Portfolio Optimization in Secondary Spectrum Markets*

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Abstract

In this paper, we address the spectrum portfolio optimization (SPO) question in the context of secondary spectrum markets, where bandwidth (spectrum access rights) can be bought in the form of *primary* and *secondary* contracts. While a primary contract on a channel provides guaranteed access to the channel bandwidth (possibly at a higher per-unit price), the bandwidth available to use from a secondary contract (possibly at a discounted price) is typically uncertain/stochastic. The key problem for the buyer (service provider) in this market is to determine the amount of primary and secondary contract units needed to satisfy its uncertain user demand.

We formulate single and multi-region spectrum portfolio optimization problems as one of minimizing the cost of the spectrum portfolio subject to constraints on bandwidth shortage. Two different forms of bandwidth shortage constraints are considered, namely, the demand satisfaction rate constraint, and the demand satisfaction probability constraint. While the SPO problem under demand satisfaction rate constraint is shown to be convex for all density functions, the SPO problem under demand satisfaction probability constraint is not convex in general. We derive some sufficient conditions for convexity in this case. We also discuss application of the Bernstein approximation technique to approximate a non-convex demand satisfaction probability constraint by a convex constraint. The SPO problems can therefore be solved efficiently using standard convex optimization techniques. We then consider a discrete version of the SPO problem, in which the primary and secondary contracts can bought/sold in discrete units. We study the submodularity property of the discrete SPO problem and discuss a branch-and-bound algorithm algorithm to solve it efficiently. Finally, we perform a thorough simulation-based study of the single-region and the multiple-region problems for different choices of the problem parameters, and provide key insights regarding the portfolio composition. We provide several insights about the scaling behavior of the unit prices of the secondary contracts, as the stochastic characterization of the bandwidth available from secondary contracts change.

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I. Introduction and Background

The number of users of the wireless spectrum, as well as the demand for bandwidth per user, has been growing at an enormous pace in recent years. Since spectrum is limited, its effective management is vitally important to meet this growing demand. The spectrum available for public use can be broadly categorized into the unlicensed and licensed zones. In the unlicensed part of the spectrum, any wireless device is allowed to transmit. To use the licensed part, however, license must be obtained from appropriate government authority – the Federal Communications Commission (FCC) in the United States, for example – for the exclusive right to transmit in a certain block of the spectrum over the license time period, typically for a fee. While spectrum management in licensed bands has mostly been controlled by responsible government bodies, the need for bringing market based reform in spectrum trading is being increasingly recognized [1], [2], [3]. In order to achieve spectrum-usage efficiency, spectrum markets should allow dynamic trading of spectral resources and derived contracts of different risk-return characteristics. Providers can then choose to buy/sell one or more of these spectrum contracts depending on the level of service they wish to provide to their customers.

We consider a spectrum market in which a wireless service provider (buyer) can purchase spectrum access rights from another provider (seller) in the form of two types of spectrum contracts: *primary contract* and *secondary contract*. Typically, the buyer will be a smaller local or regional provider, buying access rights over its operational area from a larger regional or national provider which acts as the seller, although the framework and results that we present in this paper does not make any such assumption. Primary contract offers unrestricted access rights on a channel – a specific channel or one of a set of channels "owned" by the seller. On the other hand, secondary contract offers restricted access rights on a channel or a set of channels – it provides access to the "leftover" bandwidth on the channel(s) that the primary users of the channel(s) do not need at that specific time. At their core, primary and secondary contracts differ in the risk-return tradeoff that they provide. A primary contract represents a risk-free contract in terms of its bandwidth return characteristics, while the secondary contract is inherently risky in terms of the bandwidth it can provide. Primary contracts would generally be more expensive (in terms of cost per unit contract), since they provide full access rights. Secondary contracts would typically be cheaper due to their riskiness. These two contracts represent two fundamental forms of spectrum access contracts – analogous to bonds and stocks in terms of the risk characteristics. In financial markets, it is well known that bonds and stocks help investors achieve their desired risk-return tradeoff on investment.

Similarly, we envisage that the wireless service providers can efficiently tradeoff the level of service they wish to provide against their cost by using these two types of contracts.

A key challenge for a provider in this market is to determine an appropriate mix (i.e. a portfolio) of primary and secondary contracts that can provide the desired level of service to its users at a low cost. We formulate and study this *Spectrum Portfolio Optimization (SPO)* problem from the perspective of a buyer. In standard financial portfolio optimization, the objective is to maximize the expected portfolio return while satisfying a constraint on the variance of return. In the spectrum market context, minimizing the cost of the portfolio is a more reasonable objective. Furthermore, the constraint in the SPO problem can be specified meaningfully in two ways – either in terms of the expected bandwidth shortage, or in terms of the probability of bandwidth shortage. We refer to these constraints as the *demand satisfaction rate constraint* and the *demand satisfaction probability constraint*, respectively. We study the SPO problem under the two constraints separately.

The technical contributions of this paper are as follows. Firstly, we show that the SPO problem under demand satisfaction rate constraint is convex under any assumptions on the user demand and the bandwidth return distributions. Secondly, we show that the SPO problem under demand satisfaction probability constraint is not convex in general, and also derive sufficient conditions on the demand density functions for convexity to hold. The motivation behind showing convexity of the optimization problems is that convex problems can be solved efficiently using standard techniques such as gradient descent and Newton's methods, whereas there are no general techniques for solving non-convex problems efficiently. We also discuss application of the Bernstein convex approximation technique in cases where the demand satisfaction probability constraint is non-convex; this technique aproximates a non-convex probability constraint by a convex expectation constraint. In the next step, we extend the SPO problem and the convexity results to a multiple-region scenario, where the buyer's portfolio is intended to serve a set of disjoint geographical locations, each having its own user demand, using available primary and secondary contracts that provide access rights only over subsets of all locations of interest. We then consider an integer programming formulation of the SPO problem, since the primary and secondary contracts can be bought/sold only in discrete units in an operational spectrum market. We show that the discrete SPO problem is not submodular and discuss a branch-and-bound algorithm algorithm to solve it efficiently. Finally, we perform a detailed simulation-based study of the single and the multiple-region SPO problems and provide insights about the portfolio composition and the price characteristics of the secondary contracts.

Economics of spectrum allocation and auction mechanisms have been discussed widely in the literature [4], [5], [6], [7]. Spectrum sharing games and/or pricing issues have been considered in [8], [9], [10], [11]. Discussions and recommendations for transition to spectrum markets and secondary markets for spectrum trading have emerged [12], [13], [14]. In [14], the authors consider a spectrum secondary market analogous to the stock market for dynamically trading their channel holdings. The proposed auction-based market mechanism is shown to improve user performance and spectrum utilization. However, a clear design of the contract types and tradeoff analysis using portfolio theory have not been considered before. In [15], the authors propose a wireless spectrum market with two types of contracts, namely, the long-term and the short-term contract, and study the structural properties of the optimal dynamic trading strategy. Unlike the short-term contract defined in [15], the amount of bandwidth available for access from a secondary contract is a random variable. Moreover, the problem addressed in this paper is the spectrum portfolio optimization question over a single period and is different from the multi-period trading question considered in [15].

Portfolio optimization problem has been studied extensively in finance since the development of the mean-variance optimization framework in [16]. Several attempts have been made to improve the model and the risk measure [17], [18], [19], [20]. In [20], the authors propose a new measure of risk, namely, the expected shortfall and show that the problem of minimizing expected shortfall subject to a linear equality constraint is convex. The expected shortfall function considered in [20] measures the shortfall of return with respect to the α-quantile of the return distribution. But the demand satisfaction rate constraint that we consider measures the shortfall of the bandwidth return relative to a stochastic quantity, and is therefore different from the shortfall function in [20]. However, we are still able to make use of some of their analysis techniques to our problem. Probabilistic constraints have not been studied much, until recently in [21] and [22]. In [21], the authors study probabilistically constrained linear programs and present conditions for convexity of the constraint. While we apply some of their results in our context, we also provide additional conditions for convexity on the SPO problem with the demand satisfaction probability constraint.

The novelty of our contribution stems from the following aspects. Though the notion of primary and secondary users and their spectrum access rights have been extensively discussed recently, our modeling of these access rights as bond-like riskless and stock-like risky contracts, and the rigorous formulation of the spectrum portfolio optimization problem, are novel. Convexity of various versions of the portfolio optimization question have been studied in the finance and optimization literature; however, very limited results exist on the specific demand satisfaction constraints that appear meaningful in the spectrum access context. We provide several interesting results for the SPO problem with such constraints in this paper. The formulation and analysis of the multi-region SPO problem, and the insights obtained from our numerical studies, also constitute novel contributions of this work.

The rest of the paper is organized as follows. In Section II, we formally define the SPO problems under demand satisfaction rate and probability constraints. In Sections III and IV, we study the convexity of the two SPO problems under the two types of constraints. In Section VI, we consider a discrete version of the SPO problem and study its submodularity properties. In Section V, we study the multiple-region SPO problem. In Section VI, we study the SPO problem with discrete portfolio constraints. Finally, in Section IX-A, we present the simulation results.

II. Spectrum Portfolio Optimization Problem Formulation

In this section, we formally define the spectrum portfolio optimization (SPO) problem for a single region. The formulation and discussion of the multi-region SPO problem is deferred to Section V. Although not necessary for the mathematical formulation or subsequent analytical treatment of the SPO problem, it is easy to motivate the development of the framework by considering a (secondary) spectrum market in which N "higher level" spectrum providers are selling access contracts in the form of primary and secondary contracts to other "lower level" providers. These seller spectrum providers will typically be large providers (like VerizonWireless, AT&T, and Sprint in the US for example) who have directly leased spectrum from the governing body (like FCC), and might want to offer their excess bandwidth in the form of primary and secondary contracts. The buyers of the contracts can be smaller, possibly local or smaller regional wireless spectrum service providers who are trying to obtain bandwidth at the cheapest price to serve their user (customer) demand. We assume that primary and secondary contracts can be obtained in multiple units. Without loss of generality, we can assume that each unit of primary contract provides exclusive access to 1 unit of bandwidth in some channel that the seller provider operates on. On the other hand, each unit of secondary contract provides exclusive access to bandwidth that is a random variable varying between 0 and 1 unit. While this assumption is for the ease of exposition, it can be easily generalized. A simple way to view this setting would be to consider a seller provider having C units of bandwidth, offering C units of primary and C units of secondary contracts. If in any time slot, the primary contract holders in totality use $\alpha < C$ units of bandwidth, each unit of secondary contract has access to $0 < (C-\alpha)/C < 1$ units of bandwidth. A buyer holding x units of secondary contracts with this seller provider will then have access to $x(C-\alpha)/C$ units of bandwidth in that time slot.

Note that we are associating contracts – primary or secondary – with the seller providers, not specific channels. *All primary contracts (no matter which seller provider provides it) can be considered equivalent, since they offer the same bandwidth return (one unit, guaranteed).* This also argues for the fact that they must be priced the same; without loss of generality, we assume that the cost of one unit of any primary contract is unity. Secondary contracts offered by different seller providers will differ from one another, depending on the access pattern of the primary members of the seller provider, and their price per unit will also differ. However, since each unit of secondary contract offers an average return of less than one unit bandwidth, and have some risk associated with the return, the price per unit for each secondary contract should be less than unity (the price of a unit of primary contract).

With this abstraction, the SPO problem can be viewed in the context of a market where a single type of primary contract, and N different types of secondary contracts, are being offered.[1] Each unit of primary contract sold in the secondary spectrum market offers guaranteed access to 1 unit of bandwidth at a cost of 1. The secondary contract offered by the provider i can be described by the pair (p_i, B_i), where, p_i is the unit price of the secondary contracts offered by the i^{th} seller provider and B_i is the random variable (varying between 0 and 1) characterizing the bandwidth return from one unit of secondary contract of the i^{th} provider. From the above discussion, $p_i < 1, \forall i$.

In the following, we assume that each seller provider has a large pool of available bandwidth, and so any amount of primary or secondary contract units can be bought from the providers. This is for ease of exposition, and can be easily generalized by incorporating into the SPO problem additional upper bounds on the number of primary and secondary contract units available from a seller provider.

Now we are ready to formally define the SPO problem from the perspective of a single buyer provider. The buyer's objective is to create a spectrum portfolio consisting of primary and secondary contract units from the N seller providers in order to provide service to its customer base. Let $x_i, 1 \leq i \leq N$ denote the amount of secondary contract units purchased from the i^{th} seller provider. Since the primary contracts offered by all the N providers are identical, we only need to keep track of the total amount of primary contract units bought, which we denote by x_0. We assume a relaxation that $x_0, x_1, .., x_N$ are non-negative real numbers, not necessarily integers. Let the vector $\overline{x} = (x_0, x_1, .., x_N)$, denote the buyer's spectrum

[1]Note that the basic portfolio optimization question in financial markets, while considering multiple risky (stock) assets, assumes only a single risk-free (bond) asset, for similar reasons.

portfolio. The buyer wishes to satisfy its customers' demand for bandwidth using the spectrum portfolio, $\overline{x}$. The customer demand is modeled as a random variable Q, as it is often unknown in advance. The bandwidth return or the actual units of bandwidth available from a spectrum portfolio $\overline{x}$, is uncertain, due to presence of the secondary contracts. The bandwidth return of the portfolio $\overline{x}$, $B(\overline{x})$, is defined as $B(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times B_i$.

Since the bandwidth return and the demand are stochastic, it is impossible or highly expensive to construct a portfolio that always offers enough bandwidth to satisfy the customer demand. However, it is desirable to construct portfolios with low levels of bandwidth shortage. Let us define $S(\overline{x}) = Q - B(\overline{x}$. Then the bandwidth shortage of a portfolio, denoted by $S(\overline{x})^+$, is given as $S(\overline{x})^+ = \max(S(\overline{x}), 0) = \max(Q - B(\overline{x}), 0)$. Note that the shortage, $S(\overline{x})^+$, is also a stochastic quantity as both Q and $B(\overline{x})$ are random variables.

The spectrum portfolio optimization (SPO) problem for the buyer is to find the least costly portfolio with low levels of bandwidth shortage. The SPO objective is

$$\text{minimize } C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i. \tag{1}$$

The constraint on bandwidth shortage can be specified either in terms of expected shortage or probability of shortage. Therefore, we consider two versions of constraints for the SPO problem – the *Demand Satisfaction Rate (DSR)* constraint, and the *Demand Satisfaction Probability (DSP)* constraint, as expressed below:

$$\textit{DSR Constraint:} \qquad E[S(\overline{x})^+] \;<\; \delta; \tag{2}$$

$$\textit{DSP Constraint:} \qquad Pr(S(\overline{x}) > 0) \;<\; \epsilon. \tag{3}$$

Here $C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i$ is the cost of the spectrum portfolio $\overline{x}$. The DSR constraint ensures that the expected amount of bandwidth shortage is below a certain acceptable level δ. On the other hand, the DSP constraint bounds the probability of shortage to a low value ϵ. Note here that $Pr(S(\overline{x}) > 0)$ is the same as $Pr(S(\overline{x})^+ > 0)$. We devote the following sections to the study of the SPO problem under these two types of constraints.

An alternative formulation (version) of the problem would have been to reverse the constraint and objective functions. In particular, we could minimize the bandwidth shortage probability (in expectation or probability) subject to maximum limit on the cost of the spectrum portfolio. The fundamental complexity/computability of the optimal solution, does not change with this reversal however, as the optimal solution of one version of the problem could be translated to the optimal solution of the other in polynomial time. Furthermore, the main complexity of the SPO problem comes from that of the non-linear functions in (2) and (2), the function in (1) being is a simple linear function. The main theoretical results that we show in this paper - on the convexity of the SPO problem under the DSR and DSP constraints - are equally applicable to the alternative (reversed) version of the problem as well.

Finally, note that our cost function in (1) assumes that the price per unit bandwidth remains the same irrespective of the quantity bought. In practice, however, the per-unit price would vary with the quantity bought. Since bandwidth is a limited quantity, is is reasonable to assume that the marginal price increases with the quantity purchased [23]. In other words, if $P_i(x_i)$ denotes the total price to be paid for buying x_i units of commodity, we can assume that P_i is an increasing convex function in its argument. Then the cost function in (1) can be written as $C(\overline{x}) = x_0 + \sum_{i=1}^{N} P_i(x_i)$, which is a convex function. Therefore, in that case too, the complexity of the problem is dictated by that of the functions in the constraints (2) and (2), which is what we address in this paper. For the sake of simplicity, however, in the rest if the paper we assume that $P_i(x_i)$ is linear in x_i, as given in (1).

III. SPO under Demand Satisfaction Rate (DSR) Constraint

In this section, we study the properties of the SPO problem under demand satisfaction rate constraint, and provide the expressions for certain useful quantities that can be utilized to compute the optimal portfolio solution efficiently. The objective function of the SPO problem (Equation 1) is linear and therefore convex. The demand satisfaction rate function (i.e. $E[S(\overline{x})^+]$), however, is non-linear in $\overline{x}$. Borrowing from the analysis techniques in [20], we show below that $E[S(\overline{x})^+]$ is also convex in $\overline{x}$. This implies that the feasibility set represented by the DSR constraint (Equation 2) is also convex, and therefore the SPO problem under DSR constraint is a convex problem.

Theorem 1: $E[S(\overline{x})^+]$ is convex in $\overline{x}$.

Proof: We show that the Hessian of the function $E[S(\overline{x})^+]$ is positive semi-definite. We obtain the gradient and Hessian of $E[S(\overline{x})^+]$ as follows.

Let $g(\overline{x}) = E[S(\overline{x})^+] = E[S(\overline{x}) \times I(S(\overline{x}) > 0)]$, where $I(\cdot)$ is an indicator function and $S(\overline{x}) = Q - (x_0 + \sum_{i=1}^{N} x_i \times B_i)$. Also let random vector $\overline{B} = [B_1\ B_2\ ...\ B_N]$. We first obtain $\frac{\partial g(\overline{x})}{\partial x_i}$, for $i = 1$ to

N. Given i, define $u = Q - x_0 - \sum_{j \neq i} x_j \times B_j$ and $v = B_i$.[2] Note that $S(\overline{x}) = u - x_i v$. Now,

$$g(\overline{x}) = \int_0^\infty \int_{x_i v}^\infty (u - x_i v) f_{U,V}(u, v) du dv,$$

where $f_{U,V}$ denotes the joint density function of the random variables U and V.

$$\begin{aligned}
\frac{\partial g(\overline{x})}{\partial x_i} &= \frac{\partial}{\partial x_i} \int_0^\infty \int_{x_i v}^\infty (u - x_i v) f_{U,V}(u, v) du dv \\
&= \int_0^\infty \int_{x_i v}^\infty (-v) f_{U,V}(u, v) du dv \\
&= -E[B_i \times I(S(\overline{x}) > 0)]. \qquad (4)
\end{aligned}$$

$\frac{\partial g(\overline{x})}{\partial x_0}$ can be obtained similarly by defining $u = Q - \sum_j x_j \times B_j$.

$$\begin{aligned}
\frac{\partial g(\overline{x})}{\partial x_0} &= \frac{\partial}{\partial x_0} \int_{u=x_0}^\infty (u - x_0) f_U(u) du \\
&= \int_{u=x_0}^\infty (-1) f_U(u) du \\
&= -E[I(S(\overline{x}) > 0)]. \qquad (5)
\end{aligned}$$

In the above, f_U denotes the density function of the random variable U. We next obtain the Hessian of the shortfall constraint, i.e. $\nabla^2 g(\overline{x})$, using a similar approach. First, we find $\frac{\partial^2 g(\overline{x})}{\partial x_k \partial x_i}$, where $k \neq i$ and $k, i \geq 1$. Define $u = Q - x_0 - \sum_{j \neq i,k} x_j \times B_j$, $v = B_k$, $w = B_i$, which have the joint density $f_{U,V,W}(.,.,.)$. Now, $S(\overline{x}) = u - x_k v - x_i w$ and

$$\begin{aligned}
\frac{\partial g(\overline{x})}{\partial x_i} &= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty w I(S(\overline{x}) > 0) f_{U,V,W}() du dv dw. \\
\frac{\partial^2 g(\overline{x})}{\partial x_k \partial x_i} &= \frac{\partial}{\partial x_k} \int_0^\infty w \int_0^\infty \int_{x_k v + x_i w}^\infty f_{U,V,W} du dv dw \\
&= f_{S(\overline{x})}(0) E[B_i B_k | S(\overline{x}) = 0]
\end{aligned}$$

$\frac{\partial^2 g(\overline{x})}{\partial x_0^2}$ can be obtained by defining $u = Q - \sum_{j \neq k} x_j \times B_j$, $w = B_k$, for some k.

$$\begin{aligned}
\frac{\partial^2 g(\overline{x})}{\partial x_0^2} &= -\frac{\partial}{\partial x_0} \int_{-\infty}^\infty \int_{x_0 + x_k w}^\infty f_{U,W}(u, w) du dw \\
&= -\int_{-\infty}^\infty (-1) f_{U,W}(x_0 + x_k w, w) dw \\
&= f_{S(\overline{x})}(0) \int f_{\overline{B}|S(\overline{x})}(\overline{b}|0) d\overline{b} = f_{S(\overline{x})}(0),
\end{aligned}$$

[2]Note that the variables u and v depend on i; we drop the suffix i for simplicity of notation.

where $f_{\overline{B}}$ is the joint density function of the bandwidth return vector $\overline{B}$. Similarly, we can show that: $\frac{\partial^2 g(\overline{x})}{\partial x_0 \partial x_k} = f_{S(\overline{x})}(0) E[B_k | S(\overline{x}) = 0]$ and $\frac{\partial^2 g(\overline{x})}{\partial x_k^2} = f_{S(\overline{x})}(0) E[B_k^2 | S(\overline{x}) = 0]$. Thus, the Hessian of the constraint can be written as,

$$\nabla^2 g(\overline{x}) = f_{S(\overline{x})}(0) \times E[\overline{A}\overline{A}^T | S(\overline{x}) = 0], \tag{6}$$

where $\overline{A} = [1\ B_1\ B_2\ \dots\ B_N]^T$.

Since $f_{S(\overline{x})}(0) \geq 0$ and $E[\overline{A}\overline{A}^T | S(\overline{x}) = 0]$ is positive semi-definite, $\nabla^2 E[S(\overline{x})^+]$ is also positive semi-definite. Therefore, $E[S(\overline{x})^+]$ is convex.

IV. SPO UNDER DEMAND SATISFACTION PROBABILITY (DSP) CONSTRAINT

Next, we study the convexity properties of the SPO problem under the DSP constraint. We first show that the DSP constraint is non-convex, without any assumptions on the distribution of the demand Q and the bandwidth return variables B_i. Later, we present the conditions under which the constraint and therefore the SPO problem becomes convex.

A. Non-convexity of SPO

We present an example where the feasible set of the SPO problem under the DSP constraint (Equation 3) is non-convex. Consider a simple case, when there are two secondary contracts, i.e $N = 2$. Let the B_1 and B_2 be uniformly distributed between 0 and 1. Let Q have a triangular density function given by, $f_Q(q) = 2 \times q, 0 \leq q \leq 1$. Note that $Pr(S(\overline{x}) > 0) = Pr(B(\overline{x}) < Q)$, where $S(\overline{x}) = Q - B(\overline{x})$ and $B(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times B_i$. Consider the portfolio vectors $\overline{x}_1 = (0, 1, 0)$, $\overline{x}_2 = (0, 0, 1)$. We have $Pr(S(\overline{x}_1) > 0) = Pr(B_1 < Q) = \frac{2}{3} = Pr(S(\overline{x}_2) > 0)$. Choose $\epsilon = 0.67$, and denote the feasibility set by $\mathcal{X}_{0.67} = \{\overline{x} : Pr(S(\overline{x}) > 0) < 0.67$. We see that $\overline{x}_1, \overline{x}_2 \in \mathcal{X}_{0.67}$. However, for the convex combination, $\overline{x}_3 = \frac{1}{2} \times \overline{x}_1 + \frac{1}{2} \times \overline{x}_2$, $Pr(S(\overline{x}_3) > 0) = Pr(\frac{1}{2} \times B_1 + \frac{1}{2} \times B_2 < Q) = \frac{17}{24} > 0.67$. That is, $\overline{x}_3 \notin \mathcal{X}_{0.67}$. So, the feasibility set is not convex in general.

B. Conditions for convexity

For a given ϵ, denote the feasibility set (from (3)) by $\mathcal{X}_\epsilon = \{\overline{x} : Pr(S(\overline{x}) > 0) < \epsilon\}$. Using existing literature, we derive sufficient conditions for convexity of the feasibility set $\mathcal{X}_\epsilon$.

Theorem 2: $\mathcal{X}_\epsilon$ is convex if the random vector $\overline{B} = [B_1\ B_2\ \dots B_N]^T$ and the demand Q have log-concave and symmetric density functions, and $0 \leq \epsilon \leq 0.5$.

Proof: We invoke the results from [21] to show this. From [21], we know that the function $Pr(\overline{x}^T a < b)$ is quasi-concave, if the joint density function of the random vector a and the random variable b are log-concave and symmetric. This result readily applies to our case, by rewriting the constraint (Equation 3) as $Pr(-B(\overline{x}) < -Q) \geq 1-\epsilon$. Specifically, the function $Pr(-B(\overline{x}) < -Q)$ is quasi-concave if the joint density of the random vector $\overline{B}$ and Q are log-concave and symmetric. This implies that the feasibility set $\mathcal{X}_\epsilon = \{\overline{x} : Pr(-B(\overline{x}) < -Q) \geq 1-\epsilon\}$, is convex.

Theorem 3: $\mathcal{X}_\epsilon$ is convex if the random vector $-\overline{B} = [B_1 \ B_2 \ ...B_N]^T$ and the demand Q have a joint normal distribution, and $0 \leq \epsilon \leq 0.5$.

Proof: We invoke the results from [24] to show this. From Theorem 3 of [24], we know that the set $\mathcal{X}_\epsilon = \{\overline{x} : Pr(\overline{x}^T a < b) \geq p\}$ is convex, if the density function of the random vector a and the random variable b is jointly normal. This result can be applied by rewriting the DSP constraint (Equation 3) as $Pr(-B(\overline{x}) < -Q) \geq 1-\epsilon$.

We also derive another condition for convexity, which only requires a non-increasing assumption on the distribution function of Q, and none on the bandwidth return variables B_i.

Theorem 4: $Pr(S(\overline{x}) > 0)$ is convex if the CDF of the demand, F_Q, is a concave function.

Proof:

Consider portfolios $\overline{y} = (y_0, y_1, y_2, ..., y_N)$, and $\overline{z} = (z_0, z_1, z_2, ..., z_N)$. Let $\overline{x} = \lambda\overline{y} + (1-\lambda)\overline{z}$. Now,

$$\begin{aligned} Pr(S(\overline{x}) > 0) &= Pr(S(\lambda\overline{y} + (1-\lambda)\overline{z}) > 0) \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b}) P(Q > x_0 + \sum_{i=1}^{N} x_i \times b_i) d\overline{b} \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)) d\overline{b}. \end{aligned}$$

Here F_Q is the distribution function of the demand Q. Now, $x_0 = \lambda y_0 + (1-\lambda) z_0$ and $x_i = \lambda y_i + (1-\lambda) z_i$.

$$\begin{aligned} Pr(S(\overline{x}) > 0) &= \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(\lambda y_0 + (1-\lambda) z_0 + \sum_{i=1}^{N} (\lambda y_i + (1-\lambda) z_i) \times b_i)) d\overline{b}. \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(\lambda(y_0 + \sum_{i=1}^{N} y_i b_i) + (1-\lambda)(z_0 + \sum_{i=1}^{N} z_i \times b_i))) d\overline{b}. \end{aligned}$$

If F_Q is a concave function we get the inequality,

$$\begin{aligned} Pr(S(\overline{x}) > 0) &\leq \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - \lambda \times F_Q(y_0 + \sum_{i=1}^{N} y_i b_i) + (1-\lambda) F_Q(z_0 + \sum_{i=1}^{N} z_i \times b_i))d\overline{b} \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b})(\lambda + 1 - \lambda - \lambda \times F_Q(y_0 + \sum_{i=1}^{N} y_i b_i) + (1-\lambda) F_Q(z_0 + \sum_{i=1}^{N} z_i \times b_i))d\overline{b} \\ &= \lambda \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(y_0 + \sum_{i=1}^{N} y_i b_i))d\overline{b} + (1-\lambda) \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(z_0 + \sum_{i=1}^{N} z_i b_i))d\overline{b} \\ &= \lambda Pr(S(\overline{y}) > 0) + (1-\lambda) Pr(S(\overline{z}) > 0) \end{aligned} \tag{7}$$

Corollary 1: $Pr(S(\overline{x}) > 0)$ is convex if $f_Q^{'} \leq 0$ everywhere.

Proof:

$$\begin{aligned} Pr(S(\overline{x}) > 0) &= Pr(Q > B(\overline{x})) \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b}) P(Q > x_0 + \sum_{i=1}^{N} x_i \times b_i) d\overline{b} \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b})(1 - F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i))d\overline{b}. \end{aligned}$$

Here F_Q is the distribution function of the demand Q. Since $f_{\overline{B}}(\overline{b}) \geq 0$ and independent of $\overline{x}$, we see that $Pr(S(\overline{x}) > 0)$ is convex if $F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)$ is concave in $\overline{x}$ for all $\overline{b}$. The second order derivatives of $F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)$ are given by,

$$\begin{aligned} \frac{\partial^2 F_Q(.)}{\partial x_0^2} = f_Q^{'}, \quad \frac{\partial^2 F_Q}{\partial x_i^2} = f_Q^{'} \times b_i^2, \\ \frac{\partial^2 F_Q}{\partial x_i x_j} = f_Q^{'} \times b_i \times b_j, \quad \frac{\partial^2 F_Q}{\partial x_0 x_i} = f_Q^{'} \times b_i. \end{aligned}$$

For any $\overline{z} \in R^{N+1}$, $\overline{z}^T \nabla^2 F_Q(.) \overline{z} = f_Q^{'}(.) \times (z_0 + \sum_{i=1}^{N} z_i \times b_i)^2$. Therefore, $Pr(S(\overline{x}) > 0)$ and hence $\mathcal{X}_\epsilon$ is convex, if $f_Q^{'}(.) \leq 0$. This concludes the proof.

In the proof above, Q and $\overline{B}$ are assumed to be independent of each other. If not, the sufficient condition for convexity is, $f_{Q|\overline{B}}^{'}(.|\overline{b}) \leq 0$ everywhere, $\forall \overline{b}$. It can be shown that the gradient of the DSP constraint, $Pr(S(\overline{x}) > 0)$, is given by,

$$\frac{\partial Pr(S(\overline{x}) > 0)}{\partial x_0} = -f_{S(\overline{x})}(0)$$

,

$$\frac{\partial Pr(S(\overline{x}) > 0)}{\partial x_k} = -f_{S(\overline{x})}(0) \times E[B_k | S(\overline{x}) = 0].$$

We use the above expressions, when we solve the SPO problem numerically in Section IX-A.

Theorems 2 and 3 covers important distributions such as the Gaussian, log-normal, and the uniform density functions (both $\overline{B}$ and Q must follow some symmetric, log-concave distribution, although they need not be the same distribution). Theorem 4 covers concave and other asymmetric decreasing density functions for Q that are not included in Theorem 2 (the distribution of $\overline{B}$ can be arbitrary).

Remark 1: Let $N = 1$. If Q is deterministic, then the DSP constraint reduces to a linear constraint. In this case, the optimal portfolio consists of entirely primary or entirely secondary contracts. The optimal portfolio is $(Q, 0)$, if $\epsilon < F_{B_1}(p_1)$ and $(0, \frac{Q}{F_{B_1}^{-1}(\epsilon)})$, if $\epsilon >= F_{B_1}(p_1)$, where F_{B_1} is the cumulative distribution function of B_1.

Figure 1a shows the empirical distribution (cumulative) of the total daily traffic of a Verizon Wi-Fi HotSpot network from [25]. Note that the shape of the cumulative distribution function matches well with a concave distribution function also shown in the figure. The concave function used for fitting is $1 - e^{-0.4(x-7.9)}$. Similarly, Figure 1b shows the traffic distribution of a large US-based cellular network (Refer Figure $1a$ of [26]). Here, we see that the cellular traffic distribution can be approximated by a log-normal distribution. From Theorems 2, 3, and 4, we know that the SPO problem is convex if the distribution function for the demand is concave or log-normal. Therefore, we can formulate the SPO problem for empirical distributions as a convex programs and study the nature of optimal portfolio, after approximating the empirical distributions with concave or log-normal distributions.

C. Convex Approximation

In practice, the demand and bandwidth return distributions may not satisfy the properties stated in Theorems 2 and 3, leading to the DSP constraint being non-convex. The SPO problem under the DSP constraint is also non-convex in such cases. However, several approximation techniques (inner as well as outer) have been developed in order to approximate a non-convex probability constraint to a convex contraint. In this paper, we specifically consider the Bernstein approximation technique developed in [22].

Bernstein approximation finds a convex inner approximation to the original probability constraint such that it is computationally tractable. The probability of shortage is upper bounded by the expected value of a (suitably defined) function of the shortage. The (non-convex) probability constraint is then replaced

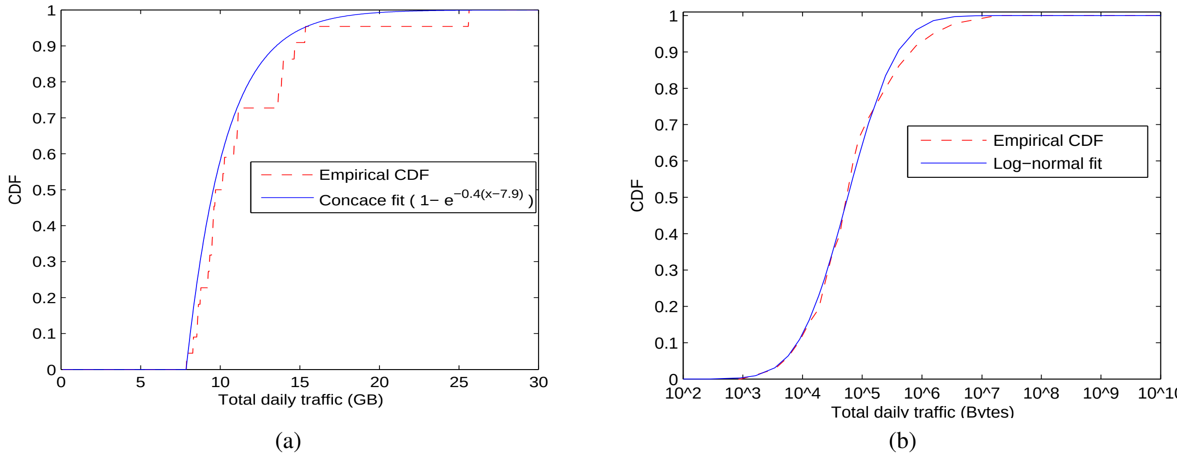


Fig. 1: a) Empirical distribution of traffic from a Wi-Fi hotspot along with a concave fit, b) Empirical distribution of cellular traffic with a log-normal fit.

by a (convex) expectation constraint. The SPO problem under Bernstein approximation can be stated as,

$$\begin{aligned} \underset{x,t>0}{\text{minimize}} \quad & C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i \\ \text{subject to} \quad & \inf_{t>0}[\Psi(x,t) - t\epsilon] \leq 0 \end{aligned}$$

where $\Psi(x,t) = tE\left[\psi(t^{-1}S(\bar{x}))\right]$ and $\psi : \mathcal{R} \rightarrow \mathcal{R}$ is a non-negative valued, non-decreasing convex function (called the *generating function*) such that $\psi(z) > \psi(0) = 1$ for any $z > 0$. We consider two generating functions - a piecewise linear generating function $\psi(z) = [1+z]_+$, and the exponential generating function $\psi(z) = e^z$. Note that ϵ is the bound on demand shortage probability (Equation 3).

V. SPO over Multiple Regions

Spectrum contracts typically come with clauses that restrict the use of the spectrum to certain geographical regions. This could be due to licensing or coverage limitations of the seller provider. For example, a seller provider may only have the license to use a part of the spectrum in certain regions (say certain counties or states in the United States), and not others. Alternatively, the base stations of the seller provider may only cover certain sub-areas of the overall area of interest to the buyer, which can span multiple regions. This adds additional complexity to the SPO problem, since the spectrum portfolio should satisfy the buyer provider's requirements for each of these regions. In this section, we formulate the SPO problem over multiple regions and argue that the results for the single region problem extend to multi-region case as well.

Let us assume that the buyer of spectrum contracts operates over a set of K disjoint geographical regions. The buyer's objective is to construct a portfolio of spectrum contracts in order to satisfy the user demand in each of the K regions. Denote the set of regions by $\mathcal{R}$, i.e, $\mathcal{R} = \{1, 2, .., K\}$. Let there be M primary and N secondary contracts in the market. Let z_i, p_j denote the unit price of i^{th} primary contract and j^{th} secondary contract, respectively. Let $\mathcal{R}_i^p \subset \mathcal{R}, 1 \leq i \leq M$ denote the set of regions in which the i^{th} primary contract is valid. Similarly, let $\mathcal{R}_j^s \subset \mathcal{R}, 1 \leq j \leq N$ denote the set of regions in which the j^{th} secondary contract is valid. The user demand for each region is uncertain, denoted by the random variable Q_k, $1 \leq k \leq K$.

The multi-region SPO problem under DSR constraint can be stated as follows:

$$\text{Minimize} \quad C(\overline{x}) = \sum_{i=1}^{M} y_i \times z_i + \sum_{j=1}^{N} x_j \times p_j, \tag{8}$$

$$E[\{Q_k - \sum_{i \in \mathcal{C}_k^p} y_i - \sum_{j \in \mathcal{C}_k^s} x_j \times B_{jk}\}^+] < \delta_k \quad \forall k, \tag{9}$$

$$E[\sum_{k=1}^{K} \{Q_k - \sum_{i \in \mathcal{C}_k^p} y_i - \sum_{j \in \mathcal{C}_k^s} x_j \times B_{jk}\}^+] < \delta. \tag{10}$$

Here $\{y_1, ..., y_M, x_1, ..., x_N\}$ denotes the spectrum portfolio. $\mathcal{C}_k^p$ and $\mathcal{C}_k^s$ denote the set of primary and secondary contracts that are valid in the k^{th} region ($1 \leq k \leq K$), respectively. $\mathcal{C}_k^p$ and $\mathcal{C}_k^s$ can be obtained from $\mathcal{R}_i^p, 1 \leq i \leq M$ and $\mathcal{R}_i^s, 1 \leq j \leq N$. Note that $\mathcal{C}_k^p \subset \{1, 2, .., M\}$ and $\mathcal{C}_k^s \subset \{1, 2, .., N\}$. The random variable B_{jk} represents the bandwidth return of the j^{th} secondary contract in the k^{th} region. For the multiple region problem, there are totally $K + 1$ inequality constraints; one DSR constraint for each of the K regions and one overall DSR constraint for all the regions. The LHS of the $(K+1)^{th}$ constraint is simply the summation of the LHS of the first K constraints. However, note that $\sum_{k=1}^{K} \delta_k > \delta$, else the last constraint would be redundant; typically, the buyer provider may want have $\delta_k > \delta/K,$ for each k. The motivation of both types of constraints (per-region as well as overall) is as follows. While the buyer provider would be interested in the ensuring a certain DSR over its overall customer base, it may also want to ensure a certain DSR (possibly a smaller normalized DSR than the overall DSR) is ensured in each of its regions of operation, to avoid excessive customer dissatisfaction in each individual region. The SPO problem under DSP constraint can be defined similarly as above, but by replacing the expectation constraints with the corresponding probability constraints, and δ_k and δ by ϵ_k and ϵ, respectively.

For both the SPO problems, we see that the k^{th} constraint ($1 \leq k \leq K$) is similar to the constraint for the single region problem ((2) and (3)) except for the presence or absence of few variables inside the two summations. First, consider the SPO problem under DSR constraint (8-(10)). Let the k^{th} rate constraint be denoted by g_k; g_k involves only some of the y_i and x_j variables. It can be rewritten as,

$$E[\{Q_k - \sum_{1 \leq i \leq M} y_i \times I(i \in C_k^p) - \sum_{1 \leq j \leq N} x_j \times B_j^{'}\}^+] < \delta_k, \tag{11}$$

where $B_j^{'} = B_{jk}$, if $j \in \mathcal{C}_k^s$, else $B_j^{'} = 0$. $I(i \in C_k^p)$ is the indicator function for the set C_k^p. Now, the proof technique for the single-region problem can be readily extended to show that g_k is convex in y_i, x_j. The final constraint (g_{K+1}) is also convex, since it is the sum of several convex functions. Therefore, the feasible set for this problem is convex, since the intersection of several convex sets is convex. Similarly, the feasible set for the multiple-region SPO problem under DSP constraint is also convex, if the density functions of all the random parameters involved are log-concave and symmetric, or the demand variables Q_k have non-decreasing density functions.

VI. SPO under Discrete Portfolio Constraints

In practice, the primary and secondary contracts can be bought and sold only in discrete units. In such scenarios, the SPO problem is represented as a discrete (or integer) program. Despite the discreteness (integrality) requirements in the variables, discrete (integer) programs derived from convex problem can often be solved efficiently [27]. That is however not the case with the SPO problem, however, as we argue in this section. We will first argue that the SPO problem is NP-hard. We then argue that it is not submodular either. These results essentially imply that it is unlikely that an efficient solution to the SPO problem exists when the allocations are constrained to take a discrete set of values.

NP-hardness of discrete-SPO (both under DSR and DSP constraints) can be established by reduction from the NP-hard Knapsack problem. Next, lets us assume that that the portfolio $\overline{x}$ is constrained to be an integer vector. Now let us consider the special case where the return from all secondary contracts, B_i, as well as the customer demand, Q, are deterministic. Then the DSR constraint (2) reduces to $Q - x_0 - \sum_{i=1}^{N} x_i \times B_i < \delta$, or $x_0 + \sum_{i=1}^{N} x_i \times B_i > Q - \delta$. In the same setting, the DSP constraint (3) becomes $x_0 + \sum_{i=1}^{N} x_i \times B_i > Q$. The problem of minimizing the objective in (1) subject to constraint (2) or (3) is then equivalent to the *minimization version* of the unbounded Knapsack problem [28]. In the minimization version of the Knapsack problem, the objective of the standard (maximization version) Knapsack problem is replaced by minimization, and the inequality in the constraint is reversed. Since the minimization version of the Knapsack problem can be transformed into an equivalent maximization version in polynomial time, and the maximization version (standard) unbounded Knapsack problem is known to be NP-hard [29], it follows that the integral versions of the DSR and DSP problems are NP-hard as well.

In the discrete domain, the equivalent of convexity is the submodularity property. For minimization of submodular functions under integrality constraints, efficient algorithms exist (that can attain a solution in pseudo-polynomial time, for example) [27]. However, we show next that the DSR and DSP constraints are not submodular.

Consider a function $g(\overline{x}) : \mathcal{Z}^n \rightarrow \mathcal{R}$. From Theorems 7.7, 7.20, and 7.21 of [27], the function g is submodular if and only it satisfies the discrete midpoint convexity defined below:

$$g(p) + g(q) \geq g(\left\lceil \frac{p+q}{2} \right\rceil) + g(\left\lfloor \frac{p+q}{2} \right\rfloor), \text{ For any p,q} \in \mathcal{Z}^n \tag{12}$$

Theorem 5: The probability of shortfall $Pr(S(\overline{x}) > 0)$ and the expected shortfall $E[S(\overline{x})^+]$ are not submodular.

Proof: We provide counter examples demonstrating that both the probabilty of shortfall as well as expected shortfall violate the discrete midpoint convexity property.

Consider the case where there is a single primary and a single secondary contract. Also, consider the DSR function, $g_1(\overline{x} : E(S(\overline{x}))$. Let the demand Q and the bandwidth return B_1 be deterministic. Note that the DSR constraint is convex for any probability density function. Let $Q = 5$ and $B_1 = 1$.

Consider portfolios $p = (1, 4)$ and $q = (4, 1)$. Now,

$$g_1(p) = E[max(Q - 1 - 4 \times B_1, 0)] = 0,$$

$$g_1(q) = E[max(Q - 4 - 1 \times B_1, 0)] = 0$$

But,

$$g_1(\left\lceil \frac{p+q}{2} \right\rceil) = g_1((3,3)) = E[max(Q - 3 - 3 \times B_1, 0)] = 0,$$

$$g_1(\left\lfloor \frac{p+q}{2} \right\rfloor) = g_1((2,2)) = E[max(Q - 2 - 2 \times B_1, 0)] = 1$$

Thus, we find that the discrete midpoint convexity is violated.

Next, consider the DSP function $g_2(\overline{x} : Pr(S(\overline{x}) > 0)$. As before let B_1 be deterministic with a value of 1. However, let Q be uniformly distributed $\mathcal{U}(0,5)$. These density functions satisfy sufficient conditions for the convexity of the DSP constraint. Now, it can be quickly calculated that,

$$g_2(p) = g_2(q) = 0,$$
$$g_1(\left\lceil \frac{p+q}{2} \right\rceil) = 0, g_1(\left\lfloor \frac{p+q}{2} \right\rfloor) = 0.2$$

Thus, we find the DSP function also violates the discrete midpoint convexity condition for submodularity.

In view of the above negative results on the efficient computability of the discrete-SPO problem, we discuss a branch-and-bound algorithm for solving the problem. The same dynamic problem algorithm is used to compute solution to the SPO problem under discreteness constraints in our evaluation section (Section IX-A). It is worth noting that application of dynamic programming to solve similar problems have been discussed in prior literature. In particular, the integral-SP problem under the DSP constraint is closely related to the *stochastic Knapsack problem*, for which a dynamic programming solution approach is described in [30]. In the stochastic Knapsack problem as discussed in [30], objective and contraint functions are reversed: the non-linear shortage probability function is set as the objective, while the linear cost function constitutes a constraint. In the Appendix, we describe a branch-and-bound algorithm that is tuned to the two versions of the SPO problem that we consider in this paper.

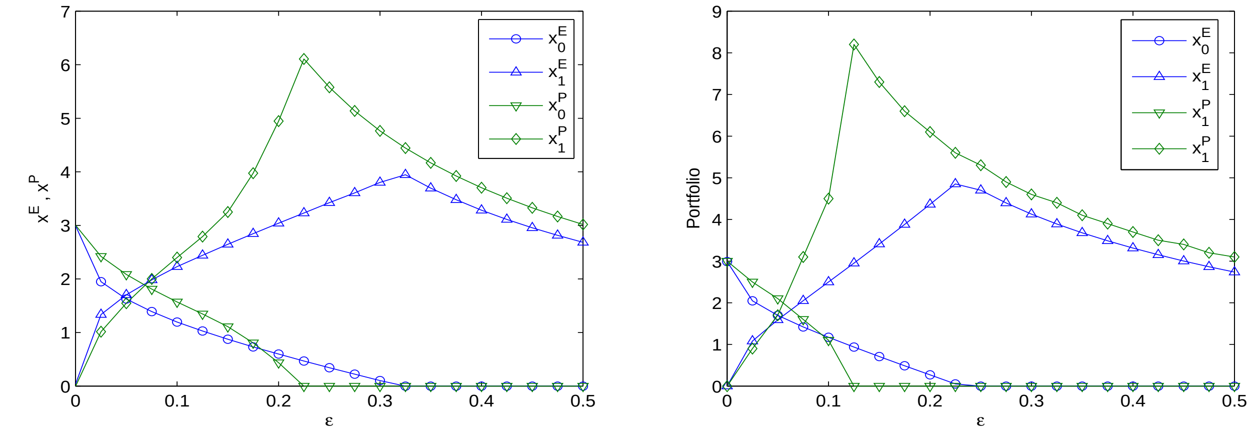
VII. Numerical Evaluation

We numerically solve the SPO problems using Matlab to study the characteristics of the spectrum portfolio in a wide range of scenarios. Our goal is examine how the parameters of the problem, namely, the price of the secondary contracts, the bandwidth return distributions, and the constraints (ϵ, δ) influence the portfolio composition. The results for the single-region SPO problems are presented in sections VII-A and VII-B, while the results for the multiple-region problem are presented in section VII-C.

A. Single Primary and Single Secondary contract

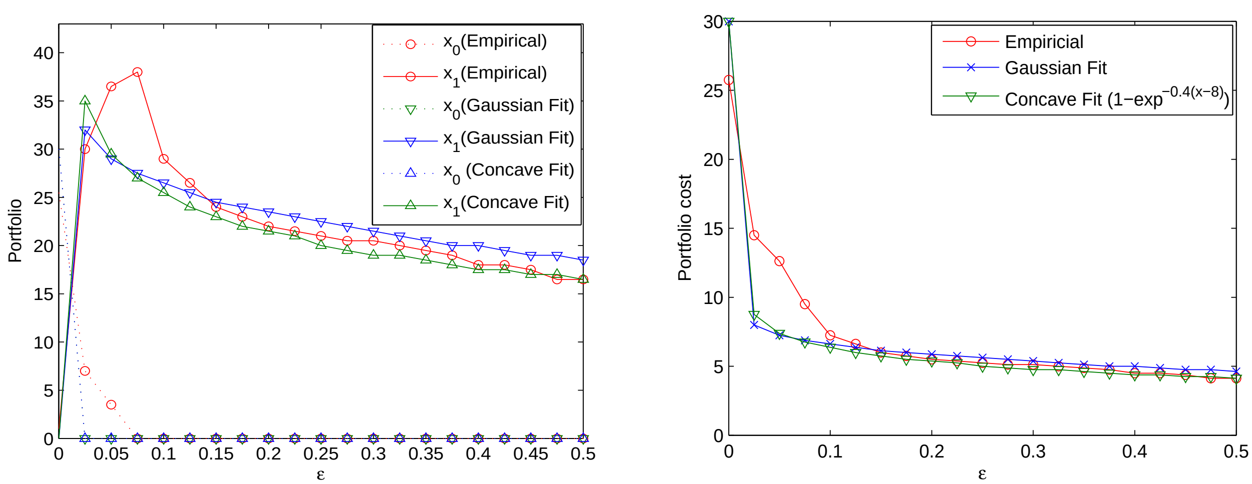
We first consider the simplest case of there being a single secondary contract seller in the market. The bandwidth return B_1 and the demand Q are assumed to have truncated normal distributions. B_1 has a mean of 0.5, while the demand Q has a mean of 1.5. The distribution of Q is restricted to the interval $[0, 3]$. We obtain optimal portfolio when the key parameters of the problem (ϵ, δ, p_1) are changed.

Figure 2a shows the spectrum portfolio composition for different choices of the DSR constraint (δ) and DSP constraint (ϵ), respectively. The unit price of the secondary contract, $p_1 = 0.25$. In the figure,



(a) (b)

Fig. 2: Number of primary (x_0^E, x_0^P) and secondary (x_1^E,x_1^P) contract units in the optimal portfolio for the SPO problem under DSR and DSP constraint. a) Demand and the secondary bandwidth return are independent, b) Demand and the bandwidth return are correlated.



(a) (b)

Fig. 3: Number of primary and secondary contract units in the optimal portfolio under DSP constraint for empirical and fitted distributions.

$\overline{x}^E = \{x_0^E, x_1^E\}$ and $\overline{x}^P = \{x_0^P, x_1^P\}$ denote the portfolios for SPO problems with DSR and DSP constraints, respectively. As expected, when $\delta = \epsilon = 0$, we observe that the portfolio consists of primary contract units only. This is due to the fact that the secondary contracts having stochastic returns introduce bandwidth shortage (or demand violation) even if they are bought in large quantities. Moreover, the number of primary contract units in both the cases is equal to the maximum possible demand (i.e 3). As the constraint (ϵ, δ) is relaxed, we find that the number of primary contract units reduces sharply until it becomes zero. On the other hand the number of secondary contract units (x_1^E, x_1^P) increases initially, but starts decreasing as soon as the number of primary contract units becomes zero. This can be explained as follows: As the constraint (ϵ, δ) is increased from zero, it becomes unnecessary to meet the demand with probability one. Therefore, total cost of the portfolio can be reduced, by reducing the number of primary contract units, while adding the requisite amount of secondary contract units to keep the demand violation below the desired value. This happens until the number of primary contract units becomes zero. Beyond this point, the only way to reduce the cost is to reduce the number of secondary contract units, which can be reduced as ϵ, δ increase.

In Figure 2a, we assume that the primary demand Q and the bandwidth return B_1 are independent. We know that a secondary contract provides access to unused or leftover channels with the seller provider. Therefore, the bandwidth return of the secondary contract would be negatively correlated with the seller's own customer demand. Moreover, the traffic demand of different providers would have similar temporal characteristics. Hence, we can expect negative correlation between the buyer's own demand Q and the bandwidth return B_1. Figure 2b shows the optimal portfolio composition for the SPO problem under DSR and DSP constraint. The degree of correlation between Q and B_1 was set to 0.5. Note that the overall trend in portfolio remains similar Figure 2a. Later in this section, we vary the degree of correlation and observe the changes in portfolio composition.

Empirical distributions: Next, we study the sensitivity of the portfolio composition to changes in the distribution of the demand (Q) and the bandwidth return (B_1). We obtain the empirical distribution of the total daily traffic of a Verizon Wi-Fi HotSpot network from [25] (Refer to Figure 12 of [25]) and consider this distribution for the user demand Q. From this, we compute the distribution of B_1 as $f_{B_1}(b) = f_Q(\beta(1-b))$, $0 \le b \le 1$, since bandwidth availability is related negatively to the user demand (the scaling factor β is used for normalization). The results for SPO problem under DSP constraint are shown in Figure 3a. We also solve the SPO problem with Q modeled as gaussian and exponential concave distribution that approximate the empirical distribution. Refer Figure 1a for the concave disctribution. In both cases, the distribution for B_1 is obtained as as $f_{B_1}(b) = f_Q(\beta(1-b))$. The SPO problem under DSP constraint is convex when distribution of Q is concave. Some small differences notwithstanding, we see that the general trend in the optimal portfolio composition for the empirical and fitted distributions is the same. Additionally, we ran simulations with uniform distribution and observed similar results. Therefore, in the following we only present the results for (truncated) Gaussian distributions.

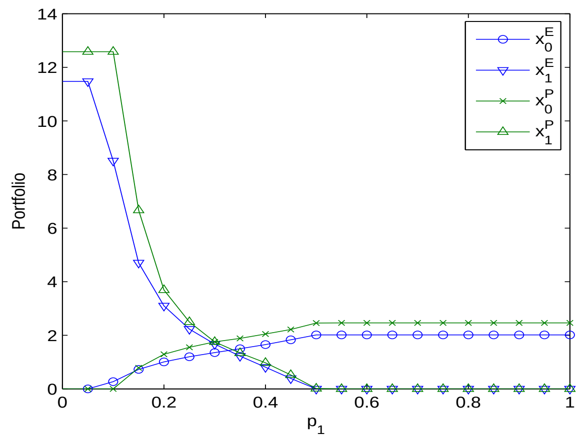


Fig. 4: Optimal spectrum portfolio composition for different choices of the unit price of secondary contract.

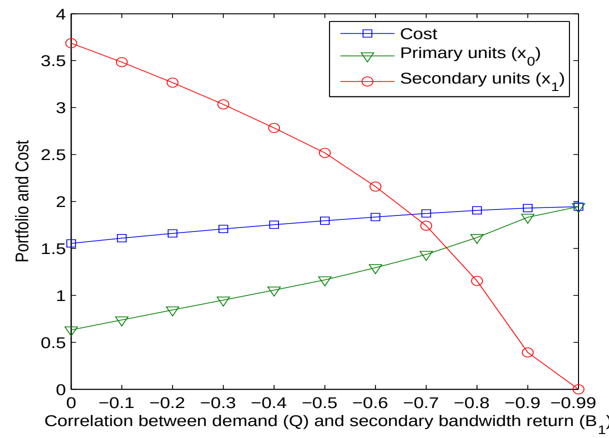


Fig. 5: Correlation between demand and secondary return.

Effect of Correlation: Figure 5 shows the effect of correlation (negative) between the demand Q and the bandwidth return B_1 of the single secondary contract. Q and B_1 have truncated and jointly gaussian distribution. We obtain the portfolio composition for increasing negative correlation between Q and B_1. When the correlation is high, the bandwidth available from the secondary contract tends to be low with high probability when the demand Q is high. This increases the possibility of shortage. Therefore, we find that the amount of primary units in the portfolio increases, as they are risk-free.

ϵ	0.1	0.2	0.3	0.4	0.5
x_0^A	14	12	10	12	8
x_0^O	4	10	6	2	0

(a) Primary units

ϵ	0.1	0.2	0.3	0.4	0.5
x_1^A	0	2	4	0	6
x_1^O	16	2	8	14	16

(b) Secondary units

ϵ	0.1	0.2	0.3	0.4	0.5
$Pr(S(\overline{x}) > 0)$	0	0.13	0.14	0.14	0.18
$Cost^A$	14	13	12	12	11
$Cost^O$	12	11	10	9	8
% Dev	17	18	20	33	37

(c) Total cost

TABLE I: SPO problem solutions with and without Bernstein approximation to the DSP constraint, $p = 0.5$

Convex Approximation: Next, we numerically evaluate the performance of the Bernstein approximation in cases where the DSP constraint is non-convex. For this study, we consider a single primary and a single secondary contract. The demand and the bandwidth return have empirical distributions discussed earlier. The disitributions do not satisfy Theorems 2 and 3 and hence the DSP constraint may not be convex. We solve the Bernstein approximation (Section IV-C) to the SPO problem and obtain the optimal portfolio (x_0^A, x_1^A) and cost $(Cost^A)$. We also solve the original problem (without any approximations) through brute-force search (Equation 1 and 3) and obtain the solution $((x_0^O, x_1^O), Cost^O)$. The results are summarized in Table I. We only present the results for the linear generating function, since we observed better approximation using the linear generating function than the exponential generating function. Tables Ia, Ib, and Ic, show respectively, the primary units (x_0), the secondary units (x_1), and the portfolio cost $(Cost)$ of the two solutions. From the percentage deviation values shown in Table Ic, we observe that the cost of the approximate solution is within $20-40\%$ of the optimum in most cases. However, Tables Ia and Ib show that the portfolio composition could be significantly different for some values of ϵ. The price of secondary contract was chosen close to the mean bandwidth return (0.61) under the empirical distribution.

$\hat{Q}$	$x_0^{Int} - x_0^*$	$x_1^{Int} - x_1^*$	Cost (% dev)
2	0	0.4552	8.2095
4	0	0.2456	1.9256
8	0.0391	0.2029	1.2545
16	0.0393	0.2022	0.5927
32	0.0393	0.2025	0.2886
64	0.0392	0.2028	0.1423
128	0.0391	0.2030	0.0707

(a) Demand is determinstic

$\overline{Q}$	$x_0^{Int} - x_0^*$	$x_1^{Int} - x_1^*$	Cost (% dev)
2	0	0.4621	10.1825
4	0.2954	-0.7026	3.5430
8	1.1720	-3.9428	2.3836
16	-0.1282	1.3377	1.2468
32	1.2444	-4.0581	0.6808
64	0.0559	0.6854	0.3354
128	-0.0441	0.2946	0.0219

(b) Demand is random

TABLE II: SPO under DSR constraint and discrete portfolio assumption

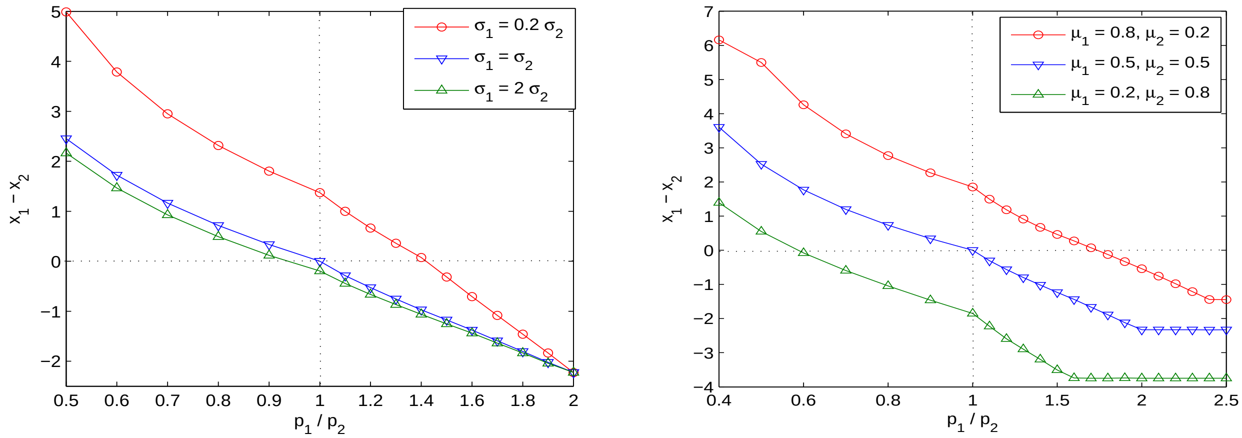
Discrete Constraints: Finally, we solve the SPO problem under discrete constraints using the branch and bound algorithm discussed in Appendix. We obtain the optimal portfolio under integral portfolio constraints and compare it with the non-integer optimum. Table II shows the results for the DSR constraint. For the results shown in Table IIa, the demand is assumed to be deterministic quantity $(\hat{Q})$, but the bandwidth return B_1 is a gaussian variable. And for the results shown in Table IIb, both the demand and the bandwidth return B_1 are gaussian random variables. In Table IIa, we increase $\hat{Q}$ and in Table IIb, we increase the mean demand $\overline{Q}$. Note that the portfolio increases in both cases. We did not observe any trend in the difference between the integer and the non-integer optimal portfolio solutions for increasing values of $\hat{Q}$ or $\overline{Q}$. However, the percentage difference in cost due to integral restrictions seems to reduce in both cases. Therefore, when the portfolios are large, integral restrictions on portfolio does not affect the optimal portfolio cost significantly.

B. Single Primary and Two Secondary contracts

We next consider two types of secondary contracts and study how the price and bandwidth return characteristics of a contract affects the choice of the secondary contract. We only present results on the SPO problem under the DSR constraint, as the results for the DSP constraint are broadly similar in nature. As before, the demand has normal distribution between 0 and 3. The price of the single primary contract is 1. The bandwidth returns of the two secondaries, B_1 and B_2, have normal distribution between 0 and 1, but with different mean and variance.

We obtain the optimal portfolio $\overline{x} = \{x_0, x_1, x_2\}$ as the ratio of the unit prices of the two secondaries, i.e. $\frac{p_1}{p_2}$, is increased. The results are shown in Figures 6a and 6b. For the results shown in Figure 6a, B_1 and B_2 have same mean (of 0.5) but different variances. Figure 6a shows $x_1 - x_2$ as $\frac{p_1}{p_2}$ is increased from 0.5 to 2. Each of the three curves corresponds to a fixed choice of the variance (σ_1, σ_2) of the bandwidth returns. Consider the curve corresponding to the variance choice $\sigma_1 = 0.2\sigma_2$. We find that $x_1 - x_2 > 0$, until $\frac{p_1}{p_2} \leq 1.4$. This implies that the contribution of the first secondary contract units to the overall portfolio is higher than that of the second contract even if the unit price of the first contract is higher than the unit price of the second contract. This is clearly due to the fact that B_1 has lesser variance than B_2. However, if $\frac{p_1}{p_2} > 1.4$, the second contract units are more, since it is much lesser priced. On the other hand, when $\sigma_1 = 2\sigma_2$, the first secondary contract is preferred over second contract, only if it costs lesser than the second contract. These results suggest that secondary contracts that have lower variance of bandwidth return can be priced higher than those with higher variances, provided they have the same mean bandwidth return. Moreover, it was observed that the portfolio consisted of non-zero units of both the secondary contracts for price ratios shown, i.e. $x_1 \neq 0, x_2 \neq 0,$ for $0.5 \leq \frac{p_1}{p_2} \leq 2$. This suggests that it is cost efficient to buy a mix of secondary contract units from multiple sellers, instead of just one, provided their prices are not very different. x_1 or x_2 became zero only when $\frac{p_1}{p_2}$ is either too high or too low, respectively.

Next, consider Figure 7. Although the two secondary contracts are sold by different providers, we can expect positive correlation between their returns B_1 and B_2. When the coustomer demand for one seller provider tends to low, it is likely that the demand for other provider is also low. Hence, the amount of secondary units available from the two providers would be comparable. The optimal portfolio and cost for different levels of correlation is shown in Figure 7. As the correlation between the secondary returns, B_1 and B_2 increase, the two secondary contracts behave like a single secondary contract, but with increased



(a) (b)

Fig. 6: a)Relative contribution of the two secondary contract units as the ratio of the unit prices of the two secondary contracts is increased. Each curve corresponds to a fixed choice of the variance of B_1 and B_2, b) Relative contribution of the two secondaries. Each curve corresponds to a fixed choice of the mean of B_1 and B_2.

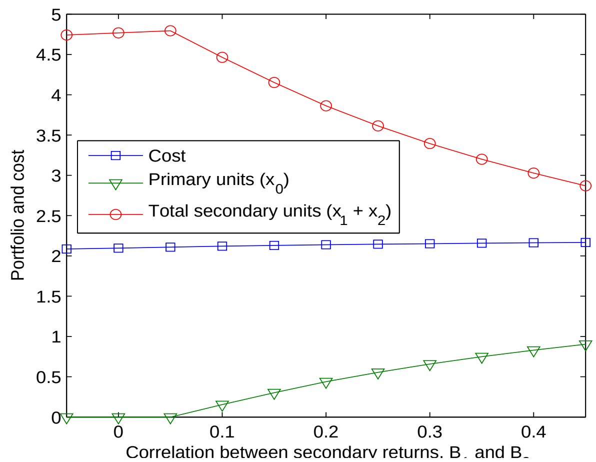


Fig. 7: Correlation between secondary returns.

variance in the banwidth return. Due to increased riskiness in the bandwidth available from secondary contracts, the portfolio shifts to those with increased primary units.

Figure 6b shows $x_1 - x_2$ vs $\frac{p_1}{p_2}$, for different choices of means of B_1 and B_2, keeping the variance fixed at 0.1. When $\mu_1 = 0.8$ and $\mu_2 = 0.2$, we find that $x_1 - x_2 > 0$ as long as $\frac{p_1}{p_2} \leq 1.75$. That is, the secondary contract with 4 times higher mean bandwidth return is preferred even at 75% higher price. We also observe that the secondary contract with lesser mean is preferred only if it has lower price (For the curve with $\mu_1 = 0.2, \mu_2 = 0.8$, $x_1 - x_2 > 0$ only for $\frac{p_1}{p_2} \leq 0.6$). Figures 6a and 6b suggest that the mean as well as the variance of the bandwidth return of a secondary contract play important roles in determining the unit price of the secondary contract.

C. Multiple Regions

For the multiple-region problem, we consider two simulation scenarios. In the first scenario (Scenario A), there are totally K regions, $K+1$ primary contracts, and $K+1$ secondary contracts. The i^{th} primary and secondary contract, where $1 \leq i \leq K$, is valid in the i^{th} region only. In other words, the first K primary and secondary contracts are single-region contracts each valid in one of the K regions. However, the $K+1^{th}$ primary and secondary contract is valid over all the K regions. The first K secondary contracts are identical in terms of their bandwidth return distributions and unit prices. The prices of all the single-region secondary contracts, $p_1, p_2.., p_K$, are set to 1. We examine the composition of secondary contract units in the optimal portfolio, when the price of the K-region secondary contract, i.e. p_{K+1}, changes. The bandwidth return variables (B_i, $1 \leq i \leq K+1$) follow truncated normal distribution with mean 0.5 and variance 0.25. The prices of all the primary contracts is set to a large value such that the portfolio consists of only secondary contract units.

Figure 8a shows the simulation results for Scenario A when $K = 2$ (the effect of larger values of K is considered later). We only show the results for the multiple-region SPO problem under DSR constraints, since the results were similar for the DSP constraint. It was observed that the total number of primary contract units is zero as expected, i.e. $y_1 = y_2 = .. = y_{K+1} = 0$. Moreover, all the single-region secondary contracts contributed equal units to the portfolio, i.e. $x_1 = x_2 = .. = x_K$. Therefore, we plot x_{K+1} and x_1 for different price ratios $\frac{p_{K+1}}{p_1}$, where $K = 2$. When the price ratio $\frac{p_{K+1}}{p_1} < 2$, we find the portfolio consists of higher quantity of $(K+1)^{th}$ secondary contract units compared to the single-region secondary contract, i.e. $x_{K+1} > x_1$. However, when $\frac{p_{K+1}}{p_1} \geq 2$, single-region secondary contracts are preferred over the K-region contracts ($x_{K+1} < x_1$).

In the second scenario (Scenario B), we consider four geographical regions. There are four single-region and four K-region contracts of both primary and secondary type ($K > 1$). Each K-region contract covers K regions out of the four regions, symmetrically. Unlike the previous simulation setup, we have multiple K-region contracts in this setup and there is overlap in the regions covered by these contracts. We fix the price of each single-region secondary contract (p_1) to 1. All the K-region secondary contracts have the same price denoted by p_K. We again increase the price of the K-region secondary contract p_K and observe the optimal portfolio.

The simulation results for Scenario B is shown in Figure 8b for $K = 2$. The amount of K-region secondary contract units x_K and the single-region secondary contract units x_1 in the optimal portfolio are

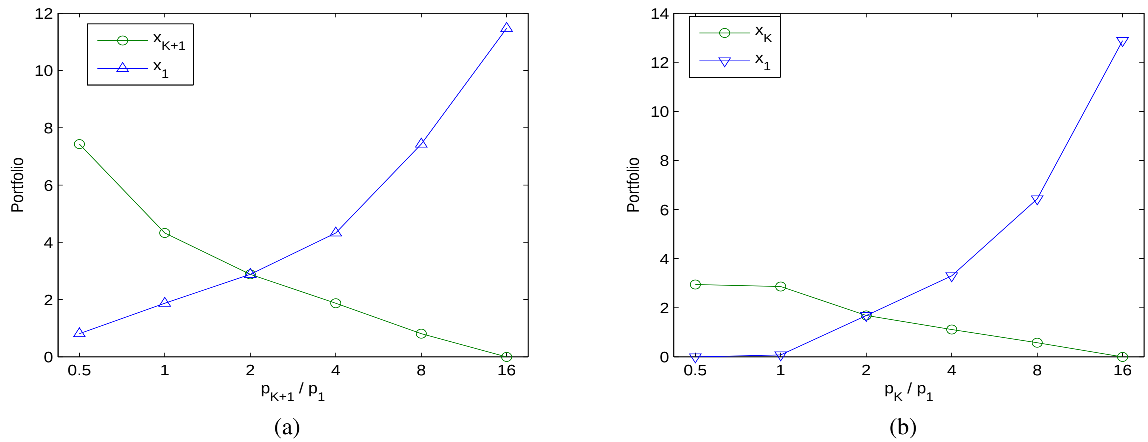


Fig. 8: Optimal portfolio composition when the ratio of the unit price of the K-region secondary contract to that of the single-region secondary contract is increased; (a) Scenario A (b) Scenario B.

shown. We again observe similar behaviour in x_K and x_1 when compared to Figure 8a. That is, K-region secondary contracts have higher weightage in the optimal portfolio whenever the price ratio $\frac{p_{K+1}}{p_1} \leq 2$. Figures 8a and 8b show that to compete fairly in the market, the 2-region secondary contracts can be priced twice of that of the single-region contracts. This "pricing advantage" of the multi-region contracts is not undue however, as they cover twice the area of the single-region contracts. In general, offering spectrum contracts over a larger area implies larger licensing cost for the seller provider; moreover, the infrastructure investment and operational costs that the seller provider incurs will also be proportional to the area covered.

Next, we consider Scenario A and solve the SPO problem for higher values of K. Figure 9 shows the simulation results for $K = 2, 3, 4$. We now plot $x_{K+1} - x_1$ for different values of the price ratio $\frac{p_{K+1}}{p_1}$. For each K, when the price ratio $\frac{p_{K+1}}{p_1} < K$, we find the portfolio consists of higher quantity of $(K+1)^{th}$ secondary contract units compared to the single-region secondary contract, i.e. $x_{K+1} - x_1 > 0$. However, when $\frac{p_{K+1}}{p_1} \geq K$, the single-region secondary contracts are preferred over the K-region contract ($x_{K+1} - x_1 < 0$, if $\frac{p_{K+1}}{p_1} \geq K$). Therefore, we find that the provider (seller) of K-region secondary contract can scale up its price upto a factor of K and still enjoy preference over the single-region contracts offered by smaller providers. This happens due to the fact that the provider can either buy one unit of the K-region secondary contract or one unit from each of the K single-region secondary contracts to provide the same service over the K regions at the same cost. The portfolio shifts completely in favor of single-region contracts only when the price, p_{K+1}, is too high. For the above choice of parameters, x_{K+1} became zero when $\frac{p_{K+1}}{p_1} > 12, 18$, and 24, respectively, for $K = 2, 3$, and 4. That is, when the price of K-region

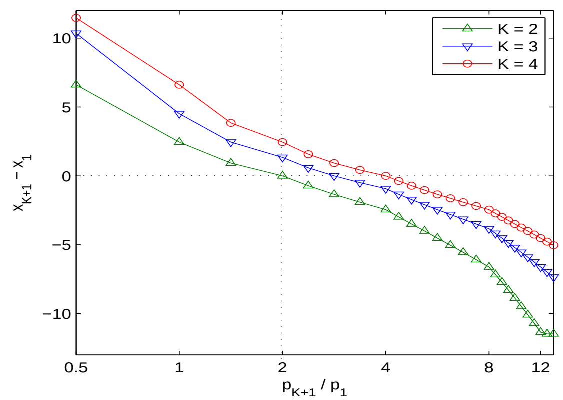


Fig. 9: Optimal portfolio composition for Scenario A when the ratio of the unit price of the K-region secondary contract to that of the single-region secondary contract is increased.

secondary contract is roughly $6K$ times (or higher) the price of the single-region secondary contract, the portfolio no longer consists of K-region secondary contract units.

VIII. Conclusion

In this paper, we have proposed a secondary spectrum market with two types of spectrum contracts – primary and secondary – and formulated the spectrum portfolio optimization (SPO) problem in this context. The two types of contracts vary in their risk-return tradeoffs, as well as their prices, and allows buyers (local or small regional providers, for example) to balance their cost with customer satisfaction level. We provide results and conditions on the convexity of the SPO problem, under both demand satisfaction rate (expectation) and demand satisfaction probability constraints. Convexity of the problems allows us to compute the optimal portfolios efficiently; we also provide expressions for the gradient that can be used for this purpose. We have also shown that convexity of the demand satisfaction constraint implies convexity of the efficient frontier, for both types of constraints. These results naturally extend to scenarios where the contracts are associated with a spatial dimension, and each contract can only provide coverage to a certain set of regions (which can differ across contracts). The convexity properties however do not extend to the integer programming formulation, where the spectrum contracts can be bought/sold only in discrete units. We have shown that the SPO problem under such constraints is not submodular.

We have used our formulation and results to compute and study the properties of optimal spectrum portfolio in a wide range of simulation scenarios. Numerical experimentation with truncated gaussian, uniform, and empirically obtained distributions (of bandwidth availability and subscriber demand), have shown that the general nature of the variations in the optimal portfolio structure and cost, with respect to variations in key parameters like prices and customer satisfaction levels, remain similar across distributions. The composition of the optimal spectrum portfolio is also strongly influenced by the relative prices of the primary and secondary contracts, and in the multi-region case, the relative prices of the single-region and multi-region contracts. Finally, the discrete SPO problem is solved efficiently using a branch and bound algorithm to show that integrality restrictions on the portfolio does not affect the optimal portfolio cost significantly when the portfolios are large.

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IX. Appendix

A. Branch and Bound Algorithm for DSR and Convex DSP under discrete portfolio constraints

In this appendix, we provide a branch-and-bound algorithm to solve the DSR and the convex DSP problem under discrete portfolio constraints, which was also used in Section to evaluate our solutions under discreteness constraints on the spectrum portfolio. The basic idea behind branch-and-bound algorithms for solving mixed-integer non-linear programs (MINLP), is to relax the integrality restrictions on the original problem. If the solution to the relaxed problem is integral, then this is solution to original problem. However, if some variables (say y) are non-integers, then two sub-problems are created by adding bounds $y \leq [y]$ and $y \geq [y] + 1$ (where $[y]$ is the largest integer not greater than y). The process is repeated until an integer solution is found. For convex MINLPs, several global optimization algorithms have been proposed. In this paper, we implement the algorithm proposed in [31] by adapting it to our problem context. In [31], the authors solve a sequence of quadratic programming problems before branching; it is not required that the quadratic problems at the intermediate stages be solved optimally. The SPO problem (when the discreteness constraints are relaxed) can be stated as,

$$P\colon \quad \text{minimize} \quad C(\overline{x}) \tag{13}$$

$$\begin{aligned} \text{subject to:} \quad g(\overline{x}) - \alpha &\leq 0, \\ \overline{x} &\in \mathcal{S}, \end{aligned}$$

where $C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i$ is the porfolio cost, and $g(\overline{x})$ represents expected shortfall ($E[S(\overline{x})^+]$) or probability of shortfall ($Pr(S(\overline{x}) > 0)$), for the DSR and DSP problems, respectively. $\alpha = \delta$ for the DSR problem and $\alpha = \epsilon$ for the DSP problem. The set $\mathcal{S}$ does not take into account the discrete (integrality) assumptions on the portfolio.

Then the branch and bound algorithm requires solving the following quadratic program (QP^k) iteratively, so as to divide the original problem into sub-problems:

$$QP^k\colon \quad \text{minimize} \quad C^{(k)} + \nabla C^{(k)^T}\overline{d} + \frac{1}{2}\overline{d}^T W^{(k)}\overline{d} \tag{14}$$

$$\begin{aligned} \text{subject to:} \quad g^{(k)} + \nabla g^{(k)^T}\overline{d} &\leq 0, \\ C^{(k)} + \nabla C^{(k)^T}\overline{d} &\leq U_{bb} - \theta, \\ \overline{x}^{(k)} + \overline{d} &\in \hat{\mathcal{S}}, \end{aligned}$$

where $C^{(k)} = C(\overline{x}^{(k)})$ and $W^{(k)} = \nabla^2 C^{(k)} + \lambda \nabla^2 g^{(k)}$. $\hat{\mathcal{S}}$ denotes the feasibility set after adding bounds for non-integral solution variables. U_{bb} denotes current uppper bound on the objective function and θ denotes the optimality tolerance of the branch-and-bound algorithm. Intially, U_{bb} is set to ∞ and updated whenever an integer solution is found at an intermediate step. QP^k is solved to obtain the increment $\overline{d}$ from the current solution ($\overline{x}^{(k)}$).

For the SPO problem, $\nabla C^{(k)} = [1 \ p_1 \ p_2 ... \ p_N]^T$ and $\nabla^2 C^{(k)} = 0$. For the DSR constraint, the gradient ($\nabla g^{(k)}$) and the hessian ($\nabla^2 g^{(k)}$) can be found using Equations (5), (4), and (6). For the DSP constraint, the gradients can be written as,

$$\begin{aligned}\frac{\partial Pr(S(\overline{x})>0)}{\partial x_0} &= -f_{S(\overline{x})}(0)\\ \frac{\partial Pr(S(\overline{x})>0)}{\partial x_k} &= -f_{S(\overline{x})}(0)\times E[B_k|S(\overline{x})=0]\\ &= -\int w\times f_{U,W}(x_0+x_kw,w)dw,\end{aligned}$$

where $U = Q - \sum_{j\neq k} x_jB_j$. The Hessian of the probability of shortfall, $Pr(S(\overline{x})>0)$, can be computed (approximately) numerically, using the above equations.