ECSE-4730: Computer Communication Networks (CCN)

Network Layer Performance Modeling & Analysis

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Rensselaer

RSVP

- ***Network Layer Performance Modeling & Analysis***
  - Part I: Essentials of Probability
  - Part II: Inside a Router
  - Part III: Network Analysis

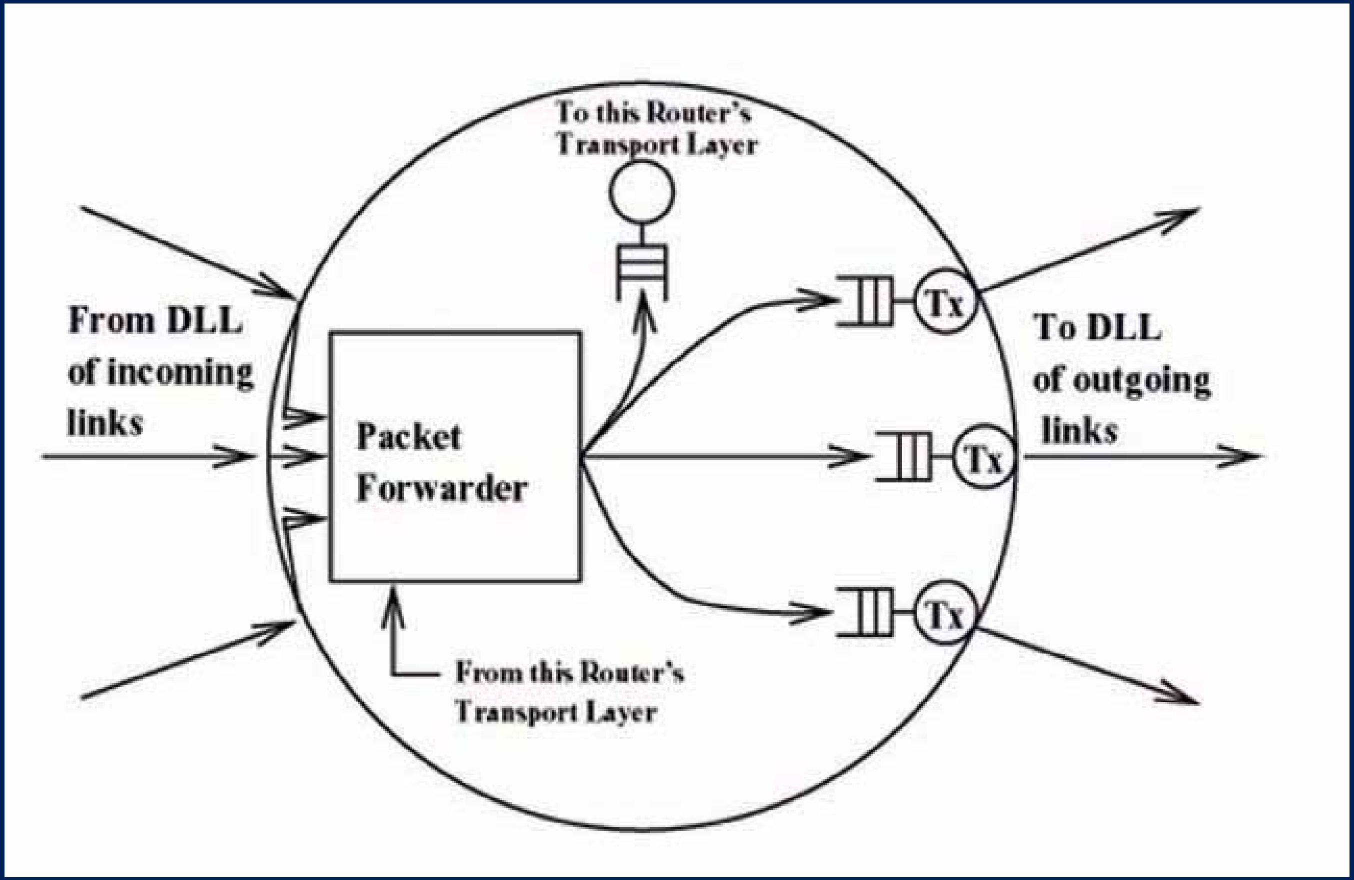
Network Layer Performance Modeling & Analysis: Part I Essential of Probability

- Motivation
- Basic Definitions
- Modeling Experiments with Uncertainty
- Random Variables: Geometric, Poisson, Exponential

Network Layer Performance Modeling & Analysis: Part I Essential of Probability

- Read any of the probability references, e.g. Ross, Molloy, Papoulis, Stark and Wood
- Check out the WWW version of the notes: ***http://networks.ecse.rpi.edu/~vastola/pslinks/perf/node1.html***

Motivation for learning Probability in CCN



Basic Definitions

- Think of Probability as modeling an experiment
- The Set of all possible outcomes is the *sample space*: S
- Classic "Experiment": Tossing a die:
  S = {1,2,3,4,5,6}
  - Any subset A of S is an *event*: A = {*the outcome is even*} = {2,4,6}

Basic Operation of Events

- **For any two events A, B the following are also events:**

$\bar{A} =$ A complement = {all possible outcomes not in A}

$A \cup B =$ A union B = {all outcomes in A or B or both}

$A \cap B =$ A intersect B = {all outcomes in A and B}

- **Note $\bar{S} = \varnothing$, the empty set.**
- **If AB $= \varnothing$, then A and B are mutually exclusive.**

Basic Operation of Events

- Can take many unions:

$$A_1 \cup A_2 \cup \ldots \cup A_n$$

- Or even infinite unions:

$$A_1 \cup A_2 \cup \ldots \bigcup_{n=1}^{\infty} A_n$$

- Ditto for intersections

Probability of Events

- P is the Probability Mass function if it maps each event A, into a real number $P(A)$, and:
  - i.) $P(A) \geq 0$ for every event $A \subseteq S$
  - ii.) $P(S) = 1$
  - iii.) If A and B are mutually exclusive events then, $P(A \cup B) = P(A) + P(B)$

Probability of Events

…In fact for any sequence of pair-wise-mutually-exclusive events $A_1, A_2, A_3, \ldots$ (i.e. $A_i A_j = 0$ for any $i \neq j$) **we have**

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Other Properties

- $P(\overline{A}) = 1 - P(A)$
- $P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(AB)$
- $A \subseteq B \Rightarrow P(A) \leq P(B)$

Conditional Probability

- $P(A|B)$ **= (conditional) probability that the outcome is in *A* <u>given</u> that we know the outcome in *B***

$$P(A|B) = \frac{P(AB)}{P(B)} \qquad P(B) \neq 0$$

- **Example: Toss one die.**

$$P(i = 3 \,|\, \text{i is odd}) =$$

- **Note that:** $P(AB) = P(B)P(A|B) = P(A)P(B|A)$

Independence

- Events A and B are <u>independent</u> if $P(AB) = P(A)P(B)$.
- <u>Example</u>: A card is selected at random from an ordinary deck of cards. A=event that the card is an ace. B=event that the card is a diamond.

$$P(AB) =$$

$$P(A) = \qquad\qquad P(B) =$$

$$P(A)P(B) =$$

Independence

- **Event *A* and *B* are <u>independent</u> if** $P(AB) = P(A)\,P(B)$.
- **Independence does NOT mean that *A* and *B* have "nothing to do with each other" or that *A* and *B* "having nothing in common".**

Independence

- **Best intuition on independence is:**

  ***A* and *B* are independent if and only if** $P(A \mid B) = P(A)$ **(equivalently,** $P(B \mid A) = P(B)$**)**

  i.e. if and only if knowing that *B* is true doesn't change the probability that *A* is true.

- **Note: If *A* and *B* are independent <u>and</u> mutually exclusive, then** $P(A)=0$ **or** $P(B) = 0$.

Random Variables

- A <u>random variable</u> *X* maps each outcome *s* in the sample space *S* to a real number *X(s)*.
- <u>Example:</u> A fair coin is tossed 3 times.

S={(TTT),(TTH),(THT),(HTT),(HHT),(HTH),(THH),(HHH)}

Random Variables

- Let X be the number of heads tossed in 3 tries.

$X(TTT)=$	$X(HHT)=$
$X(TTH)=$	$X(HTH)=$
$X(THT)=$	$X(THH)=$
$X(HTT)=$	$X(HHH)=$

- So $P(X=0)=$ $P(X=1)=$
  $P(X=2)=$ $P(X=3)=$

Random Variable as a Measurement

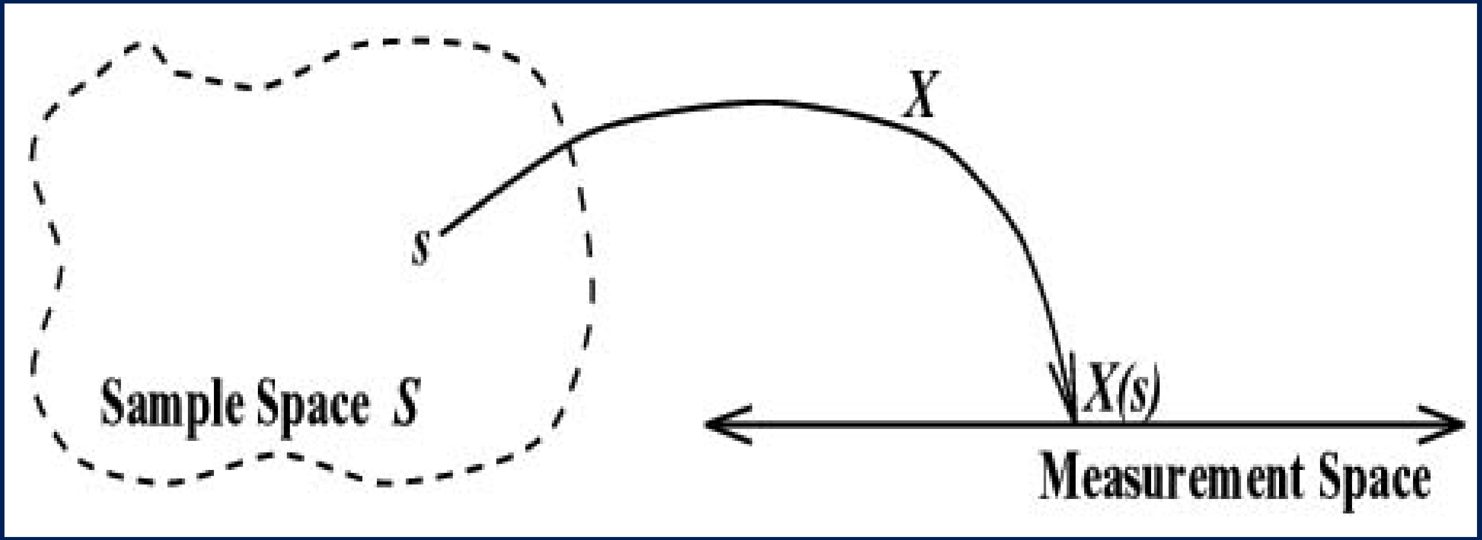
- Think of much more complicated experiments
  - A chemical reaction.
  - A laser emitting photons.
  - A packet arriving at a router.

Random Variable as a Measurement

- We cannot give an exact description of a sample space in these cases, but we can still describe specific measurements on them
  - The temperature change produced.
  - The number of photons emitted in one millisecond.
  - The time of arrival of the packet.

Random Variable as a Measurement

- Thus a random variable can be thought of as a measurement on an experiment



Probability Mass Function for a Random Variable

- The probability mass function (PMF) for a (discrete valued) random variable *X* is:

$$P_X(x) = P(X = x) = P(\{s \in S \mid X(s) = x\})$$

- Note that $P_X(x) \geq 0$ for $-\infty < X < \infty$
- Also for a (discrete valued) random variable *X*

$$\sum_{x=-\infty}^{\infty} P_X(x) = 1$$

Cumulative Distribution Function

- The <u>cumulative distribution function</u> (CDF) for a random variable *X* is

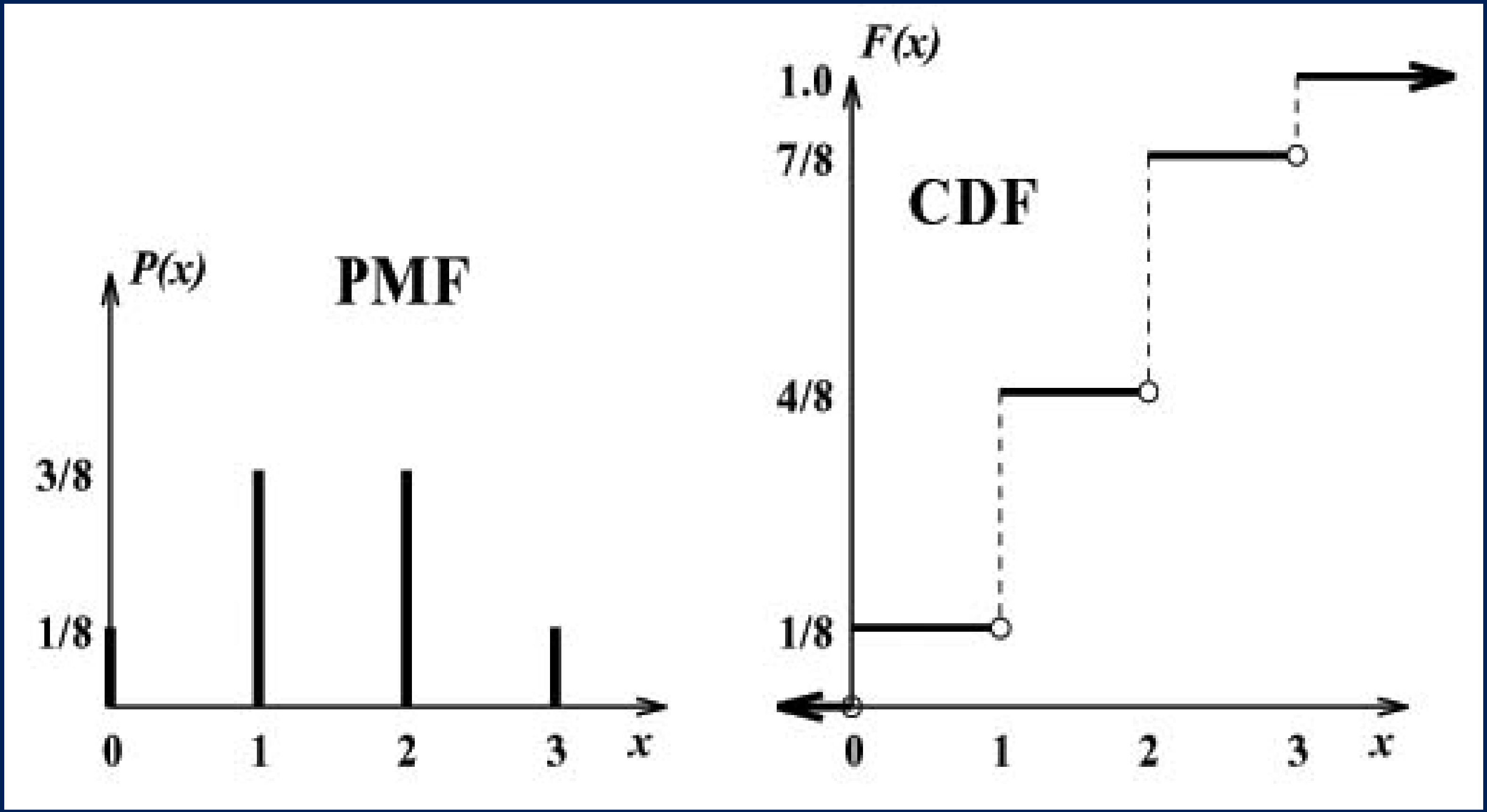
$$F_X(x) = P(X \leq x) = P(\{s \in S \mid X(s) \leq x\})$$

- Note that $F_X(x)$ is non-decreasing in *x*, i.e.

$$x_1 \leq x_2 \Rightarrow F_x(x_1) \leq F_x(x_2)$$

- Also $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to \infty} F_x(x) = 1$

PMF and CDF for the 3 Coin Toss Example



Expectation of a Random Variable

- The <u>expectation</u> (average) of a (discrete-valued) random variable *X* is

$$\overline{X} = E(X) = \sum_{x=-\infty}^{\infty} xP(X = x) = \sum_{-\infty}^{\infty} xP_X(x)$$

- Three coins example:

$$E(X) = \sum_{x=0}^{3} xP_X(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

Important Random Variables: Bernoulli

- The simplest possible measurement on an experiment:

  Success (*X = 1*) or failure (*X = 0*).

- Usual notation:

$$P_X(1) = P(X = 1) = p \qquad P_X(0) = P(X = 0) = 1 - p$$

- *E(X)*=

Important Random Variables: <u>Binomial</u>

- **Let X = the number of success in n independent Bernoulli experiments (or trials).**

  $P(X=0) =$

  $P(X=1) =$

  $P(X=2)=$

- **In general,** $P(X = x) =$

Important Random Variables: Binomial

- **Exercise: Show that**

$$\sum_{x=0}^{n} P_X(x) = 1 \quad \textbf{and} \quad E(X) = np$$

Important Random Variables: Geometric

- **Let X = the number of independent Bernoulli trials until the first success.**

  $P(X=1) = p$

  $P(X=2) = (1-p)p$

  $P(X=3) = (1-p)^2p$

- **In general,**

$$P(X = x) = (1 - p)^{x-1} p \qquad \text{for } x = 1,2,3,\ldots$$

Important Random Variables: Geometric

- **Exercise: Show that**

$$\sum_{x=1}^{\infty} P_X(x) = 1 \quad \text{and} \quad E(x) = \frac{1}{p}$$

Important Random Variables: Poisson

- A Poisson random variable X is defined by its PMF:

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

Where λ > 0 is a constant

- Exercise: Show that

$$\sum_{x=0}^{\infty} P_X(x) = 1$$ and $E(X) = \lambda$

Important Random Variables: Poisson

- Poisson random variables are good for counting things like the number of customers that arrive to a bank in one hour, or the number of packets that arrive to a router in one second.

Continuous-valued Random Variables

- **So far we have focused on discrete(-valued) random variables, e.g. *X(s)* must be an integer**
- **Examples of discrete random variables: <u>number</u> of arrivals in one second, <u>number</u> of attempts until sucess**

Continuous-valued Random Variables

- A continuous-valued random variable takes on a range of real values, e.g. $X(s)$ ranges from 0 to ∞ as s varies.
- Examples of continuous(-valued) random variables: time when a particular arrival occurs, time between consecutive arrivals.

Continuous-valued Random Variables

- A discrete random variable has a "staircase" CDF.
- A continuous random variable has (some) continuous slope to its CDF.

Continuous-valued Random Variables

- Thus, for a continuous random variable *X*, we can define its <u>probability density function</u> (pdf)

$$f_x(x) = F'_X(x) = \frac{dF_X(x)}{dx}$$

- Note that since $F_X(x)$ is non-decreasing in *x* we have

$$f_X(x) \geq 0 \quad \text{for all } x.$$

Properties of Continuous Random Variables

- From the Fundamental Theorem of Calculus, we have

$$F_X(x) = \int_{-\infty}^{x} f_x(x)dx$$

- In particular,

$$\int_{-\infty}^{\infty} fx(x)dx = F_X(\infty) = 1$$

- More generally,

$$\int_a^b f_X(x)dx = F_X(b) - F_X(a) = P(a < X \le b)$$

Expectation of a Continuous Random Variable

- The expectation (average) of a continuous random variable X is given by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Note that this is just the continuous equivalent of the discrete expectation

$$E(X) = \sum_{x=-\infty}^{\infty} x P_X(x)$$

Important Continuous Random Variable: Exponential

- Used to represent time, e.g. until the next arrival
- Has PDF $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

  for some $\lambda > 0$
- Show $\int_0^\infty f_X(x)dx = 1$ and $E(X) = \frac{1}{\lambda}$
  - Need to use integration by Parts!

Exponential Random Variable

- The CDF of an exponential random variable is:

$$F_X(x) = \int_0^x f_X(\bar{x})d\bar{x} = \int_0^x \lambda e^{-\lambda \bar{x}} d\bar{x}$$

$$= \left[-e^{-\lambda \bar{x}} \right]_0^x = 1 - e^{-\lambda x}$$

- So

$$P(X > x) = 1 - F_X(x) = e^{-\lambda x}$$

Memoryless Property of the Exponential

- An exponential random variable *X* has the property that “the future is independent of the part”, i.e. the fact that it hasn’t happened yet, tells us nothing about how much longer it will take.

$= e^{-\lambda s}$

- In math terms

$$P(X > s + t \mid X > t) = P(X > s) \qquad \text{for } s, t > 0$$

Memoryless Property of the Exponential

- **Proof:** $P(X > s+t \mid X > t) = \dfrac{P(X > s+t, X > t)}{P(X > t)}$

$$= \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \quad = e^{-\lambda s} = P(X > s)$$