An Inversion Method for the Cone-Beam Transform

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ABSTRACT

This paper presents an alternative formulation for the cone-beam projections given an arbitrary source trajectory and detector orientation. This formulation leads to a new inversion formula. As a special case, the inversion formula for the spiral source trajectory is derived.

Keywords: Cone-beam inversion, arbitrary source trajectory

1. INTRODUCTION

In many medical and industrial CT applications, it is desirable to collect data rapidly and reconstruct accurate images in real time. The development of wide area detectors has enabled rapid collection of data and created a need for fast and real time reconstruction algorithms based on the cone-beam transform. In particular, it is desirable to develop reconstruction algorithms that invert the cone-beam transform during data collection process.

A milestone in the area of cone-beam reconstruction algorithms is the development of a filtered backprojection (FBP) type exact inversion method by Katsevich[1–3] . Changing the order of the filtering and backprojection, Zou and Pan[4–6] presented a backprojection filtered (BPF) type new exact inversion method. Following these works, other exact reconstruction algorithms were developed in[7–10] . A unified analysis of Katsevich's, and Zou and Pan's inversion methods based on Tuy's inversion formula[11] is presented in[12, 13] .

In this work, we present a new inversion method for the cone-beam transform. We formulate cone-beam transform using the rotation and translation operators in $\mathbb{R}^3$. Using change of variables and well known Fourier relationship of the X-ray transform, we derive a new inversion formula for the cone-beam transform along an arbitrary source trajectory. As a special case, we derive the inversion formula for the spiral source trajectory.

The rest of the paper is organized as follows: In Section 2, cone-beam transform is presented. In Section 3, the Fourier transform relationship between the function and its cone-beam transform is presented. In Section 4, the proposed inversion formula for the cone-beam transform is derived. In Section 5, the inversion formula for spiral source trajectory is derived. Finally, Section 6 concludes the paper.

2. CONE-BEAM TRANSFORM

Let $\boldsymbol{h} \in \mathbb{R}^3$ denote the location of the source. The cone-beam transform of the attenuation map f is defined by[14]

$$(Df)(\boldsymbol{h}, \boldsymbol{\theta}) = \int_0^\infty f(\boldsymbol{h} + t\boldsymbol{\theta})dt, \tag{1}$$

where $\boldsymbol{\theta}$ is the direction of the X-ray beam, an element of the unit sphere S^2. The attenuation map f is a real valued function defined over $\mathbb{R}^3$. If the source travels along a curve, i.e. $\boldsymbol{h} = \boldsymbol{h}(\gamma)$, $\gamma \in \mathbb{R}$, then $(Df)(\boldsymbol{h}(t), \boldsymbol{\theta})$ is called the cone-beam transform of f along $\boldsymbol{h}(\gamma)$.

We will express $(Df)(\boldsymbol{h}(t), \boldsymbol{\theta})$ in an alternative way, which explicitly separates the detector and source parameters. We would like to write (2) using Dirac delta function. To do this, first we assume

$$(Df)(\boldsymbol{h}, \boldsymbol{\theta}) = \int_0^\infty f(\boldsymbol{h} + t\boldsymbol{\theta})dt = \int_{-\infty}^\infty f(\boldsymbol{h} + t\boldsymbol{\theta})dt. \tag{2}$$

Medical Imaging 2006: Physics of Medical Imaging, edited by Michael J. Flynn, Jiang Hsieh,
Proceedings of SPIE Vol. 6142, 61424C, (2006) · 1605-7422/06/$15 · doi: 10.1117/12.647810

Let $\boldsymbol{e_i}$ be the unit vector with its i^{th} component equal to 1, and R_i be the rotation matrix with respect to x_i-axis, for $i = 1, 2, 3$, i.e.

$$\boldsymbol{e_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \quad \boldsymbol{e_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \quad \boldsymbol{e_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{3}$$

$$R_1(\zeta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & -\sin\zeta \\ 0 & \sin\zeta & \cos\zeta \end{bmatrix}, \quad R_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix},$$

$$R_3(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Let $l_{0,0}$ be the line passing from the origin in the direction of $\boldsymbol{e_1}$. Then, $\boldsymbol{x} \in \mathbb{R}^3$ is on $l_{0,0}$ if and only if it is orthogonal to both $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$, i.e.

$$l_{0,0} = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{x} \cdot \boldsymbol{e_2} = 0, \quad \boldsymbol{x} \cdot \boldsymbol{e_3} = 0\}. \tag{5}$$

Hence, the integral of f along $l_{0,0}$ can be written as

$$\int_{\boldsymbol{x} \in l_{0,0}} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \delta(\boldsymbol{x} \cdot \boldsymbol{e_2}) \delta(\boldsymbol{x} \cdot \boldsymbol{e_3}) d\boldsymbol{x} - \int_{\mathbb{R}^3} f(\boldsymbol{x}) \delta(\boldsymbol{x} \cdot [\boldsymbol{e_2}\ \boldsymbol{e_3}]) d\boldsymbol{x}. \tag{6}$$

where δ is the Dirac delta generalized function, and we used the notation $\delta(\boldsymbol{x} \cdot [\boldsymbol{u}\ \boldsymbol{v}]) = \delta(\boldsymbol{x} \cdot \boldsymbol{u})\delta(\boldsymbol{x} \cdot \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

Any element of S^2 can be represented by $R_3(\beta)R_2(\alpha)\boldsymbol{e_1}$, for $\alpha \in [-\pi/2, \pi/2]$ and $\beta \in [-\pi, \pi]$. Let $l_{\alpha,\beta}$ be the line passing from the origin in the direction of $R_3(\beta)R_2(\alpha)\boldsymbol{e_1}$, i.e.

$$l_{\alpha,\beta} = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{x} \cdot R_3(\beta)\boldsymbol{e_2} = 0, \quad \boldsymbol{x} \cdot R_3(\beta)R_2(\alpha)\boldsymbol{e_3} = 0\}. \tag{7}$$

For $R \in \mathrm{SO}(3)$, let $L(R)$ be the rotation operator defined by $L(R)f(\boldsymbol{x}) = f(R^{-1}\boldsymbol{x})$. Then,

$$\int_{\boldsymbol{x} \in l_{\alpha,\beta}} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^3} f(\boldsymbol{x})\ L(R_3(\beta)R_2(\alpha))\delta(\boldsymbol{x} \cdot [\boldsymbol{e_2}\ \boldsymbol{e_3}]) d\boldsymbol{x}. \tag{8}$$

Let l be the cone given by $l = \{\boldsymbol{x} | \boldsymbol{x} \in l_{\alpha,\beta},\ 0 \le |\alpha| < \alpha_0 \le \pi/2,\ 0 \le |\beta| < \beta_0 \le \pi\}$. We will refer α and β as the *cone parameters*, and $\boldsymbol{e_1}$ as the *center* of the cone. If the detector plane is a fixed plane with respect to the origin, (8) models the cone beam projections with source located at the origin with the normal of the detector plane in the direction of the center of the cone, $\boldsymbol{e_1}$. For the rest of the paper, we will assume that the detector plane is fixed with respect to the source location and its normal coincides with the center of the cone. Therefore, we won't distinguish the detector plane from the cone l.

Let $R_D \in \mathrm{SO}(3)$, and $R_D\boldsymbol{e_1}$ be the normal of the detector plane. We will refer R_D as the *orientation* of the detector plane with respect to the source. If the source is located at the origin, then the cone-beam projections in this setting is given by

$$\int_{\mathbb{R}^3} f(\boldsymbol{x})\ L(R_D)L(R_3(\beta)R_2(\alpha))\delta(\boldsymbol{x} \cdot [\boldsymbol{e_2}\ \boldsymbol{e_3}]) d\boldsymbol{x}. \tag{9}$$

It is straightforward to see that the cone beam parameters α and β are invariant with respect to R_D. Equivalently, α and β can be visualized as the intrinsic parameters of the detector plane, while R_D becomes an extrinsic parameter.

Let $T(\boldsymbol{h})$, $\boldsymbol{h} \in \mathbb{R}^3$ be the translation operator given by $T(\boldsymbol{h})f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{h})$. Then, we can write the cone beam projections at a point $\boldsymbol{h}$ with detector orientation R_D as

$$(Df)(\boldsymbol{h}, R_D R_3(\beta)R_2(\alpha)\boldsymbol{e_1}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\ T(\boldsymbol{h})L(R_D)L(R_3(\beta)R_2(\alpha))\delta(\boldsymbol{x} \cdot [\boldsymbol{e_2}\ \boldsymbol{e_3}]) d\boldsymbol{x}. \tag{10}$$

Thus, we define the cone-beam projections as follows:

Definition 2.1. *Given a source trajectory* $\boldsymbol{h}(\gamma) \in \mathbb{R}^3$, $\gamma \in \mathbb{R}$, *and a function* f *over* $\mathbb{R}^3$, *we define the* **cone-beam projections** Pf *of* f *along the curve* $\boldsymbol{h}(\gamma)$ *as*

$$\begin{aligned}\mathcal{P}f(\alpha,\beta,\gamma) &= \int_{\mathbb{R}^3} f(\boldsymbol{x})\ T(\boldsymbol{h}(\gamma))L(R_D(\gamma))\ L(R_3(\beta)R_2(\gamma))\delta(\boldsymbol{x}\cdot[\boldsymbol{e_2}\ \boldsymbol{e_3}])\,d\boldsymbol{x} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{x})\delta(R_2^{-1}(\alpha)R_3^{-1}(\beta)R_D^{-1}(\gamma)(\boldsymbol{x}-\boldsymbol{h}(\gamma))\cdot[\boldsymbol{e_2}\ \boldsymbol{e_3}])\,d\boldsymbol{x}, \end{aligned} \tag{11}$$

where $\alpha \in (-\alpha_0, \alpha_0) \subset (-\pi/2, \pi/2)$, $\beta \in (-\beta_0, \beta_0) \subset (-\pi, \pi)$, *are the* **cone parameters**, $\gamma \in (\gamma_{\min}, \gamma_{\max}) \subset \mathbb{R}$ *is the* **trajectory-detector parameter**, *and* $R_D(\gamma) \in \mathrm{SO}(3)$ *is the relative orientation of the detector array with respect to the source position.*

In practice, α_0 and β_0 are determined by the size of detector array. With out loss of generality, we assume that $\alpha_0 = \pi/2$ and $\beta_0 = \pi$.

3. THE FOURIER RELATIONSHIP

By (11), the cone-beam projection operator $\mathcal{P}$ is a linear integral operator with kernel

$$K(\boldsymbol{x};\alpha,\beta,\gamma) = K_2(\boldsymbol{x};\alpha,\beta,\gamma)K_3(\boldsymbol{x};\alpha,\beta,\gamma), \tag{12}$$

where

$$\begin{aligned} K_2(\boldsymbol{x};\alpha,\beta,\gamma) &= \delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma) - t_2(\beta,\gamma)) \\ K_3(\boldsymbol{x};\alpha,\beta,\gamma) &= \delta(\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma) - t_3(\alpha,\beta,\gamma)), \end{aligned} \tag{13}$$

with

$$\boldsymbol{\theta_2}(\beta,\gamma) = R_D(\gamma)R_3(\beta)\boldsymbol{e_2} \tag{14}$$

$$t_2(\beta,\gamma) = \boldsymbol{h}(\gamma)\cdot\boldsymbol{\theta_2}(\beta,\gamma) \tag{15}$$

$$\boldsymbol{\theta_3}(\alpha,\beta,\gamma) = R_D(\gamma)R_3(\beta)R_2(\alpha)\boldsymbol{e_3} \tag{16}$$

$$t_3(\alpha,\beta,\gamma) = \boldsymbol{h}(\gamma)\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma). \tag{17}$$

Hence, we rewrite (11) as

$$\mathcal{P}f(\alpha,\beta,\gamma) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma) - t_2(\beta,\gamma))\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma) - t_3(\alpha,\beta,\gamma))d\boldsymbol{x}. \tag{18}$$

Using the identity

$$\delta(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\rho}d\rho, \tag{19}$$

we can rewrite (20) as

$$\mathcal{P}f(\alpha,\beta,\gamma) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\left[\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho_2[\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma)-t_2(\beta,\gamma)]}d\rho_2\right]\left[\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho_3[\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma)-t_3(\alpha,\beta,\gamma)]}d\rho_3\right]d\boldsymbol{x} \tag{20}$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^5} f(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\rho_2[\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma)-t_2(\beta,\gamma)]+\mathrm{i}\rho_3[\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma)-t_3(\alpha,\beta,\gamma)]}d\rho_2 d\rho_3 d\boldsymbol{x} \tag{21}$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^5} f(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot[\rho_2\boldsymbol{\theta_2}(\beta,\gamma)+\rho_3\boldsymbol{\theta_3}(\alpha,\beta,\gamma)]}\mathrm{e}^{-\mathrm{i}\rho_2 t_2(\beta,\gamma)}\mathrm{e}^{-\mathrm{i}\rho_3 t_3(\alpha,\beta,\gamma)}d\rho_2 d\rho_3 d\boldsymbol{x} \tag{22}$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \tilde{f}(-[\rho_2\boldsymbol{\theta_2}(\beta,\gamma)+\rho_3\boldsymbol{\theta_3}(\alpha,\beta,\gamma)])\mathrm{e}^{-\mathrm{i}\rho_2 t_2(\beta,\gamma)}\mathrm{e}^{-\mathrm{i}\rho_3 t_3(\alpha,\beta,\gamma)}d\rho_2 d\rho_3, \tag{23}$$

and treat $\mathcal{P}$ as a Fourier integral operator[15] , where $\tilde{f}$ is the Fourier transform of f defined by

$$\tilde{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\omega}} d\boldsymbol{x}. \tag{24}$$

We will use the following lemma and its corollary to compute f from its cone-beam projections.

LEMMA 3.1. *Let $\boldsymbol{\theta_1}, \boldsymbol{\theta_2} \in S^2$, such that $\boldsymbol{\theta_1} \perp \boldsymbol{\theta_2}$. Given an integrable function f over $\mathbb{R}^3$, define $\mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2)$ by*

$$\mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \delta(\boldsymbol{x}\cdot\boldsymbol{\theta_1} - t_1)\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2} - t_2)\ d\boldsymbol{x}. \tag{25}$$

Then,

$$\begin{aligned}\mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) &= \int_{\mathbb{R}^3} f(\boldsymbol{x}) \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi_1(\boldsymbol{x}\cdot\boldsymbol{\theta_1} - t_1)} e^{i\xi_2(\boldsymbol{x}\cdot\boldsymbol{\theta_2} - t_2)}\, d\xi_1\, d\xi_2\, d\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i\xi_1(\boldsymbol{x}\cdot\boldsymbol{\theta_1} - t_1)} e^{i\xi_2(\boldsymbol{x}\cdot\boldsymbol{\theta_2} - t_2)}\, d\boldsymbol{x}\, d\xi_1\, d\xi_2.\end{aligned} \tag{26}$$

Proof. The first line of Equation 26 follows directly by treating $\delta(\boldsymbol{x})$ as a Fourier integral operator and substituting the Fourier transform of $\delta(\boldsymbol{x})$. To show the equality of the first and the second lines, let us first look at the inverse Fourier transform of $\mathbf{P}f$ with respect to t_1 and t_2.

$$\begin{aligned}&\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) \mathrm{e}^{\mathrm{i}\xi_1 t_1} \mathrm{e}^{\mathrm{i}\xi_2 t_2}\, dt_1\, dt_2 \\ &\qquad = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) \delta(\boldsymbol{x}\cdot\boldsymbol{\theta_1} - t_1)\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2} - t_2)\ d\boldsymbol{x}\, \mathrm{e}^{\mathrm{i}(\xi_1 t_1 + \xi_2 t_2)}\, dt_1\, dt_2 \\ &\qquad = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t_1\boldsymbol{\theta_1} + t_2\boldsymbol{\theta_2} + t_3(\boldsymbol{\theta_1} \times \boldsymbol{\theta_2}))\, dt_3\, \mathrm{e}^{\mathrm{i}(\xi_1 t_1 + \xi_2 t_2)}\, dt_1\, dt_2.\end{aligned}$$

Let $\boldsymbol{x} = t_1\boldsymbol{\theta_1} + t_2\boldsymbol{\theta_2} + t_3(\boldsymbol{\theta_1} \times \boldsymbol{\theta_2})$, $t_1 = \boldsymbol{x}\cdot\boldsymbol{\theta_1}$, $t_2 = \boldsymbol{x}\cdot\boldsymbol{\theta_2}$, $d\boldsymbol{x} = dt_1\, dt_2\, dt_3$, hence 26becomes

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) \mathrm{e}^{\mathrm{i}\xi_1 t_1} \mathrm{e}^{\mathrm{i}\xi_2 t_2}\, dt_1\, dt_2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot(\xi_1\boldsymbol{\theta_1} + \xi_2\boldsymbol{\theta_2})}\, d\boldsymbol{x}. \tag{27}$$

Taking the Fourier transform of both sides with respect to ξ_1 and ξ_2, we have

$$\mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) = \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot(\xi_1\boldsymbol{\theta_1} + \xi_2\boldsymbol{\theta_2})}\, d\boldsymbol{x} \mathrm{e}^{-\mathrm{i}\xi_1 t_1} \mathrm{e}^{-\mathrm{i}\xi_2 t_2}\, d\xi_1\ d\xi_2. \tag{28}$$

□

COROLLARY 3.2. *Let $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$ and $\mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2)$ be given as in Lemma 3.1. Then,*

$$\int_{\mathbb{R}^2} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) e^{i(t_1\rho_1 + t_2\rho_2)} dt_1\, dt_2 = \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot(\rho_1\boldsymbol{\theta_1} + \rho_2\boldsymbol{\theta_2})}\, d\boldsymbol{x}. \tag{29}$$

If there is there is $\rho_1, \rho_2 \in \mathbb{R}$ such that $\boldsymbol{\omega}(\rho_1, \rho_2, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = \rho_1\boldsymbol{\theta_1} + \rho_2\boldsymbol{\theta_2}$ then

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^5} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) e^{i(t_1\rho_1 + t_2\rho_2)} dt_1 dt_2\, e^{-i\boldsymbol{\omega}\cdot\boldsymbol{y}} d\boldsymbol{\omega}. \tag{30}$$

Proof. Taking the integral of both sides of (26), we obtain

$$\int_{\mathbb{R}^2} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) e^{i(t_1\rho_1 + t_2\rho_2)} dt_1\, dt_2 \tag{31}$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot(\xi_1\boldsymbol{\theta_1} + \xi_2\boldsymbol{\theta_2})}\, d\boldsymbol{x} e^{-it_1(\xi_1-\rho_1)+t_2(\xi_2-\rho_2)}\, dt_1\, dt_2\, d\xi_1\ d\xi_2 \tag{32}$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot(\xi_1\boldsymbol{\theta_1} + \xi_2\boldsymbol{\theta_2})}\, d\boldsymbol{x}\, \delta(\xi_1 - \rho_1)\delta(\xi_2 - \rho_2)\, d\xi_1\, d\xi_2 \tag{33}$$

$$= \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot(\rho_1\boldsymbol{\theta_1} + \rho_2\boldsymbol{\theta_2})}\, d\boldsymbol{x}. \tag{34}$$

This is a well-known Fourier relation of X-ray transform (Theorem 1.1 in,[16] Section 7(b) in[17]). Thus, (30) is a direct consequence of the inverse Fourier transform. Denote $\boldsymbol{\omega} = \rho_1\boldsymbol{\theta_1} + \rho_2\boldsymbol{\theta_2}$. If $\boldsymbol{\omega}$ spans $\mathbb{R}^3$, then

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \mathbf{P}f(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, t_1, t_2) e^{i(t_1\rho_1 + t_2\rho_2)} e^{-i\boldsymbol{y}\cdot\boldsymbol{\omega}} dt_1\, dt_2 d\boldsymbol{\omega} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\boldsymbol{x}) e^{i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\omega}}\, d\boldsymbol{x}\ d\boldsymbol{\omega} \tag{35}$$

$$= \int_{\mathbb{R}^3} f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{x} = f(\boldsymbol{y}). \tag{36}$$

□

4. INVERSION OF CONE-BEAM PROJECTIONS

Recall that

$$\mathcal{P}f(\alpha, \beta, \gamma) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma) - t_2(\beta,\gamma))\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma) - t_3(\alpha,\beta,\gamma))\ d\boldsymbol{x} \tag{37}$$

$$= \mathbf{P}f(\boldsymbol{\theta_2}(\beta,\gamma), \boldsymbol{\theta_3}(\alpha,\beta,\gamma), t_2(\beta,\gamma), t_3(\alpha,\beta,\gamma)). \tag{38}$$

Now we can recover f from $\mathcal{P}f(\alpha,\beta,\gamma)$ using Corollary 3.2.

COROLLARY 4.1. *Let $\boldsymbol{\theta_2}(\beta,\gamma)$, $\boldsymbol{\theta_3}(\alpha,\beta,\gamma)$, $t_2(\beta,\gamma)$, and $t_3(\beta,\gamma)$ be given as in* (14) *and define $\boldsymbol{\omega}$ by*

$$\boldsymbol{\omega}(\rho_2, \rho_3, \alpha, \beta, \gamma) = (\omega_1, \omega_2, \omega_3)^T = \rho_2\boldsymbol{\theta_2}(\beta,\gamma) + \rho_3\boldsymbol{\theta_3}(\alpha,\beta,\gamma), \quad \rho_2,\ \rho_3 \in \mathbb{R}. \tag{39}$$

Then, by (20)*, if f is an integrable function supported within Ω,*

$$\Omega = \{\boldsymbol{x}|\boldsymbol{x} = t_2(\beta,\gamma)\boldsymbol{\theta_2}(\beta,\gamma) + t_3(\alpha,\beta,\gamma)\boldsymbol{\theta_3}(\alpha,\beta,\gamma),\ \alpha \in (-\alpha_0,\alpha_0),\ \beta \in (-\beta_0,\beta_0),\ \gamma \in (\gamma_{\min},\gamma_{\max})\}, \tag{40}$$

and $\boldsymbol{\omega}(\rho_2,\rho_3,\alpha,\beta,\gamma)$ spans $\mathbb{R}^3$, then f can be exactly reconstructed as follows:

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^3} \int_{\gamma_{\min}}^{\gamma_{\max}} \int_{-\beta_0}^{\beta_0} \int_{-\alpha_0}^{\alpha_0} \int_{\mathbb{R}^2} \mathcal{P}f(\alpha,\beta,\gamma) e^{it_2(\beta,\gamma)\rho_2} e^{it_3(\alpha,\beta,\gamma)\rho_3} e^{-i\boldsymbol{\omega}(\rho_2,\rho_3,\alpha,\beta,\gamma)\cdot\boldsymbol{y}} \left| \frac{\partial(\omega_1,\omega_2,\omega_3,t_2,t_3)}{\partial(\rho_2,\rho_3,\alpha,\beta,\gamma)} \right| d\rho_2\, d\rho_3\, d\alpha\, d\beta\, d\gamma. \tag{41}$$

Proof. This result is a direct consequence of Corollary 3.2 and change of variables.

Since f is supported within Ω and

$$\mathcal{P}f(\alpha,\beta,\gamma) = \mathbf{P}f(\boldsymbol{\theta_2}(\beta,\gamma), \boldsymbol{\theta_3}(\alpha,\beta,\gamma), t_2(\beta,\gamma), t_3(\alpha,\beta,\gamma)), \tag{42}$$

the integration with respect to t_2, t_3 can be considered as an integration over $\mathbb{R}^2$ and then by change of variables (30) can be written as

$$
\begin{aligned}
f(\boldsymbol{y}) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\mathbf{P}f(\boldsymbol{\theta_2},\boldsymbol{\theta_3},t_2,t_3)\mathrm{e}^{\mathrm{i}(t_2\rho_2+t_3\rho_3)}\mathrm{e}^{-\mathrm{i}\boldsymbol{y}\cdot\boldsymbol{\omega}}dt_2\,dt_3 d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^3}\int_{\gamma_{\min}}^{\gamma_{\max}}\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0}\int_{\mathbb{R}^2}\mathcal{P}f(\alpha,\beta,\gamma)\mathrm{e}^{\mathrm{i}(t_2(\beta,\gamma)\rho_2+t_3(\alpha,\beta,\gamma)\rho_3-\boldsymbol{y}\cdot\boldsymbol{\omega}(\rho_2,\rho_3,\alpha,\beta,\gamma))} \\
&\qquad\times\left|\frac{\partial(\omega_1,\omega_2,\omega_3,t_2,t_3)}{\partial(\rho_2,\rho_3,\alpha,\beta,\gamma)}\right| d\rho_2\,d\rho_3\,d\alpha\,d\beta\,d\gamma, \qquad (43)
\end{aligned}
$$

where $\left|\frac{\partial(\omega_1,\omega_2,\omega_3,t_2,t_3)}{\partial(\rho_2,\rho_3,\alpha,\beta,\gamma)}\right|$ is the absolute value of the determinant of the Jacobian.

□

Equation 41 gives a weighted backprojection reconstruction formula where the weights are determined by the integral

$$
\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}t_2(\beta,\gamma)\rho_2}\mathrm{e}^{\mathrm{i}t_3(\alpha,\beta,\gamma)\rho_3}\ \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}(\rho_2,\rho_3,\alpha,\beta,\gamma)\cdot\boldsymbol{y}}\left|\frac{\partial(\omega_1,\omega_2,\omega_3,t_2,t_3)}{\partial(\rho_2,\rho_3,\alpha,\beta,\gamma)}\right| d\rho_2\,d\rho_3. \qquad (44)
$$

The condition $\boldsymbol{h}(\gamma)\in C^1(\mathbb{R})$ is required in order $\left|\frac{\partial(\omega_1,\omega_2,\omega_3,t_2,t_3)}{\partial(\rho_2,\rho_3,\alpha,\beta,\gamma)}\right|$ to be well-defined. Corollary 4.1 provides a support condition between the function and the source trajectory. To reconstruct a function f given $\mathcal{P}f$ over the source trajectory $\boldsymbol{h}(\gamma)$, f should be supported within Ω defined as in Corollary 4.1 and $\boldsymbol{h}(\gamma)$ should be $C^1(\mathbb{R})$. Conversely, in order to reconstruct a function f with known support supp$f$, $\boldsymbol{h}(\gamma)$ must be C^1 and for all $\boldsymbol{x}\in$ supp$f$ and α, β , there must exists $\gamma\in\mathbb{R}$ such that $\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta)=t_2(\beta,\gamma)$ and $\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta)=t_3(\alpha,\beta,\gamma)$.

5. SPIRAL SOURCE TRAJECTORY

A spiral source trajectory is given by $\boldsymbol{h}(\gamma)=-aR_3(\gamma)\boldsymbol{e_1}+h(\gamma)\boldsymbol{e_3}$, where $a\in\mathbb{R}^+$ is the radius of the spiral, and $h(\gamma)\in\mathbb{R}$ is an increasing function governing the escalation of the spiral. The difference between $h(\gamma)$ and $h(\gamma+2\pi)$ is called the *pitch*. If the pitch changes with γ then the spiral is said to have a dynamic pitch. In practice, $h(\gamma)$ is usually chosen as $h(\gamma)=h\frac{\gamma}{2\pi}$, for some fixed $h>0$. In our case, we assume the pitch is fixed and is equal to h. For the rest of this Section, we will assume that $h(\gamma)$ is $h\frac{\gamma}{2\pi}$.

Due to the instrumental design, the relative orientation of the detector plane with respect to source is given by $R_D(\gamma)=R_3(\gamma)$. Then, the cone-beam projections over $\boldsymbol{h}(\gamma)$ are given by

$$
\mathcal{P}f(\alpha,\beta,\gamma)=\int_{\mathbb{R}^3}f(\boldsymbol{x})\,\delta(R_2^{-1}(\alpha)R_3^{-1}(\beta)(R_3^{-1}(\gamma)\boldsymbol{x}+a\boldsymbol{e_1}-h(\gamma)\boldsymbol{e_3})\cdot[\boldsymbol{e_2}\ \boldsymbol{e_3}])\,d\boldsymbol{x}. \qquad (45)
$$

Let K denote the kernel of the cone-beam transform for the spiral trajectory given in Equation 45. Using the definition of $\delta(\boldsymbol{x}\cdot[\boldsymbol{u}\ \boldsymbol{v}])$, K can be factored out as

$$
K(\boldsymbol{x};\alpha,\beta,\gamma)=K_2(\boldsymbol{x};\alpha,\beta,\gamma)K_3(\boldsymbol{x};\alpha,\beta,\gamma), \qquad (46)
$$

where

$$
\begin{aligned}
K_2(\boldsymbol{x};\alpha,\beta,\gamma)&=\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_2}(\beta,\gamma)-t_2(\beta,\gamma)) \qquad (47)\\
K_3(\boldsymbol{x};\alpha,\beta,\gamma)&=\delta(\boldsymbol{x}\cdot\boldsymbol{\theta_3}(\alpha,\beta,\gamma)-t_3(\alpha,\beta,\gamma))
\end{aligned}
$$

with

$$
\boldsymbol{\theta_2}(\beta,\gamma)=R_3(\beta+\gamma)\boldsymbol{e_2} \qquad (48)
$$

$$
t_2(\beta,\gamma)=a\sin\beta \qquad (49)
$$

$$
\boldsymbol{\theta_3}(\alpha,\beta,\gamma)=R_3(\beta+\gamma)R_2(\alpha)\boldsymbol{e_3} \qquad (50)
$$

$$
t_3(\alpha,\beta,\gamma)=-a\cos\beta\sin\alpha+h\frac{\gamma}{2\pi}\cos\alpha. \qquad (51)
$$

Then, $\boldsymbol{\omega} = \rho_2 \boldsymbol{\theta_2} + \rho_3 \boldsymbol{\theta_3}$ is given by

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \rho_2 \begin{bmatrix} -\sin(\beta+\gamma) \\ \cos(\beta+\gamma) \\ 0 \end{bmatrix} + \rho_3 \begin{bmatrix} \cos(\beta+\gamma)\sin\alpha \\ \sin(\beta+\gamma)\sin\alpha \\ \cos\alpha \end{bmatrix}. \tag{52}$$

Determinant of the Jacobian is computed to be

$$\left| \frac{\partial(\omega_1, \omega_2, \omega_3, t_2, t_3)}{\partial(\rho_2, \rho_3, \alpha, \beta, \gamma)} \right| = \left| \frac{a\cos\beta\cos\alpha\,(-\rho_3\,h + 2\pi\,a\cos\beta\cos\alpha\,\rho_2 + h\gamma\,\sin\alpha\,\rho_2)}{2\pi} \right|. \tag{53}$$

Note that, if f is supported within a cylinder of radius a_0, $0 < a_0 < a$, in order to reconstruct f, β_0 should at least be equal to $\arcsin\left(\frac{a_0}{a}\right) < \pi/2$, with $\beta_0 \in (0, \pi/2)$, or conversely a_0 should be less than $a\sin\beta_0$. We will assume that this is the case, which makes both $\cos\alpha$ and $\cos\beta$ positive. Thus, the weight function takes the form

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}t_2(\beta,\gamma)\rho_2}\mathrm{e}^{\mathrm{i}t_3(\alpha,\beta,\gamma)\rho_3}\ \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}(\rho_2,\rho_3,\alpha,\beta,\gamma)\cdot\boldsymbol{y}} \left| \frac{a\cos\beta\cos\alpha\,(-\rho_3\,h + 2\pi\,a\cos\beta\cos\alpha\,\rho_2 + h\gamma\,\sin\alpha\,\rho_2)}{2\pi} \right| d\rho_2\,d\rho_3 \\
&= a\cos\beta\cos\alpha \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\rho_2(t_2(\beta,\gamma)-\boldsymbol{y}\cdot\boldsymbol{\theta_2})}\mathrm{e}^{\mathrm{i}\rho_3(t_3(\alpha,\beta,\gamma)-\boldsymbol{y}\cdot\boldsymbol{\theta_3})} \left| \rho_3\,\frac{h}{2\pi} + \partial_\alpha t_3(\alpha,\beta,\gamma)\rho_2 \right| d\rho_2\,d\rho_3 \\
&= a\cos\beta\cos\alpha \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho_2(t_2(\beta,\gamma)-2\pi\partial_\alpha t_3(\alpha,\beta,\gamma)[t_3(\alpha,\beta,\gamma)-\boldsymbol{y}\cdot\boldsymbol{\theta_3}]/h-\boldsymbol{y}\cdot\boldsymbol{\theta_2})} d\rho_2 \frac{h}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho_3(t_3(\alpha,\beta,\gamma)-\boldsymbol{y}\cdot\boldsymbol{\theta_3})}\ |\rho_3|\ d\rho_3 \\
&= a2\pi\cos\beta\cos\alpha\,\delta(t_2(\beta,\gamma) - 2\pi\partial_\alpha t_3(\alpha,\beta,\gamma)[t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3}]/h - \boldsymbol{y}\cdot\boldsymbol{\theta_2}) \frac{h}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho_3(t_3(\alpha,\beta,\gamma)-\boldsymbol{y}\cdot\boldsymbol{\theta_3})}\ |\rho_3|\ d\rho_3 \\
&= -a2h\cos\beta\cos\alpha\,\frac{\delta(t_2(\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_2} - 2\pi\partial_\alpha t_3(\alpha,\beta,\gamma)[t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3}]/h)}{(t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3})^2}
\end{aligned} \tag{54}$$

We have used the identities[18]

$$\delta(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\rho} d\rho, \tag{55}$$

and

$$\frac{1}{t^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\rho}(-\pi)|\rho| d\rho. \tag{56}$$

Note that the argument of the Dirac delta function in (54) can be zero provided that the support of the function satisfies the condition of Corollary 4.1, i.e.

$$\mathrm{supp} f \subset \Omega \Longrightarrow \{\alpha, \beta, \gamma \mid t_2(\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_2} - 2\pi\partial_\alpha t_3(\alpha,\beta,\gamma)[t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3}]/h = 0\} \neq \emptyset. \tag{57}$$

Thus, f can be reconstructed by

$$\begin{aligned}
f(\boldsymbol{y}) = \frac{ah}{4\pi^3} \int_{-\gamma_{\min}}^{\gamma_{\max}} \int_{-\beta_0}^{\alpha_0} \int_{-\alpha_0}^{\alpha_0} \mathcal{P}f(\alpha,\beta,\gamma) \frac{\cos\alpha\cos\beta}{(t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3})^2} \\
\times\,\delta\left(t_2(\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_2} - 2\pi\partial_\alpha t_3(\alpha,\beta,\gamma)[t_3(\alpha,\beta,\gamma) - \boldsymbol{y}\cdot\boldsymbol{\theta_3}]/h\right) d\alpha d\beta d\gamma.
\end{aligned} \tag{58}$$

6. CONCLUSION

We have presented a weighted backprojection reconstruction formula for the cone-beam transform (Equation 41). The weights were determined by the source trajectory and detector orientation. As a special case, we worked out the corresponding explicit inversion formula for the spiral trajectory (Equation 58). In our future work, we will investigate the relationship between our formula and other reconstruction formulas[3,5,13] for spiral and arbitrary source trajectories.

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