Problem 2 (15 points) Data and Curve Fitting

The following data was obtained from some circuit. The source voltages, $V_A$ and $V_B$ were varied and an output voltage was measured.

<table>
<thead>
<tr>
<th>$V_A$</th>
<th>$V_B$</th>
<th>$V_{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Using matrix mathematics, find the coefficients, $a$ and $b$, that give the best least squares linear fit relationship between the two inputs and the input in the following expression,

$$V_{out} = aV_A + bV_B$$

You must show your work to receive credit. Hint: This problem involves almost all the matrix manipulations we have utilized in recent experiments.

\[
X = (A^T A)^{-1} A^T b
\]

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = \left(\begin{array}{c}
1 & 0 \\
1 & 1 \\
1 & 3 \\
1 & 5
\end{array}\right)^T \left(\begin{array}{c}
1 & 0 \\
1 & 1 \\
1 & 3 \\
1 & 5
\end{array}\right)^{-1} \left(\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}\right) = \left(\begin{array}{c}
1 \\
2 \\
4 \\
8
\end{array}\right)
\]
\[
\begin{bmatrix}
\mathbf{a} \\
\mathbf{b}
\end{bmatrix} = \left( \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 3 \\
1 & 0 & 3 \\
\end{bmatrix} \right)^{-1} \begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]

\[
= \left( \begin{bmatrix}
4 & 9 \\
9 & 35
\end{bmatrix} \right)^{-1} \begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.5935 & -0.1525 \\
-0.1525 & 0.0678
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.5932 & 0.4407 & 0.1356 & -0.1695 \\
-0.1525 & -0.0847 & 0.0508 & 0.1864
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.6610 \\
1.3729
\end{bmatrix}
\]

\[
\mathbf{a} = 0.661 \\
\mathbf{b} = 1.373
\]
Problem 3. (15 points) First Order RC Circuit

A capacitor is charged using a voltage source and discharged through a resistor. The voltage source used is an AC source with square wave shape. Two full cycles of capacitor charge and discharge cycles are shown below. The plot shows how the voltage across the capacitor changes with time. Note the vertical axis is in volts and the horizontal axis is in milli-seconds. Some data points have been marked for ease of calculation (more points than what you would need). If you need to make any assumptions, please state them clearly.

Part a) (5 points) For the above plot, determine the time constant in milli-seconds.

\[ \tau = 7 \text{ ms} \]

Part b) (10 points) Design any circuit that would produce the above voltage plot. Your circuit should include a source and include a brief description of how the source behaves.

[Diagram of the circuit]

Square wave with amplitude = 1.25 V and offset = 1.25 V, time period = 100 ms (f = 10 Hz)

\[ V_{in} = 2.5 \text{ V} \]

\[ V_{out} = 0 \]

\[ R = 1.5 \text{ k}\Omega \]

\[ C = 4.7 \mu\text{F} \]

\[ \tau = 7 \text{ ms} = RC \]

Let \( R = 1.5 \text{ k}\Omega \)

\[ \Rightarrow C = 4.7 \mu\text{F} \]

Multiple correct answers possible as long as \( \tau = 7 \text{ ms} \)

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Problem 4: (20 points) Second Order RLC circuit

The second order transient response of an RLC circuit is shown below. Note the vertical axis is in volts and the horizontal axis is in milli-seconds.

Part a) (4 points) Estimate the DC steady state value of the voltage (the value of the voltage as time becomes very large).

\[ 1 \text{V} \]

Part b) (4 points) Estimate the period, \( T \), of the sinusoid.

4 cycles in 4 ms \( \Rightarrow \) \( T = 1 \text{ms} \)

Part c) (4 points) Estimate the damped frequency, \( \beta \).

\[ \beta = \frac{2\pi}{T} = 6.28 \text{ K rad/sec} \]

Part d) (4 points) Estimate the attenuation constant, \( \alpha \).

\[ \alpha = -\ln \left( \frac{v_2 - V_{\text{offset}}}{v_1 - V_{\text{offset}}} \right) = -\ln \left( \frac{1}{4} \right) = 346.57 \text{ rad/s} \]

Part e) (4 points) Estimate the resonant frequency, \( \omega_0 \).

\[ \omega_0 = \sqrt{\beta^2 + \alpha^2} = \sqrt{(6.28 \times 10^3)^2 + 346.57^2} = 6.293 \text{ K rad/s} \]
Problem 5: (15 points) Linear Regression

Part a) (5 points) Using linear regression analysis, find the best linear fit to the following three points by finding the slope 'm' and y-intercept 'b' of the linear regression. Show your handwritten work.

(-1, -2), (1, 4), and (3, 6)

Express your answer in the form \( y = mx + b \)

\[
egin{align*}
\bar{x} &= \frac{1}{3} \sum_{i=1}^{3} x_i = 1 \\
\bar{y} &= \frac{1}{3} \sum_{i=1}^{3} y_i = 2.667 \\
m &= \frac{\sum_{i=1}^{3} (x_i y_i - \bar{x} \bar{y})}{\sum_{i=1}^{3} (x_i^2 - \bar{x}^2)} = \frac{(2+2.667)+(4-2.667)+(18-8)}{(1+1)+(1-1)+(9-3)} = \frac{16}{8} = 2 \\
b &= \bar{y} - m \bar{x} = 0.667
\end{align*}
\]

\( y = 2x + 0.667 \)

Part b) (5 points) Sketch a graph to plot the data points and your best fit line.
Part c) (5 points) Using Matlab, determine the correlation coefficient for the same three data points. (-1, -2), (1, 4), and (3, 6). Provide a screenshot of Matlab work and result. Would you characterize these data points as linear?

```
>> corrcoef([-1 1 3],[-2 4 6])
ans =
    1.0000    0.9608
    0.9608    1.0000
```

Linear data set as the correlation coefficient is 0.9608
Problem 6. (15 points) Determinant and Matrix Inverse

Part a) (5 points) Find the determinant of the matrix $Z$ shown below. You must show your work to receive credit.

\[
Z = \begin{bmatrix}
2 & 1 & 1 \\
3 & 2 & 1 \\
2 & 1 & 2 \\
\end{bmatrix}
\]

\[
\det(Z) = 2 \cdot (4 \cdot 1) - 1 \cdot (6 \cdot 2) + 1 \cdot (3 \cdot 4) = 6 - 4 - 1 = 1
\]

Part b) (10 points) Using techniques applied in the experiments, determine the inverse of the matrix shown below. You must show your work to receive credit. There is more than one method and you can use any method.

\[
Z = \begin{bmatrix}
a & b & c \\
a' & b' & c' \\
a'' & b'' & c'' \\
\end{bmatrix}
\]

Matrix of cofactors:

\[
\begin{bmatrix}
A & B & C \\
D & E & F \\
G & H & I \\
\end{bmatrix}
\]

\[
A = 3 \quad E = 2 \quad I = 1 \\
B = 4x-1 \quad F = 0x-1 \\
C = -1 \quad G = -1 \\
D = 1x-1 \quad H = -1x-1 \\
\]

\[
\text{adj}(Z) = \text{Transpose of matrix of cofactors} = \begin{bmatrix}3 & -4 & -1 \\
-1 & 2 & 0 \\
-1 & 1 & 1 \\
\end{bmatrix}
\]

\[
\text{det}(Z) = 1 \quad \text{(part a)}
\]

\[
Z^{-1} = \frac{1}{\det(Z)} \text{adj}(Z) = \begin{bmatrix}3 & -1 & -1 \\
-4 & 2 & 1 \\
-1 & 0 & 1 \end{bmatrix}
\]