Capacity of Wireless Channels –
A brief discussion of some of the point-to-point capacity results and their design implications

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Slides based on Tse & Viswanath textbook. Contents based on the same, and Yeung's textbook.

Information Theory & Wireless Comm

- So far we have looked at specific communication schemes (BPSK, QAM, etc.).
- What is the optimal performance achievable on a given channel?
- Information theory provides a fundamental limit to (coded) performance.
- It succinctly identifies the impact of channel resources on performance as well as suggests new and innovative ways to communicate over the wireless channel.
- It provides the basis for the modern development of wireless communication.
- If interested in this topic, take full course on Information Theory by Prof. John Woods in Fall 2007.
- This lecture we only focus on the intuitive meaning of some of the capacity results and their implications not the derivation of the results.
- We focus here on point-to-point case. Multiuser case is also addressed by information theory.

Capacity of Wireless Channels

- Information theory was invented by Calude Shannon in 1948 to characterize the limits of reliable communication.
- Prior to Shannon, it was widely believed that the only way to achieve reliable communication over a noisy channel was to reduce the data rate.
- By reliable communication, we mean that we want to make the error probability as small as desired.
- Shannon showed that this belief is incorrect.
- Shannon: By more intelligent coding of the information, one can communicate at a strictly positive rate but at the same time with as small an error probability as desired.
- However, there is a limit to how high that rate can be: beyond a certain rate, called capacity, it is impossible to drive the error probability to zero.
- All the capacity results described here can be derived from this general theory.
- We focus on the AWGN channel and channels closely related to it (e.g. fading channel).

AWGN Channel

$$y[m] = x[m] + w[m]$$

where x[m] and y[m] are real input/output at time *m* and *w[m]* is $N(0, \sigma^2)$ Noise

- **Repetition coding**
  - Using uncoded BPSK symbols $x[m] = \pm\sqrt{P}$, the error probability is

$$Q(\sqrt{P/\sigma^2})$$

  - To reduce error probability, repeat the same symbol N times to transmit a one bit information; called repetition code of block length N.
  - Each block length has total power constraint P joules/symbol.
  - Can show that the probability is now reduced to

$$Q(\sqrt{NP/\sigma^2})$$

  - Can choose error probability as small as needed by increasing N
  - But, the data rate is only 1/N bits per symbol time.

- **Sphere Packing**
  - Repetition coding is an inefficient way of coding since it uses only two dimensions of signal space.
  - A more efficient coding should spread the codewords in all N dimensions.
  - What is the maximum number of codewords that can be packed in the signal space for a given power constraint P?

  [→Check notes for SKETCH of Sphere packing example.]
  - Only puts a bound on the max number of bits per symbol reliably communicated – does not give an achievability result.
  - Shannon also showed that a certain code, called iid Guassian code, constructed randomly, achieves any desired rate R with high probability as long as R<C where C is the upper bound just derived, hence also proving that C is the capacity.
  - Appendix B.5 in Tse and Viswanath gives a more complete and precise sphere packing argument.
  - Capacity-achieving AWGN codes have been found and implemented e.g. LDPC does.

Capacity of AWGN Channel

Capacity of AWGN channel

$$
\begin{aligned}
C_{\text{awgn}} &= \log(1+\text{SNR}) \quad \text{bits/s/Hz} \\
&= W\log(1+\text{SNR}) \quad \text{bits/s}
\end{aligned}
$$

If average transmit power constraint is $\bar{P}$ watts and noise psd is N_0 watts/Hz,

$$
C_{\text{awgn}} = W\log\left(1+\frac{\bar{P}}{N_0 W}\right) \quad \text{bits/s.}
$$

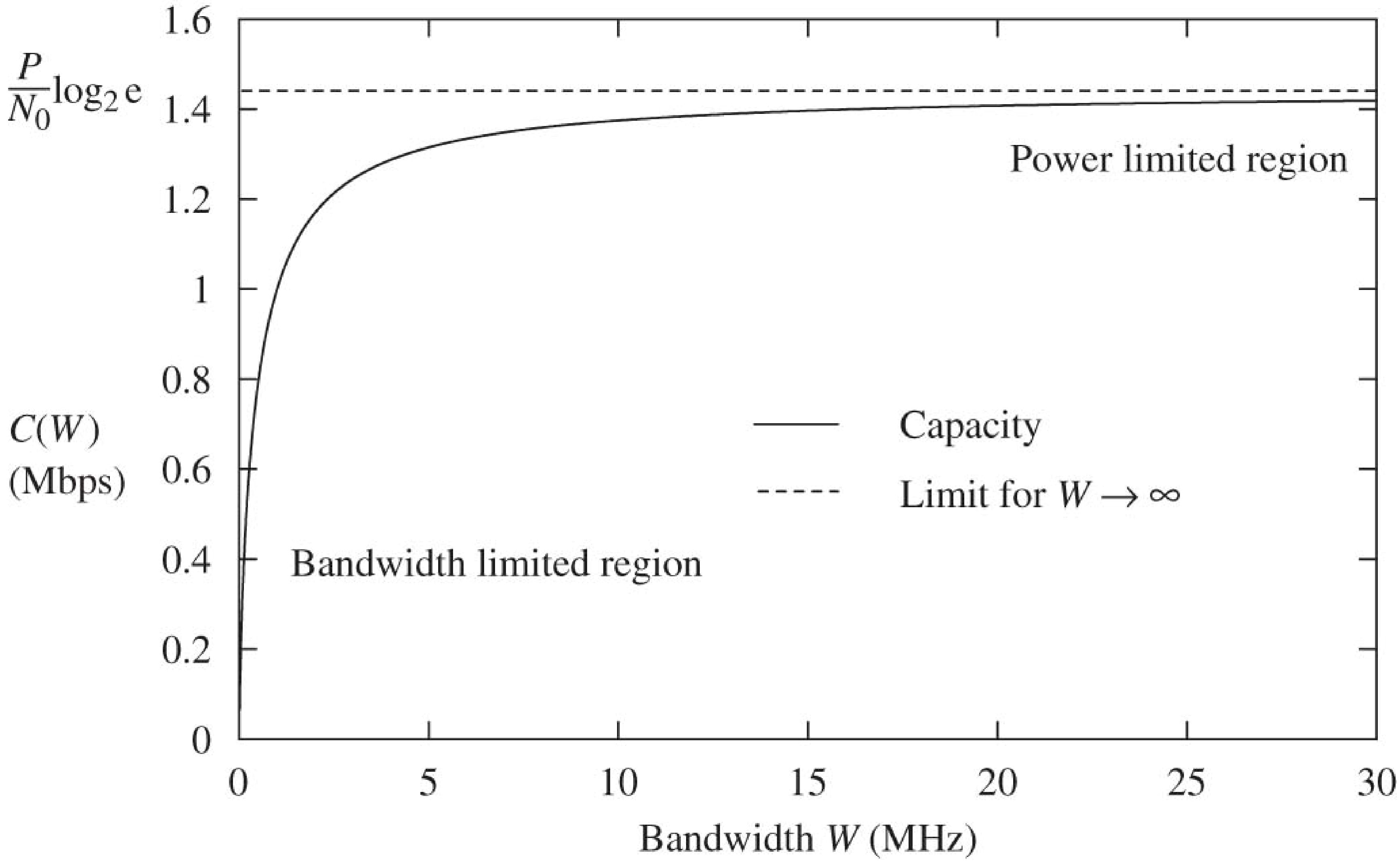
Power and Bandwidth Limited Regimes

$$C_{\mathsf{awgn}} = W \log\left(1 + \frac{\bar{P}}{N_0 W}\right)$$

$$\mathsf{SNR} = \frac{\bar{P}}{N_0 W}$$

Bandwidth limited regime $\mathsf{SNR} \gg 1$: capacity logarithmic in power, approximately linear in bandwidth.

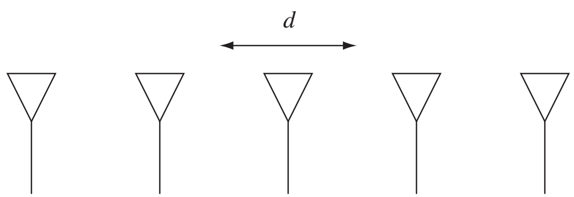
Power limited regime $\mathsf{SNR} \ll 1$: capacity linear in power, insensitive to bandwidth.


1.6
$\frac{P}{N_0}\log_2 e$
1.4
1.2
1
0.8
$C(W)$
(Mbps)
0.6
0.4
0.2
0
Power limited region
Capacity
Limit for $W \to \infty$
Bandwidth limited region
0
5
10
15
20
25
30
Bandwidth W (MHz)

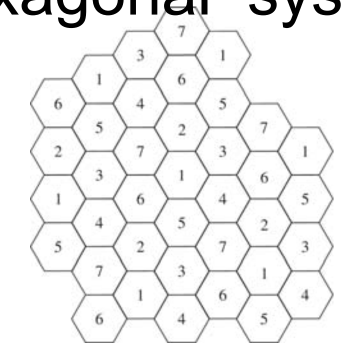
Example 1: Impact of Frequency Reuse

- System divided into cells; study the uplink of this cellular system
- Users within a cell are orthogonal
- The main parameter of interest is the reuse ratio ρ
- ρ =1 means full reuse (e.g. OFDM system). ρ <1 means a narrowband system
- W denotes the bandwidth per user within a cell
- Each user transmission occurs over a bandwidth of ρ W
- Different degree of frequency reuse allows a tradeoff between SINR and degrees of freedom (bandwidth) per user.
- Users in narrowband systems have high link SINR but small fraction of system bandwidth.
- Users in wideband systems have low link SINR but full system bandwidth.
- Capacity depends on both SINR and d.o.f. and can provide a guideline for optimal reuse.
- Optimal reuse depends on the out-of-cell interference fraction f(ρ): which depends on the reuse factor ρ and the topology of the cellular system

Linear cellular system



Hexagonal system



- It can be shown that the rate of reliable communication for a user at the edge of a cell as a function of ρ is

$$R_{½} = ½W \log(1 + SINR)$$

$$= ½W \log\left(1 + \frac{SNR}{½ + f_{½} SNR}\right)$$

- The expression of $f_{½}$ is different for hexagonal or linear cellular topology

Main conclusions

- At large SNR, the interference grows as well and the SINR peaks at $1 = f$ ½ The largest rate is

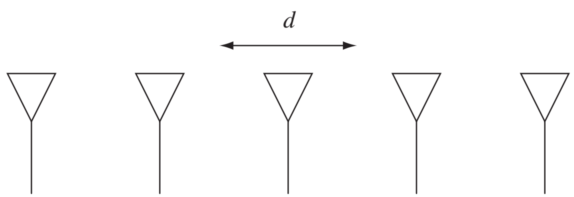
$$½W \log\left(1 + \frac{1}{f_{½}}\right)$$

(a general rule of thumb is to set the SNR such that the interference is of the same order as background noise; this guarantees that the operating SINR is close to the largest value)

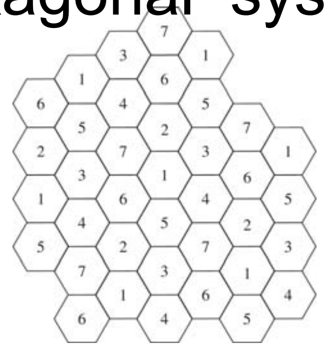
- Can show that:
  - Low /rho not recommended in this case (zero rate)
  - For hex topology, optimal reuse is \rho=1
  - For linear topology, optimal reuse is \rho=1/2 i.e. reuse the frequency every other cell (Exercise 5.5)
  - From the figures:
    - universal reuse always optimal for hex system;
    - \rho=1/2 is optimal if SNR>threshold (10 dB) for linear system

Numerical Examples

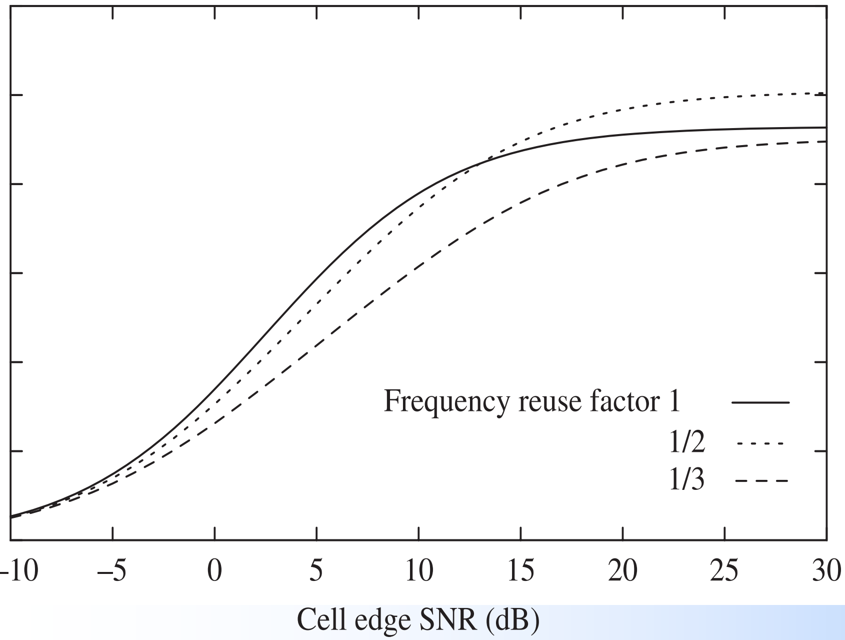
Linear cellular system

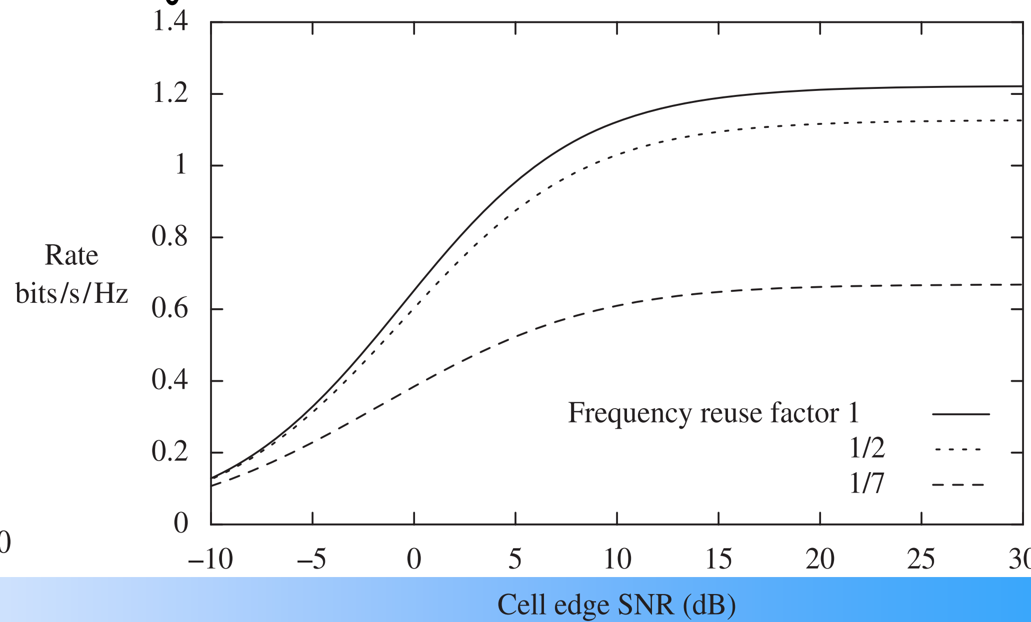


Hexagonal system



$$SNR = \frac{P}{N_0 W d^{\alpha}}$$





Linear Time Invariant Gaussian Channels

- Will first study three examples of channels that are closely related to the AWGN channel
- Their capacities can be easily computed
- Optimal codes can be easily constructed from optimal AWGN codes
- Time-invariant channels: known to both the transmitter and receiver
- These channels form a bride to the fading channels which we study next
- The channels are :
  - Single Input Multiple Output Channel
  - Multiple Input Single Output Channel
  - Frequency Selective Channel

Single Input Multiple Output (SIMO) Channel

- A channel with one transmit and L receive antennas

$$y_l[m] = h_\ell x[m] + w[m]; l = 1, \ldots, L$$

- The channel is equivalent to a single AWGN channel with received SNR equal to the norm of the channel gain vector multiplied by the awgn snr i.e.

$$C = \log\left(1 + \frac{P\|h\|^2}{N_0}\right) \text{ bits/s/Hz}$$

- Thus, multiple rcv antennas increase the effective SNR and provide a "power gain"
- E.g. L=2, |h_1|=|h_2|=1, dual receive antennas provide a factor of sqrt(2) in SINR i.e. a 3dB power gain over a single receive antenna system
- Note: The optimal receiver maximizes the output SNR by linear combining, also called *receive beamforming*

Multiple Input Single Output (MISO) Channel

- Can show that the channel is equivalent to a scalar AWGN channel with power constraint P with capacity

$$C = \log\left(1 + \frac{P\|h\|^2}{N_0}\right) \text{ bits/s/Hz}$$

- Note: the optimal tx strategy maximizes the rcvd SNR by
  - having the rcvd signals from various tx antennas add up in-phase (coherently), called *transmit beamforming,* and
  - Allocate more power to the tx antenna with the better gain
- As in SIMO, the benefit is “power gain”

Frequency-selective Channel

Time-invariant L-tap frequency selective AWGN channel:

$$y[m] = \sum_{\ell} h_{\ell} x[m-\ell] + w[m]$$ h_l 's are time-invariant.

OFDM converts it into a *parallel channel:*

A collection of N_c AWGN sub-channels, one for each sub-carrier (S#3.4.4 Tse/Viswana *with a total power constraint across the subchannels. Comm over i'th OFDM block is:*

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \qquad n = 1, \ldots, N_c.$$

$$C_{N_c} = \sum_{n=0}^{N_c-1} \log\left(1 + \frac{P_n^* |\tilde{h}_n|^2}{N_0}\right),$$

where P_n^* is the waterfilling allocation:

$$P_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+$$

with λ (Lagrange multiplier) chosen to meet the power constraint.

Can be achieved with separate coding for each sub-carrier.

- As number of sub-carriers N_c grows, the optimal power allocation converges to

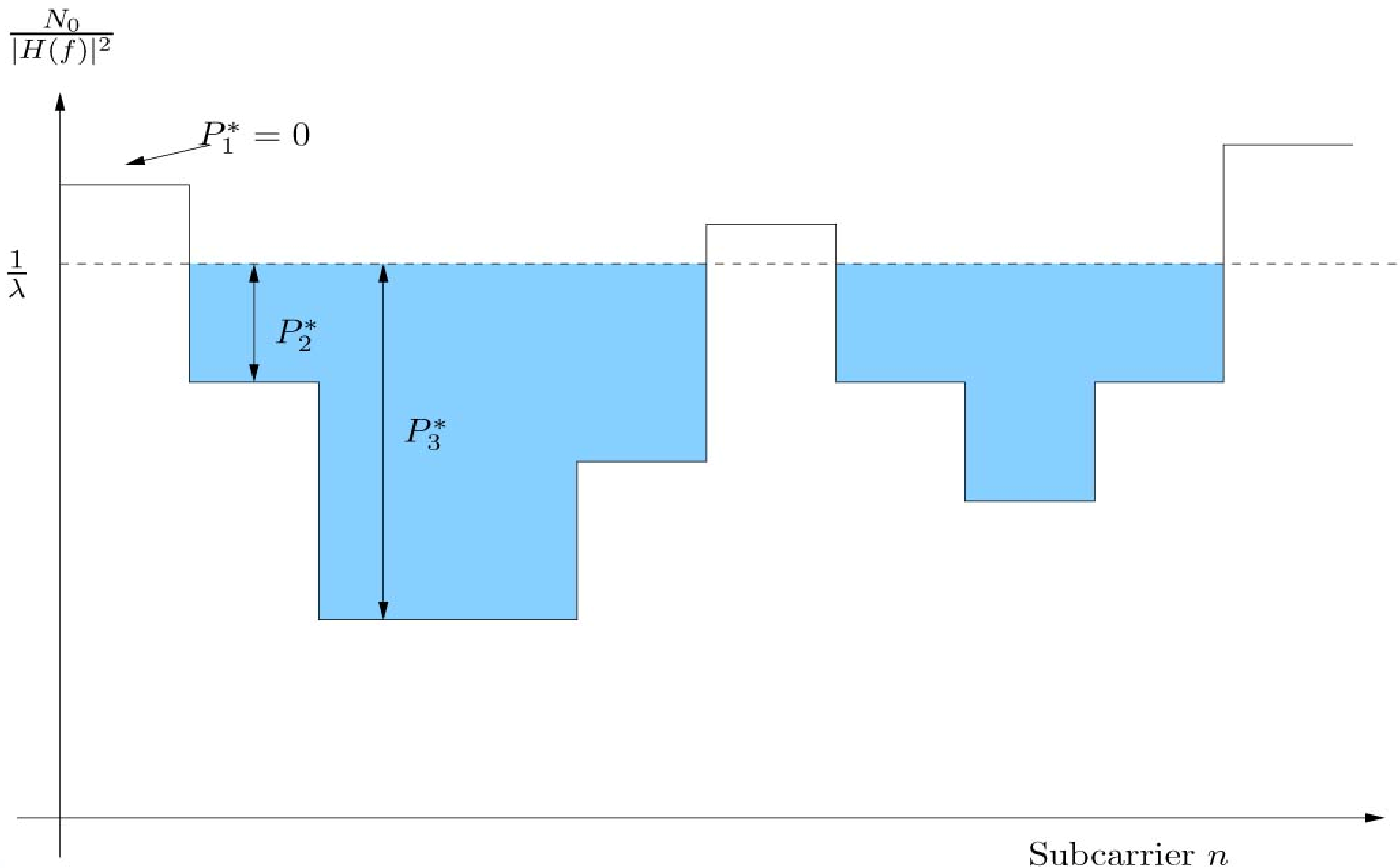
$$P^*(f) = \left(\frac{1}{\lambda} - \frac{N_0}{H(f)^2}\right)^+$$

where λ satisfies

$$\int_0^W P^*(f)\,df = P$$

and H(f) is the DFT evaluated at f=n W/N_c

Waterfilling in Frequency Domain



Capacity of Fading Channels

- Slow Fading Channels Capacity
  - Outage Capacity
  - Capacity with receive diversity
  - Capacity with transmit diversity
  - Time and frequency diversity
  - Fast fading channel
  - Transmitter side information

Slow Fading Channel

$y[m] = hx[m] + w[m]$ h random, represents the fading process

- Conditioned on a realization of the channel *h*, this is an AWGN channel with rcvd snr $|h|^2$SNR.

- Max rate of reliable comm for this channel is log(1+ $|h|^2$SNR) bits/s/Hz.

- There is no definite capacity. If transmitter encodes at R bits/s/Hz, such that R> log(1+ $|h|^2$SNR), system is in *outage*.

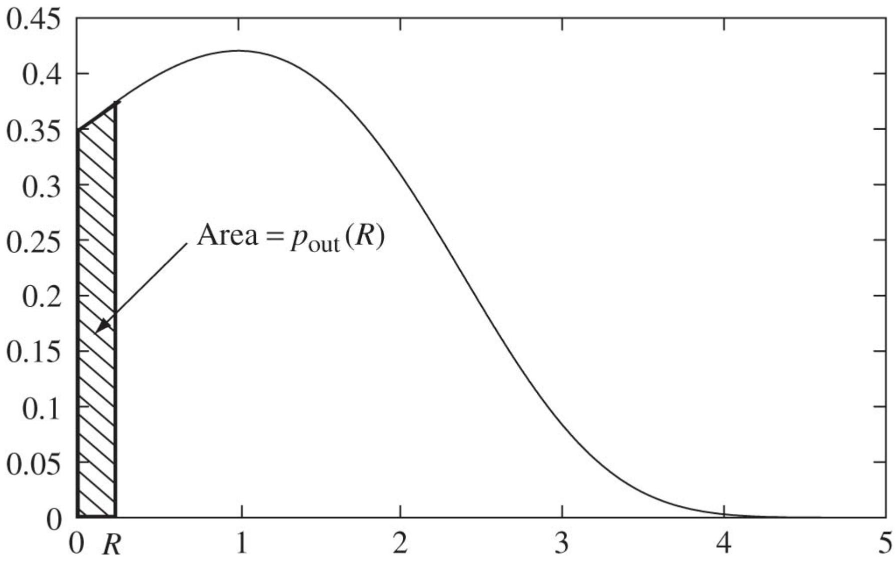
Outage probability: $p_{\text{out}}(R) = \mathcal{P}\left\{\log(1 + |h|^2 \text{SNR}) < R\right\}$

ϵ–outage capacity: C_ϵ Largest R st outage prob less than ϵ

(lower ϵ: lower C_ϵ)

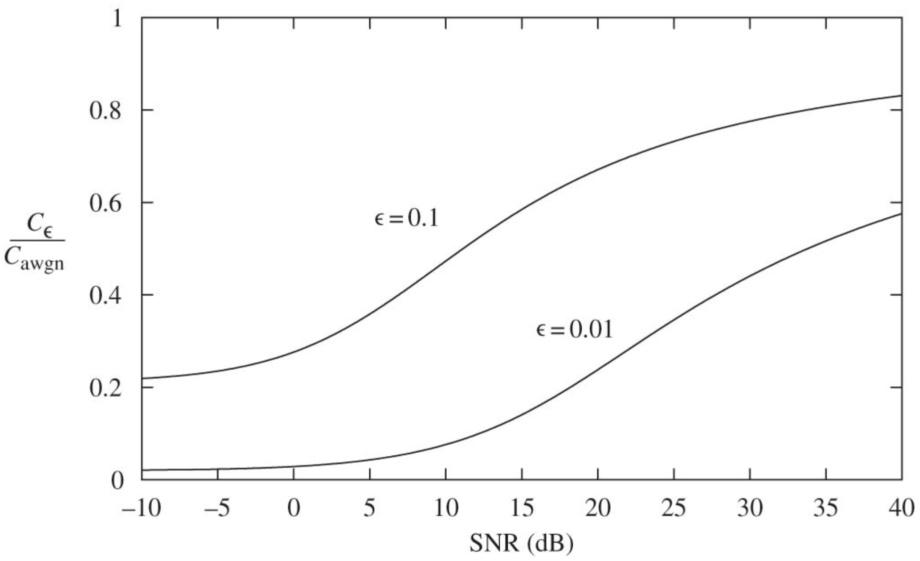
Outage for Rayleigh Channel

Pdf of log(1+|h|^{2}SNR)



$$p_{\text{out}}(R) \approx \frac{2^R - 1}{\text{SNR}}$$

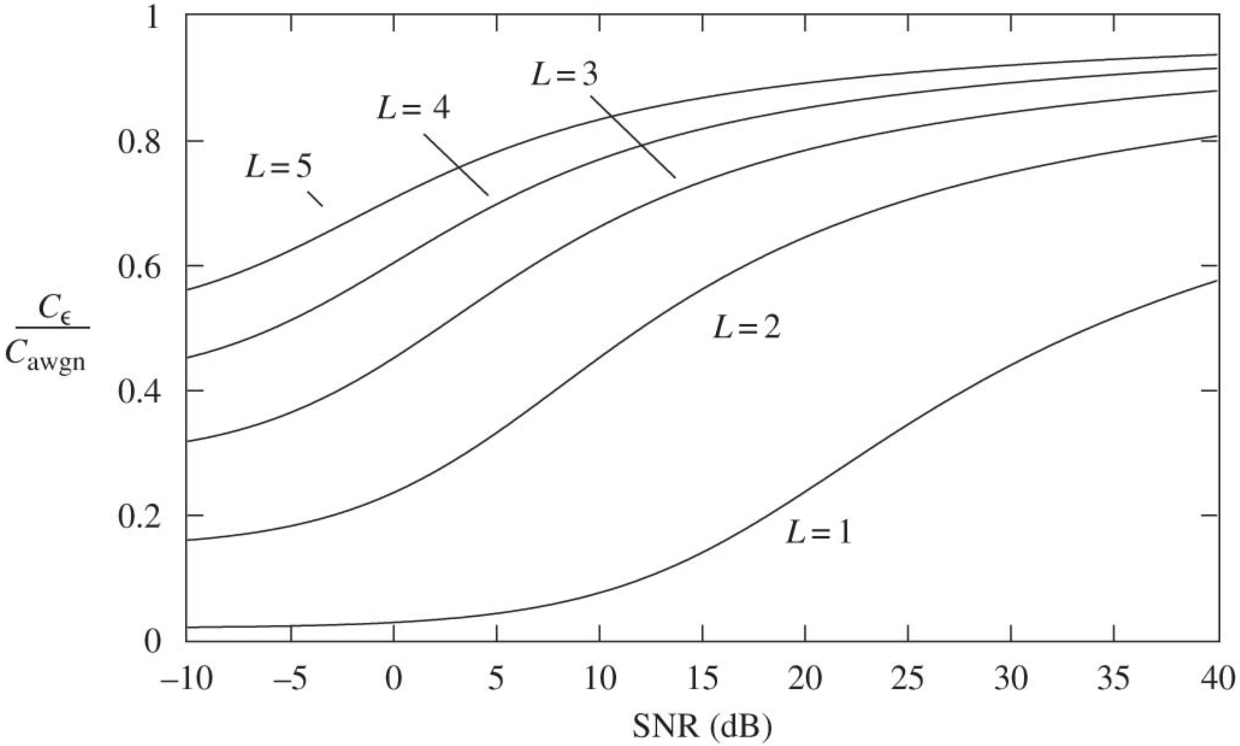
Outage cap. as fraction of AWGN cap.



Receive Diversity

L receive antennas increases SNR by jjhjj^2. Thus,

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \text{SNR}\right) < R\right\}$$



Diversity incurs power gain plus power gain.

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Transmit Diversity

Transmit beamforming (only if transmitter knows **h**):

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \mathsf{SNR}\right) < R\right\}$$

Transmit diversity without knowledge of **h:**

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \frac{\mathsf{SNR}}{L}\right) < R\right\}$$

loss of a factor L in the received SNR because the transmitter has no knowledge of the channel and is unable to beamform.

Diversity but no power gain.

Time Diversity (I)

•Exploit time-variation of the channel: in addition to coding over symbols within one coherence period, code over symbols from L such periods.

$$y_\ell = h_\ell x_\ell + w_\ell, \qquad \ell = 1, \ldots, L$$

This can be modeled as a parallel channel; each sub-channel ` represents a coherence period.

Can *always* achieve:

$$p_{\text{out}}(R) = \mathcal{P}\left\{ \frac{1}{L} \sum_{\ell=1}^{L} \log\left(1 + |h_\ell|^2 \text{SNR}\right) < R \right\}$$

- If transmitter knows the channel, can do waterfilling for each realization of the channel, hence the result is average of the capacity of each subchannel, and each subchannel can be coded separately using an AWGN capacity-achieving code.
- Otherwise, coding across the different coherence periods is now necessary.: if the channel is in deep fade in one coherence period, the information bits can still be protected if the channel is strong in other periods.

Fast Fading

- Slow fading: channel remains constant over the *entire* transmission duration of the codeword.
- Time Diversity: Achieved when the codeword length spans *several* coherence periods (outage probability improves)
- Fast fading: codeword length spans *many* coherence periods

Fast Fading Channel

Channel with L-fold time diversity:

$$p_{\text{out}}(R) = \mathcal{P}\left\{\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1+|h_\ell|^2\text{SNR}\right) < R\right\}$$

As $L \to \infty$,(with fast fading, we can indeed code over a very large number of coherence periods)

$$\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1+|h_\ell|^2\text{SNR}\right) \to E[\log(1+|h|^2\text{SNR})]$$

Fast fading channel has a definite capacity (not outage):

$$C = E[\log(1+|h|^2\text{SNR})] \quad \text{Bits/s/Hz}$$

Caveat: Tolerable delay >> coherence time.

Or, Interleave so that codeword symbols are sufficiently far apart

Capacity with Full CSI

- So far we have assumed that only the receiver can track the channel.

- Suppose now the transmitter has full channel knowledge, e.g.
  - In a TDD, assuming channel reciprocity
  - Explicitly, eg in CDMA systems through the feedback in the uplink

- What is the capacity of the channel?

Fast Fading Channel with Full CSI

This is a parallel channel, with a sub-channel for each fading state, but now we can do optimal power allocation.

$$C = \mathcal{E}\left[\log\left(1 + \frac{P^*(h)|h|^2}{N_0}\right)\right]$$

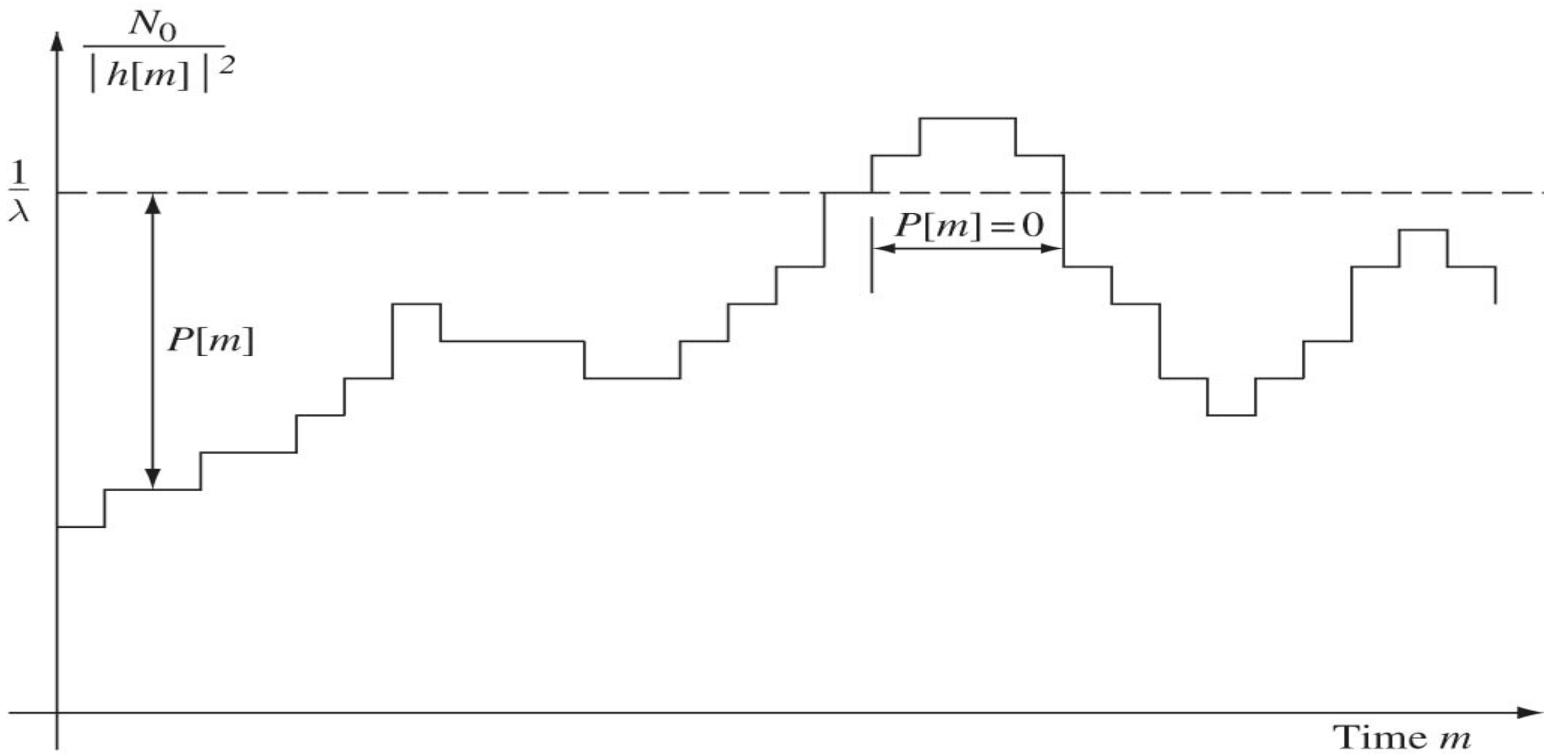
where

$$P^*(h) = \left(\frac{1}{\lambda} - \frac{N_0}{|h|^2}\right)^+ .$$

is the waterfilling power allocation as a function of the fading state (ie instantaneous channel gain h), and λ is chosen to satisfy the average power constraint.

Can be achieved with separate coding for each fading state.

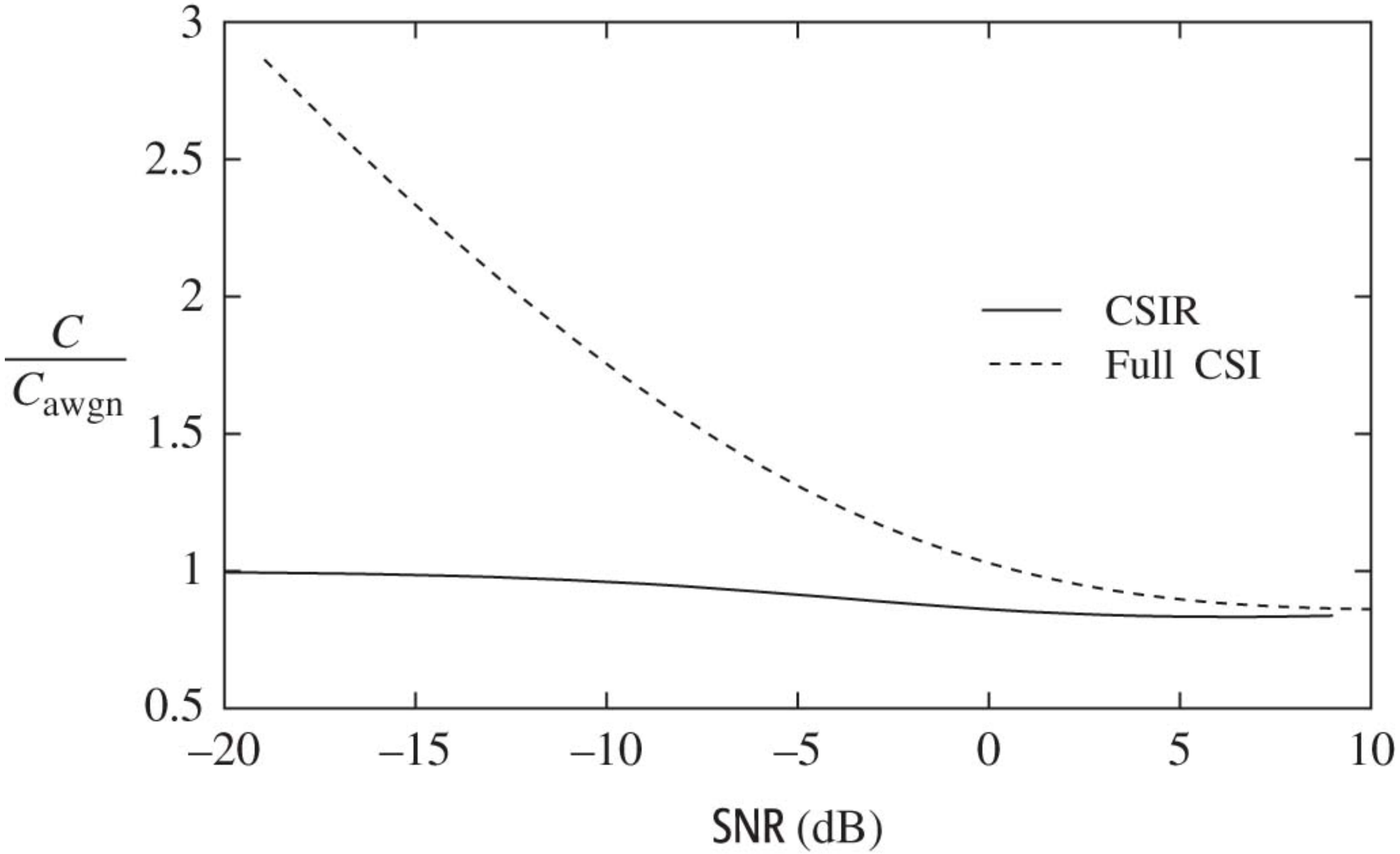
Transmit More when Channel is Good



Knowledge of state allows:

- -variable rate coding scheme, with P*(h) for each state h; also called dynamic power allocation
- -achieved with separate coding for each fading state

Performance: SNR



- At high SNR, full CSI does not provide any gain. But transmitter knowledge allows rate adaptation and simplifies coding.
- At low SNR, the capacity of full CSI is significantly larger than the CSIR capacity. This is because dynamic power allocation translates to a received power gain, and the capacity is quite sensitive to the received power (linear) in the power-limited regime.

Summary

- A slow fading channel is a source of unreliability: very poor outage capacity. Diversity is needed.
- A fast fading channel with only receiver CSI has a capacity close to that of the AWGN channel. Delay is long compared to channel coherence time.
- A fast fading channel with full CSI can have a capacity *greater* than that of the AWGN channel: fading now provides more *opportunities* for performance boost.
- The idea of *opportunistic communication* is even more powerful in multiuser situations, as we will see.