Supplementary material: Particle dynamics and multi-channel feature dictionaries for robust visual tracking

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1 Additional results

The mean CLE and success rate for each individual test sequence are shown in Tables 1 and 2 respectively. The success plots for the following attributes: background clutter (BC), motion blur (MB), fast motion (FM), occlusion (OCC), non-rigid object deformation (DEF), out-of-view (OV), and low resolution are shown in Figure 1.

2 Adaptive candidate filtering

We first derive the expression representing the number of particles to be chosen in each frame:

\[ n = \frac{1}{2v} \chi^2_{k-1,1-\delta} \approx \frac{k-1}{2v} \left( 1 - \frac{2}{9(k-1)} + \sqrt{\frac{2}{9(k-1)}} z_{1-\delta} \right)^3 \] (1)

Consider two probability distributions \( p_1 \) and \( p_2 \). The Kullback-Leibler distance \([\square]\) \( K \) between \( p_1 \) and \( p_2 \) is defined as

\[ K(p_1, p_2) = \sum_x p_1(x) \log \left( \frac{p_1(x)}{p_2(x)} \right) \] (2)

The basic idea of KLD-sampling \([\square]\) is to find the number of particles in each iteration such that the error between the true posterior probability density and the probability density approximated by the particle filter is less than \( v \) with probability \( (1 - \delta) \). At any particular iteration, suppose we draw \( n \) particles from a discrete probability distribution that has \( k \) disparate bins. Defining the vector \( \mathbf{N} = [N_1, N_2, \ldots, N_k] \) as the number of particles drawn from each bin, we can see that \( \mathbf{N} \) follows a multinomial distribution \( f_k(n,p) \), where \( p = [p_1, p_2, \ldots, p_k] \) represents the probability of each of the \( k \) bins. We can use the maximum

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Table 1: Mean center location error (in pixels) for each of the 25 test sequences. **Red** - Best, **Blue** - Second best.

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<td>29.7</td>
<td>90.5</td>
<td>30</td>
<td>97</td>
<td>96.2</td>
<td>95.5</td>
<td>15.5</td>
<td>28</td>
<td>27.7</td>
</tr>
</tbody>
</table>
Figure 1: Success plots for background clutter, fast motion, occlusion, non-rigid object deformation, out-of-view, motion blur, fast motion, and low-resolution attributes.
likelihood estimation procedure to obtain $\hat{p}$ as

$$\hat{p} = \frac{N}{n}$$

(3)

The likelihood ratio $\lambda_n$ statistic for $p$ is given by

$$\log \lambda_n = \sum_{j=1}^{k} N_j \log \frac{\hat{p}_j}{p_j}$$

(4)

Since $N_j = n\hat{p}_j$, this equation becomes

$$\log \lambda_n = n \sum_{j=1}^{k} \hat{p}_j \log \frac{\hat{p}_j}{p_j} = nK(\hat{p}, p)$$

(5)

Noting that $2\log \lambda_n$ converges in distribution to a chi-square distribution as $n \to \infty$ [4], consider the probability $P(K(\hat{p}, p) \leq \nu)$:

$$P(K(\hat{p}, p) \leq \nu) = P(2\log \lambda_n \leq 2n\nu)$$

$$= P(\chi^2_{k-1} \leq 2n\nu)$$

(6)

Using the fact that $P\left(\chi^2_{k-1} \leq \chi^2_{k-1,1-\delta}\right) = 1 - \delta$, if we choose $n$ according to the following expression:

$$2n\nu = \chi^2_{k-1,1-\delta}$$

(7)

we get

$$P(K(\hat{p}, p) \leq \nu) = 1 - \delta$$

(8)

which is exactly what we wished to achieve, hence completing the proof. We see that equation 1 follows from Equation 7.

3 Optimization problem

In each feature channel, we solve the following optimization problem:

$$\min_{x^j, \epsilon^j} \|x^j\|_1 + \|\epsilon^j\|_1$$

s.t. $\ y^j = A^j x^j + \epsilon^j$

(9)

This problem is of the general form

$$\min_{x, \epsilon} f_1(x, \epsilon)$$

s.t. $f_2(x, \epsilon) = 0$

(10)

where $f_2(x, \epsilon) = y - Ax - \epsilon$. Both $f_2(x, \epsilon)$, and $f_1(x, \epsilon)$ are continuous and convex functions in $(x, \epsilon)$, and hence the problem

$$\min_{x, \epsilon} f_1(x, \epsilon) + \frac{\xi}{2}\|f_2(x, \epsilon)\|^2_2$$

s.t. $f_2(x, \epsilon) = 0$

(11)
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has the same optimal value pair \((x^*, \epsilon^*)\) as the problem defined in Equation 10. We now eliminate the equality constraints in this problem by introducing the Lagrange multipliers. The augmented Lagrangian for this problem is

\[
L_\zeta(x, \epsilon, \rho) = f_1(x, \epsilon) + \frac{\zeta}{2} \|f_2(x, \epsilon)\|_2^2 + \rho^T f_2(x, \epsilon)
\]

(12)

The minimization problem of Equation 11 is equivalent to minimizing the augmented Lagrangian of Equation 12. Therefore, we now have

\[
(x^*, \epsilon^*) = \arg \min_{x, \epsilon} L_\zeta(x, \epsilon, \rho)
\]

(13)

The minimization problem of Equation 13 can be solved using the framework of alternating directions algorithms \([6]\). Specifically, in each iteration, we compute \(x\) and \(\epsilon\) separately, and then update \(\rho\). Formally, the optimal solution pair \((x^*, \epsilon^*)\) is computed as

\[
x_{i+1} = \arg \min_x L_\zeta(x, \epsilon_i, \rho_i)
\]

(14)

\[
\epsilon_{i+1} = \arg \min_\epsilon L_\zeta(x_{i+1}, \epsilon, \rho_i)
\]

(15)

\[
\rho_{i+1} = \rho_i + \zeta \left( f_2(x_{i+1}, \epsilon_{i+1}) \right)
\]

(16)

The sub-problem defined by

\[
\epsilon_{i+1} = \arg \min_\epsilon L_\zeta(x_{i+1}, \epsilon, \rho_i)
\]

(17)

has a closed form solution, which we derive next. Consider the definition

\[
L_\zeta(x_{i+1}, \epsilon, \rho_i) = \|x_{i+1}\|_1 + \|\epsilon\|_1 + \frac{\zeta}{2} \|f_2(x_{i+1}, \epsilon)\|_2^2 + \rho_i^T \left( f_2(x_{i+1}, \epsilon) \right)
\]

(18)

Defining \(\epsilon_d = y - Ax_{i+1}\), minimizing \(L_\zeta(x_{i+1}, \epsilon, \rho_i)\) is equivalent to

\[
\epsilon^* = \arg \min_\epsilon \left\{ \|\epsilon\|_1 + \frac{\zeta}{2} \|f_2(x_{i+1}, \epsilon)\|_2^2 + \rho_i^T \left( f_2(x_{i+1}, \epsilon) \right) \right\}
\]

\[
= \arg \min_\epsilon \left\{ \|\epsilon\|_1 + \rho_i^T (\epsilon_d - \epsilon) + \frac{\zeta}{2} (\epsilon_d - \epsilon)^T (\epsilon_d - \epsilon) \right\}
\]

\[
= \arg \min_\epsilon \left\{ \|\epsilon\|_1 + \rho_i^T \left( \epsilon_d + \frac{\rho_i}{\zeta} \right) \right\}
\]

\[
= T_{\epsilon_i} \left( \epsilon_d + \frac{\rho_i}{\zeta} \right)
\]

(19)

where \(T_{\alpha}(t)_i = \sgn(t_i) \max\{|t_i| - \alpha, 0\}, i = 1, 2, \ldots, n\). Thus, the update step for \(\epsilon_{i+1}\) has an analytic solution given by

\[
\epsilon_{i+1} = T_{\epsilon_i} \left( \epsilon_d + \frac{\rho_i}{\zeta} \right)
\]

(20)

However, the sub-problem defined by

\[
x_{i+1} = \arg \min_x L_\zeta(x, \epsilon_i, \rho_i)
\]

(21)
does not have an analytic solution, and we hence must resort to iterative schemes. To solve this problem, we use the Fast Iterative Shrinkage Thresholding Algorithm (FISTA) \cite{beck2009fast}. We first show that this optimization problem is basically the classic lasso \cite{tibshirani1996regression} problem. Defining $b' = y - \epsilon_i$, and $b'' = b' + \frac{\rho_i}{\zeta}$, we have

$$
x_{i+1} = \arg \min_x \{ \|x\|_1 + \rho_i^T (b' - Ax) + \frac{\zeta}{2} (b' - Ax)^T (b' - Ax) \}
$$

$$
x_{i+1} = \arg \min_x \left\{ \|x\|_1 + \frac{\zeta}{2} \|Ax - b'\|_2^2 \right\}
$$

(22)

Thus, we see that the problem of Equation 21 reduces to the lasso framework, which can be efficiently solved using FISTA.

References


