Solutions to Probability Related Problems

1. Let n_1 and n_2 denote the outcome in the first and second toss respectively. The two successive tosses are independent. Thus $P[n_1, n_2] = P[n_1]P[n_2]$. If we have a 3 in the first toss, the only we can have a total of 7 from 2 tosses is if we have a 4 in the second toss. Thus

$$P[n_{1,2} = i] = \frac{1}{6} \qquad\qquad i = 1, 2, \cdots, 6$$

$$\begin{aligned} P[n_1 + n_2 = 7 \mid n_1 = 3] &= P[n_2 = 4 \mid n_1 = 3] \\ &= \frac{P[n_2 = 4]P[n_1 = 3]}{P[n_1 = 3]} \\ &= \frac{(1/6)(1/6)}{1/6} = \frac{1}{6} \end{aligned}$$

2.

$$\begin{aligned} P[knows \mid correct] &= \frac{P[knows \cdot correct]}{P[correct]} \\ &= \frac{P[correct \mid knows]P[knows]}{P[correct \mid knows]P[knows] + P[correct \mid guess]P[guess]} \\ &= \frac{1 \cdot p}{1 \cdot p + \frac{1}{m}(1-p)} = \frac{mp}{1 - p + mp} \end{aligned}$$

3. Let

$$P[A] = P[\text{randomly drawn chip is from machine A}] = 0.25$$
$$P[B] = P[\text{randomly drawn chip is from machine B}] = 0.35$$
$$P[C] = P[\text{randomly drawn chip is from machine C}] = 0.40$$

$$\begin{aligned} P[D] &= P[\text{randomly drawn chip is defective}] \\ &= P[D \mid A]P[A] + P[D \mid B]P[B] + P[D \mid C]P[C] \\ &= 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.40 \\ &= 0.0345 \end{aligned}$$

4. Let

$$P[A] = P[\text{a chip meets specs}] = 0.85$$
$$P[B] = P[\text{a chip needs rework}] = 0.10$$
$$P[C] = P[\text{a chip is discarded}] = 0.05$$

We use the multinomial distribution here.

(a)

$$P[\text{all chips meet specs}] = \frac{10!}{10!0!0!}(0.85)^{10}(0.10)^0(0.05)^0 = 0.197$$

(b)

$$\begin{aligned} P[\text{all chips meet specs}] &= 1 - P[\text{no discard}] - P[1 \text{ discard}] \\ &= 1 - \binom{10}{0}(0.95)^{10}(0.05)^0 - \binom{10}{1}(0.95)^9(0.05)^1 \\ &= 1 - 0.6 - 0.315 = 0.085 \end{aligned}$$

(c)

$$P[8 \text{ meet specs, } 1 \text{ needs work, } 1 \text{ discard}] = \frac{10!}{8!1!1!}(0.85)^8(0.10)^1(0.05)^1 = 0.122$$

5. If all tickets in one lottery, then

$$P[\text{win}] = \frac{50}{100} = 0.50$$

If one ticket in each of 50 lotteries, the the probability of win per lottery is $p = 1/100 = 0.01$. But we have 50 chances. Hence

$$\begin{aligned} P[\text{at least one win}] &= 1 - P[\text{no win}] \\ &= 1 - \binom{50}{0}(0.01)^0(0.99)^{50} \\ &= 0.395 < 0.5 \end{aligned}$$

6. For proper normalization

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Integrating the three functions (over the ranges $\{-\infty, \infty\}$, $\{0, \infty\}$ and $\{0, 1\}$ respectively for the three functions) and equating it to 1 will allow us to solve for B. We get

(a)

$$B = \frac{1}{\beta\pi}$$

(b)

$$B = \frac{4}{\alpha^3\sqrt{\pi}}$$

(c)

$$B = \frac{I(b+c+2)}{I(b+1)I(c+1)}$$

where

$$I(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1} d\theta$$

7. The successive tosses of the dies are independent. On any given toss, the probability that we get a 6 is $p = 1/6$ and the probability that we do not get a 6 is $1 - p = 5/6$. Thus we need only one toss with probability p. We need two tosses if we fail the first time and succeed the second time (which happens with probability $(1-p)p$). Three tosses are required if we fail the first two times and succeed in the third try ((which happens with probability $(1-p)^2 p$) and so on. We thus have a geometric distribution with parameter p and thus the expected number of tosses is given by the expected values of this geometric distribution which is

$$E[X] == \frac{1}{p} = 6$$

8. To find the mean, evaluate the integral

$$\int_{-\infty}^{\infty} x f(x) dx$$

and show that this is infinity when you put in the limits.

9. The arrival time for the Professor is uniformly distributed over the 60 minute interval. Thus the density and distribution functions are

$$f(t) = \begin{cases} \frac{1}{60} & 0 \le t \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{60} & 0 \le t \le 60 \\ 1 & t > 60 \end{cases}$$

$$P[A] = P[T > 30] = 1 - F(30) = \frac{30}{60}$$

$$P[B] = P[T \le 31] = F(31) = \frac{31}{60}$$

$$P[AB] = P[30 < T \le 31] = F(31) - F(30) = \frac{31-30}{60} = \frac{1}{60}$$

$$P[B \mid A] = \frac{P[AB]}{P[A]} = \frac{1/60}{30/60} = \frac{1}{30}$$
$$P[A \mid B] = \frac{P[AB]}{P[B]} = \frac{1/60}{31/60} = \frac{1}{31}$$

10.

$$n_A = 3n_B \qquad\qquad n_B = 2n_C$$

$$n = n_A + n_B + n_C = 9n_C$$

$$P[C] = \frac{n_C}{n} = \frac{1}{9}$$
$$P[B] = \frac{n_B}{n} = \frac{2}{9}$$
$$P[A] = \frac{n_A}{n} = \frac{6}{9}$$

$$\begin{aligned}
E[X] &= E[X \mid A]P[A] + E[X \mid B]P[B] + E[X \mid C]P[C] \\
&= \frac{6}{9}\left(\frac{1}{5}\int_0^\infty xe^{-x/5}dx\right) + \frac{2}{9}\left(\frac{1}{6.5}\int_0^\infty xe^{-x/6.5}dx\right) + \frac{1}{9}\left(\frac{1}{10}\int_0^\infty xe^{-x/10}dx\right) \\
&= \frac{6}{9}5 + \frac{2}{9}6.5 + \frac{1}{9}10 \\
&= 5.9 years
\end{aligned}$$