

Transformer Equation Notes

This file contains a more detailed derivation of the transformer equations than the notes or the experiment 3 write-up. It will help you to understand what assumptions were needed while deriving the ideal transformer equations we use. To do the derivation, we will use the figure pictured below:

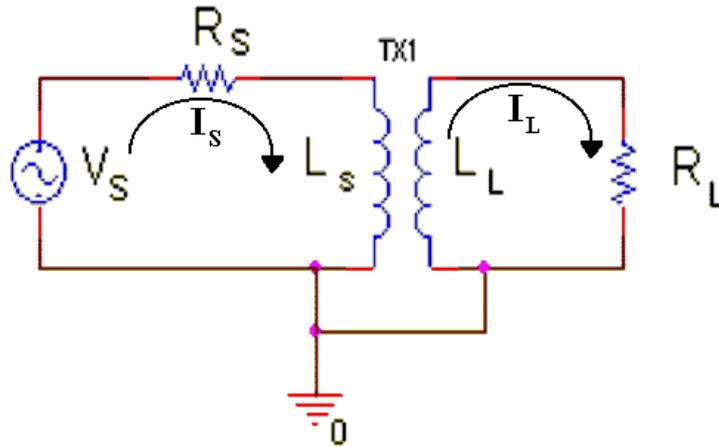


figure 1

As seen in figure 1, the transformer has two inductors: a source (or primary) inductor (L_S) and a load (or secondary) inductor (L_L). Each inductor loop is in series and has some resistance. The resistance of the source loop is represented by R_S and the resistance of the load loop is represented by R_L . The source has an input voltage of V_S . The voltage in the load loop is measured across R_L and is induced in the load inductor by the source inductor. The current through the source loop is given by I_S . The current through the load loop is I_L .

When you do any derivation, it is best to know what result you are trying to achieve. The ideal equations for a transformer in terms of the figure above are:

$$a = \frac{N_L}{N_S} = \frac{V_L}{V_S} = \sqrt{\frac{L_L}{L_S}} = \frac{I_S}{I_L} \quad Z_{in} = \frac{R_L}{a^2}$$

where a is a constant and N is the number of turns on each inductor in the transformer. Therefore, N_S is the turns on the source inductor and N_L is the number of turns on the load inductor. Z_{in} is the input impedance. It is the steady state impedance or “resistance” of the source inductor coil when it is coupled with the load coil in the transformer.

Inductor Impedance

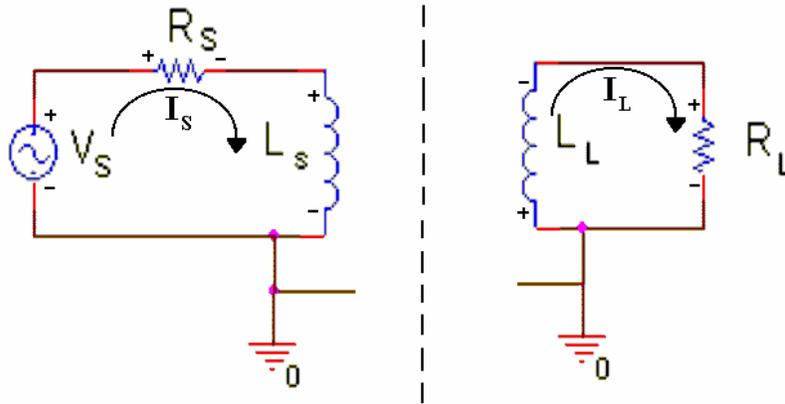


figure 2

We are going to use the loop equations for the circuit to derive our final result. If the two circuits are separated, as depicted in figure 2, the steady state loop equations are simply: $V_S = V_{RS} + V_{LS} = I_S R_S + I_S j\omega L_S$ and $0 = V_{LS} + V_{RS} \quad 0 = I_L j\omega L_L + I_L R_L$. (Note: $V=IZ$.) What effect does putting them together in a transformer have? We will assume that the effect of the transformer is to create some *mutual inductance*, M . The mutual inductance is the net amount that the source inductor pushes on the load inductor and visa versa. We do not know what M is, but we do know it is an inductance and that it works against the primary and load inductances. We can see that because the currents are going in opposite directions. This idea is pictured in figure 3 below:

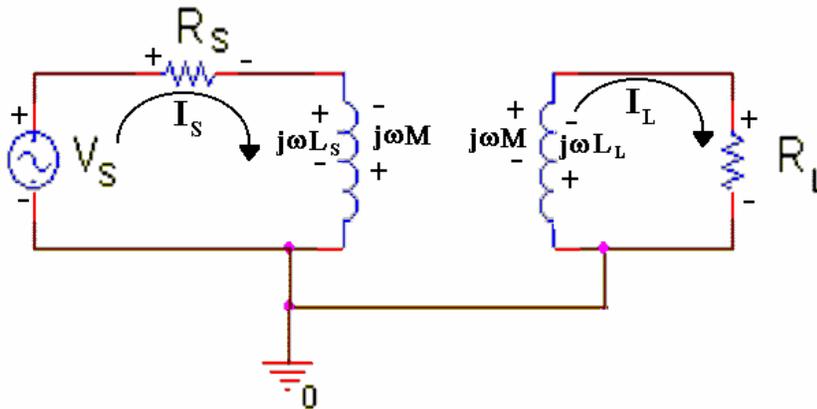


figure 3

Now we can write our loop equations again:

$$1] V_S = I_S R_S + I_S j\omega L_S - I_L j\omega M \quad \text{and} \quad 2] I_S j\omega M = I_L j\omega L_L + I_L R_L$$

We will use these equations to derive all the relationships about transformers, but first, we need to further define M . We know M depends on both L_S and L_L . We also know that in a real transformer, there will be some influence of one inductor on the other, but there will also be some losses due to the fact that they are connected through a medium (such as air or iron). Therefore, we will define M in terms of a constant, k , the *coupling*

coefficient. We will let k take on a number from 0 to 1. When k is 1, the influence of one inductor in the transformer on the other is (unrealistically) ideal. This means that all of the inductance from one pulls down on the other. We call this a perfect transformer. Mathematically, we will define M by the following expression: $M^2 = k^2 L_S L_L$. The reason for this exact choice will become evident later when we do the derivation.

Finding the Input Impedance

First we want to find an expression for Z_{in} , the net impedance of the source inductor in the transformer. This impedance is the combined influence of M and L_S . We know that whatever Z_{in} is, it must be the “resistance” of the source inductor in the circuit.

Therefore, we know the total impedance of the circuit must be.

$$Z_T = R_S + Z_{in} \quad Z_{in} = Z_T - R_S \quad Z_T = \frac{V_S}{I_S} \quad Z_{in} = \frac{V_S}{I_S} - R_S$$

If we can find an expression for $\frac{V_S}{I_S} - R_S$, then we will have the input impedance.

We can use the source loop equation to find this:

$$1] \quad V_S = I_S R_S + I_S j\omega L_S - I_L j\omega M$$

$$V_S - I_S R_S = I_S j\omega L_S - I_L j\omega M \quad \frac{V_S}{I_S} - R_S = j\omega L_S - \frac{I_L j\omega M}{I_S}$$

$$Z_{in} = j\omega L_S - \frac{I_L j\omega M}{I_S}$$

Now we have an expression for Z_{in} in terms of the source inductance and the mutual inductance, but it is still in terms of complex impedances. We can use the second loop equation (and a few assumptions) to simplify things.

$$2] \quad I_S (j\omega M) = I_L (j\omega L_L + R_L) \quad \frac{I_L}{I_S} = \frac{j\omega M}{j\omega L_L + R_L}$$

$$Z_{in} = j\omega L_S - \frac{(j\omega M)^2}{j\omega L_L + R_L} = j\omega L_S - \frac{-1\omega^2 M^2}{j\omega L_L + R_L} = j\omega L_S + \frac{\omega^2 M^2}{j\omega L_L + R_L}$$

Now our definition for M begins to make sense. Recall that we defined M with the expression: $M^2 = k^2 L_S L_L$. Before we continue, we must make our first assumption.

Assumption 1: Assume the transformer is perfectly coupled. ($k=1$)

Now we can substitute for M^2 and simplify.

$$Z_{in} = j\omega L_S + \frac{\omega^2 L_S L_L}{(j\omega L_L + R_L)}$$

$$Z_{in} = \frac{j\omega L_S(j\omega L_L + R_L) + \omega^2 L_S L_L}{(j\omega L_L + R_L)} = \frac{-\omega^2 L_S L_L + j\omega L_S R_L + \omega^2 L_S L_L}{(j\omega L_L + R_L)} = \frac{j\omega L_S R_L}{(j\omega L_L + R_L)}$$

Now we need to make our next assumption:

Assumption 2a: Assume that R_L is small compared to the value of L_L and L_S .

We could also make the following assumption with the same conclusion:

Assumption 2b: Assume we are at high frequencies.

Note that in general inductance gets bigger at high frequencies. We need one of these conditions to be true, so we can eliminate the R_L term and simplify the equation.

$$Z_{in} = \frac{j\omega L_S R_L}{j\omega L_L} = \frac{L_S R_L}{L_L} \quad \text{We next define a constant and call it } a.$$

$$a^2 \equiv \frac{L_L}{L_S} \quad a = \sqrt{\frac{L_L}{L_S}} \quad Z_{in} = \frac{R_L}{a^2} \quad \text{We now have the desired relationship for } L \text{ and } Z_{in}.$$

Current Relation

Deriving the remaining relationships is fairly straight forward. From loop equation 2 and the definition of M, we have:

$$\frac{I_L}{I_S} = \frac{j\omega M}{j\omega L_L + R_L} = \frac{j\omega k \sqrt{L_S L_L}}{j\omega L_L + R_L}$$

Again assuming that R_L has minimal contribution and that the transformer is perfectly coupled ($k=1$), this becomes:

$$\frac{I_L}{I_S} = \frac{\sqrt{L_S L_L}}{L_L} = \frac{\sqrt{L_S} \sqrt{L_L}}{\sqrt{L_L} \sqrt{L_L}} = \sqrt{\frac{L_S}{L_L}} = \frac{1}{a}$$

$$a = \frac{I_S}{I_L}. \quad \text{We now have the current relation. (Note it is the inverse of the others.)}$$

Voltage Relation

The voltage relation can apply either to the source voltage or to the input voltage depending upon the size of the source resistance R_S .

Case 1 : R_S is small

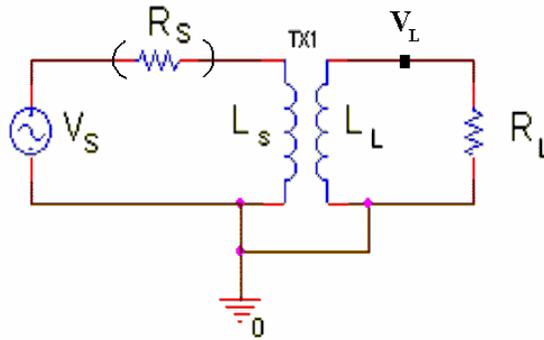


figure 4

When R_S is small, we can use the source voltage as the voltage in the relationship. First we use the loop equations for the two circuits.

$$V_S = V_{RS} + V_{LS} = V_{RS} + V_{LS} \quad V_L = V_{RL} = I_L R_L$$

If R_S is small, we can ignore the voltage drop across the resistor and this becomes:

$$V_S = V_{LS} = I_S Z_{in} \quad V_L = I_L R_L \quad \text{and} \quad \frac{V_L}{V_S} = \frac{I_L R_L}{I_S Z_{in}}$$

Now we can substitute our relationships for Z_{in} and the current ratio:

$$\frac{V_L}{V_S} = \frac{I_L R_L}{I_S Z_{in}} = \frac{R_L}{a Z_{in}} = \frac{R_L a^2}{a R_L} = a$$

$$\frac{V_L}{V_S} = a \quad \text{This is our voltage relationship.}$$

Case 2 : R_S is large

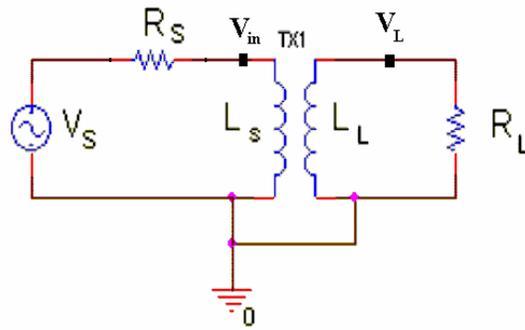


figure 5

When R_S is large, we need to use V_{in} to find the voltage relationship instead of V_S . In this case we can use Z_{in} directly.

$$V_{in} = I_S Z_{in} \quad V_L = I_L R_L \quad \frac{V_L}{V_{in}} = \frac{I_L R_L}{I_S Z_{in}} \quad \frac{V_L}{V_{in}} = a$$

Number of Turns Relationship

The turns relationship depends more on the physical attributes of the transformer than on the circuit itself. We will assume that our transformers inductance is determined by the equation given in experiment 3.

$$L_S = \frac{\mu N_S^2 \pi r^2}{d_S} \quad \text{and} \quad L_L = \frac{\mu N_L^2 \pi r_c^2}{d_L}$$

If we assume that both coils have the same radius, are wrapped around the same core, and have about the same length, then this reduces to:

$$\frac{L_L}{L_S} = \frac{\frac{\mu N_L^2 \pi r_c^2}{d_L}}{\frac{\mu N_S^2 \pi r_c^2}{d_S}} = \frac{N_L^2}{N_S^2} = a^2$$

$$a = \frac{N_L}{N_S}$$

Conclusion

The ideal transformer equations we use are sufficient to design a transformer that works under certain constraints. Your load resistance should be small relative to the impedances of the inductors in the transformer, at least at the frequency you want it to work at. Also, you should expect the transformer to have losses from the ideal value that you calculate. You can quantify those losses by experimentally determining a coupling coefficient for your transformer. Also, there are physical constraints on transformer design which influence whether or not it will behave like the equations dictate. These include the core size, the length of the coil and the core material. To check if a transformer circuit you design is working properly, you need to have consistent results for these three relationships:

$$a = \frac{V_L}{V_S} = \frac{I_S}{I_L} \quad Z_{in} = \frac{R_L}{a^2}$$