e(\lambda)=\left(I-\hat{W}_{opt}(\lambda)\hat{f}(\lambda)\right)S_x(\lambda) \quad (2.9)$$

where I denotes the identity operator.
Proof : [13]-[14].

3. TARGET REFLECTIVITY ESTIMATION

There is a vast literature on wideband signal processing, particularly in the context of communications. However, most of these studies deal with signals that can be well approximated with the narrowband wave propagation model, i.e., relative bandwidth of the signals is 10% or less. As mentioned above, the echo model in (1.1) is primarily applicable to UWB signals [15]-[16], [6]-[9]. Signal processing for UWB radar and communications is an emerging area of research. For recent publications in UWB channel estimation and waveform design, see [6]-[9], [17],[18].

The UWB wave propagation model as described above has been studied before (See, [1]-[4],[11], [19]-[20] and references therein). In [9] and [31], Naparst and Miller suggested to use the Fourier theory of the affine group and proposed a method to reconstruct the target reflectivity density function in a deterministic setting. In [4], Weiss suggested to use the wavelet transform for the image recovery in a deterministic setting. However, this approach requires that the target reflectivity function to be in the reproducing kernel Hilbert space of the transmitted wavelet. In [19]-[20], this approach extended to include affine frames. In all these studies, the received signal is modeled noise and clutter free, which is not a realistic assumption for radar return signals. Furthermore, none of these studies provided a regularized inversion for the reconstruction.

Here, we shall introduce our preliminary receiver or image reconstruction methods and discuss further research problems. The preliminary result can be stated as follows: Adaptive receiver designs (image reconstruction) that can simultaneously perform detection and estimation in the presence of nonstationary noise, and range-Doppler clutter based on the a priori target and clutter information with the following properties:

1. Optimal signal-to-clutter ratio in the mean square sense given a priori target and clutter information. Thus from information theoretic point of view, provides best detectability.
2. Provides a method of data fusion to form UWB high resolution images from low resolution multiple radars.
3. Multiple mathematical representations leading to different practical implementations.

In conventional radar engineering, the detection and estimation are performed sequentially. When detecting targets, the presence or absence of a target is generally determined by a threshold test on the energy of the received signal. Once the detection is performed, target echo is filtered using the estimator/correlator to extract target track parameters. It is straightforward to show that the estimator/correlator is an optimal filter that maximizes the signal-to-noise ratio for a given waveform in the presence of white Gaussian noise.

Here, we shall address the following problem, given the affine stationary target spectral density function $S_{\pm}^{T}$ and the additive noise/clutter spectral density function $S_{\pm}^{n}$ in range and Doppler, design a receiver filter such that the signal-to-noise ratio of the receiver output is maximized. The primary difference between this problem and the standard matched filtering problem is the effect of the target and clutter spectra (a priori information in both range and Doppler) in defining the structure of the receiver. Furthermore, we will see that the receiver design problem couples naturally with the waveform design problem. As a result, the image reconstruction can be customized to performe only for those targets that are of interest.

Observe that the UWB echo model is in fact equal to the Affine Fourier transform of the target reflectivity density function, T_W, evaluated at the transmitted pulse f, i.e., Equation (1.2) can be alternatively expressed as

$$e_W(t) = \sum_{\pm} \mathcal{F}_{\pm}(T_W) f_{\pm}(t)$$

or

$$e_W(t) = \langle T_W,\ \pi_+(a,b) f_+(t) \rangle + \langle T_W,\ \pi_-(a,b) f_-(t) \rangle \qquad (3.1)$$

where e_W is the received echo and $f = f_+ \oplus f_-$ are the orthogonal components of the transmitted pulse.

Now, assume that the unknown target reflectivity density function is a random variable $T_W(a,b)$ on the range-Doppler plane contaminated with an additive random noise/clutter $N(a,b)$ in the rage-Doppler plane. Thus, the model for the return signal is given by

$$D(a,b) = T_W(a,b) + N(a,b), \quad (a,b) \in A$$

or

$$y(t) = e_W(t) + n(t) \qquad (3.2)$$

where

$$y(t) = \langle D,\ \pi(a,b) f(t) \rangle, \quad e_W(t) = \langle T_W,\ \pi(a,b) f(t) \rangle,$$
$$n(t) = \langle N,\ \pi(a,b) f(t) \rangle. \qquad (3.3)$$

The minimum mean square error filter providing the best compromise between the clutter and the target reflectivity density function is the Wiener filter over the affine group given by

$$\mathcal{F}_{\pi_{\pm}}(W_{opt}) = (S_{\pm}^{T})[S_{\pm}^{T} + S_{\pm}^{N}]^{-1} \qquad (3.4)$$

where $S_{\pm}^{T}$ and $S_{\pm}^{N}$ are the components of the spectral density functions of the target T_W and clutter N, respectively. Affine Wiener filter can be estimated from the a priori target and clutter information. Such information is routinely compiled for air defense radar. (See, for example [21]). The affine spectra, $S_{\pm}^{T}$ and $S_{\pm}^{N}$, of the target and clutter can be estimated from such plots. One such estimate could be the periodogram like estimate given as: $\hat{S}_{\pm}^{T} = \mathcal{F}_{\pm}(T_W)\mathcal{F}_{\pm}^{\dagger}(T_W)$. Note that the Affine Wiener filter is Hermitian symmetric since both target and noise spectra, $S_{\pm}^{T}$ and $S_{\pm}^{N}$, are Hermitian, symmetric and bounded. Therefore, the minimum mean square error estimate $\hat{T}_W(a,b)$ of the target reflectivity is given by

$$\hat{T}_W(a,b) = \sum_{\pm} trace\left(\pi_{\pm}^{\dagger}(a,b)\mathcal{F}_{\pi_{\pm}}(W_{opt})\mathcal{F}_{\pi_{\pm}}(D)\delta\right). \qquad (3.5)$$

This estimate can be implemented in various forms leading to different adaptive receiver structures. Below, we describe two different implementations and describe how receiver design problem couples with the waveform design.

<u>Receiver Design 1:</u> Let $\{s_n^{\pm}(t)\}$ be a set of Schwartz class orthogonal basis for $H_{\pm}$, respectively. Then, the target reflectivity estimate in Equation (2.16) can be expressed as

$$\hat{T}_W(a,b) = \sum_{\pm}\sum_{n} \langle \mathcal{F}_{\pi_{\pm}}(W_{opt})\mathcal{F}_{\pi_{\pm}}(D)\tilde{s}_{\pm}^{n},\ \pi_{\pm}(a,b)s_{\pm}^{n} \rangle$$

where

$$\tilde{s}_{\pm}^{n}(t) = \delta s_{\pm}^{n}(t) = -\frac{i}{2\pi}\frac{ds_{\pm}^{n}}{dt}(t). \qquad (3.6)$$

Note that $y^n(t) = \mathcal{F}(D)\tilde{s}^n$ is the received echo signal if $\tilde{s}^n$ is chosen to be the transmitted signal where $\tilde{s}^n = \tilde{s}_+^n + \tilde{s}_-^n$. Then, Equation (3.6) can be reexpressed as

$$\hat{T}_W(a,b) = \sum_{n} \langle y^n,\ \mathcal{F}_{\pi}^{\dagger}(W_{opt})\pi(a,b)s^n \rangle \qquad (3.7)$$

where W_{opt} is the Affine Wiener filter as in Equation (2.8).

<u>Receiver Design 2:</u> Alternatively, we can express the estimate in Equation (2.16) in terms of the matrix elements of the linear operators for a given set of orthogonal basis functions

$$\hat{T}_W(a,b) = \sum_{n}\sum_{m}\sum_{p}\sum_{q} A_{n,m}B_{m,p}C_{p,q}E_{q,n} \qquad (3.8)$$

where

$$s^n = s_+^n \oplus s_-^n,$$
$$A_{n,m} = \langle s^n,\ \pi(a,b)s^m \rangle,\ B_{m,p} = \langle \mathcal{F}(W_{opt})s^m,\ s^p \rangle,$$
$$C_{p,q} = \langle \mathcal{F}(D)s^p,\ s^q \rangle, \text{ and } E_{q,n} = \langle \delta s^q,\ s^n \rangle \qquad (3.9)$$

are the matrix elements of the operators $\pi = \pi_+ \oplus \pi_-$, $\mathcal{F}(W_{opt}) = \mathcal{F}_+(W_{opt}) \oplus \mathcal{F}_-(W_{opt})$, and $\mathcal{F}(D) = \mathcal{F}_+(D) \oplus \mathcal{F}_-(D)$.

Algorithm for Receiver 1

1. Choose a Schwartz class orthogonal basis $\{s^n\}$ for $L^2(\mathbb{R}, dt)$.
2. Transmit the pulses $\tilde{s}^n(t) = \delta s^n(t) = -\frac{i}{2\pi}\frac{ds^n}{dt}(t)$.
3. Modife the reference signals $\{s^n\}$ by the linear transformation $\mathcal{F}_\pi^\dagger(W_{opt})\pi(a,b)$ to match to the received echo, i.e., $z^n(t) = \mathcal{F}_\pi^\dagger(W_{opt})\pi(a,b)s^n(t)$.
4. Correlate $\{z^n\}$ with $\{s^n\}$ and coherently sum, i.e., $\hat{T}_W(a,b) = \sum_n \langle y^n, \ z^n \rangle$.

Algorithm for Receiver 2

1. Choose a Schwartz class orthogonal basis $\{s^n\}$ for $L^2(\mathbb{R}, dt)$ as a set of transmitted pulses.
2. Form wideband auto- and cross ambiguity functions $A_{n,m}$ of the transmitted pulses $\{s^n\}$.
3. Compute the matrix elements, $B_{m,p}$, of the Affine Wiener filter with respect to the basis $\{s^n\}$.
4. Compute $C_{p,q}$, the inner product of the received echo $\{y^p\}$ with the transmitted pulses $\{s^n\}$.
5. Compute $E_{q,n}$, the matrix elements of the operator δ.
6. Fuse echo $\{y^p\}$ as in Equation (3.8).

The primary difference between Receiver 1 and 2 are in the choice of transmitted pulses. While in Receiver 1, the derivatives of the orthogonal basis functions are being transmitted, in Receiver 2, the orthogonal basis functions themselves are transmitted. While the Receiver 1 leads to a numerically simpler structure, designing appropriate waveforms is analytically more involved. On the other hand, Receiver 2 is a numerically less efficient implementation; however, it is less restrictive in the way transmission waveforms can be chosen.

Note that, we have not specified how we can choose the set of basis functions $\{s^n\}$. Therefore, the UWB image formation algorithms described above are valid independent of the choice of transmitted waveforms. Furthermore, the orthogonal functions or their derivatives do not need to be UWB signals. Thus, this reconstruction formula leads to a scenario where there are multiple radars operating independently, each with a limited low resolution aperture (i.e., narrowband transmission). Nevertheless, appropriate processing and fusion of data as described above from multiple narrowband radars leads to a synthetic UWB high resolution imaging.

4. CONCLUSION

In this paper, we present a mathematical framework based on the group representation theory for the modeling and estimation of ultrawideband signals. In particular, we described two adaptive receivers for the estimation of range-scale target reflectivity function in the presence of nonstationary noise and clutter. The receivers are designed such that both detection and estimation are performed simultaneously. For the significance of this issues in radar engineering, see the recent article [22].

While our discussion is focused on UWB radar, the fundamental results of our study are directly applicable to UWB sonar, medical ultrasound, nondestructive testing and UWB wireless communications.

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