Spring, 2006   ESE 601

Hybrid Systems

Homework 1

Problem 1 (20 pts)
Consider a linear autonomous system as: \( \dot{x} = Ax \) \( x \in \mathbb{R}^n \) Let \( V \) be subspace of \( \mathbb{R}^n \). The following suggests a recursive algorithm for finding the largest subspace \( U \subseteq V \) such that \( AU \subseteq U \)

- \( U_0 = V \)
- \( U_{i+1} = U_i \cap A^{-1}U_i \) where \( A^{-1}U = \{ x \mid Ax \in U \} \)

Prove that:

1. The iteration terminates after finite step means that there exist \( N \) such that for all \( i \geq N \) \( U_{i+1} = U_i \). Hint: Prove that \( U_i, i \geq 1 \), are linear spaces and consider their dimension.

2. The fix point of the iteration is invariant under \( A \) means that \( AU_N \subseteq U_N \)

3. \( U_N \) is the largest invariant subspace i.e for all subspaces \( W \subseteq V \), if \( AW \subseteq W \) then we have \( W \subseteq U_N \)

Problem 2 (15 pts)
Consider three languages \( L_1, L_2 \) and \( L_3 \) and assume that \( L_1 \) does not contain the empty string. Show that if \( L_2 = L_1L_2 \cup L_3 \) then \( L_2 = L_1^*L_3 \)

Problem 3 (20 pts)
Let \( D = (Q,A,\rightarrow,Q_0,Q_m) \) be a deterministic finite state automaton where \( Q_0 = \{ q_0 \} \). Given \( a = a_0a_1...a_{n-1} \in L(D) \) and execution of \( D \) as \( q_0a_0q_1a_1...q_n \) we define \( \delta(q_0,a) = q_n \). We call a state \( p \in Q \) is reachable iff there is some string \( w \in L(D) \), such that \( \delta(q_0,w) = p \), i.e., there is some path from \( q_0 \) to \( p \) in \( D \). Consider the following method for computing the set \( Q_r \) of reachable states (of \( D \)): define the sequence of sets \( Q_r^i \subseteq Q \), where

\[
Q_r^0 = \{ q_0 \},
\]

\[
Q_r^{i+1} = \{ q \in Q \mid \exists p \in Q_r^i, \exists a \in \Sigma, q = \delta(p,a) \}.
\]

(i) Prove by induction on \( i \) that \( Q_r^i \) is the set of all states reachable from \( q_0 \) using paths of length \( i \) (where \( i \) counts the number of transitions).

Show that it is generally false that there is an index \( i_0 \), such that \( Q_r^{i_0+1} = Q_r^{i_0} \), by giving a counter-example.
(ii) Show that $Q_{i_0}^0 = Q_r$ for some $i_0$ is generally false, by giving a counter-example.

(iii) Change the inductive definition of $Q_i^r$ as follows:

$$Q_{i+1}^{r+i} = Q_i^r \cup \{ q \in Q \mid \exists p \in Q^i_r, \exists a \in \Sigma, q = \delta(p, a) \}.$$

Prove that there is a smallest integer $i_0$ such that

$$Q_{i_0}^1 = Q_{i_0}^0 = Q_r.$$

**Problem 4 (20 pts)**

Consider the discontinuous differential equation

$$\dot{x}_1 = -\text{sgn}(x_1) + 2\text{sgn}(x_2)$$

$$\dot{x}_2 = -2\text{sgn}(x_1) - \text{sgn}(x_2)$$

where $x(0) \neq (0,0)$, and

$$\text{sgn}(z) = \begin{cases} 
1 & \text{if } z > 0 \\
-1 & \text{if } z < 0 \\
\text{undefined} & \text{otherwise}
\end{cases}$$

This system defines a hybrid automaton with four discrete modes having invariants corresponding to the four quadrants.

(a) Specify a deterministic hybrid automaton modeling the system.

(b) Prove that H has Zeno execution for every initial state, and compute the Zeno time as a function of the initial condition $(x_{1,0}, x_{2,0})$.

**Problem 5 (25 pts)**

Consider the discontinuous differential equation

$$\dot{x}_1 = -\text{sgn}(x_1)$$

$$\dot{x}_2 = -x_2$$

(a) Show that this system has a livelock.

(b) Fix an initial condition $(x_{1,0}, x_{2,0}) = (1,1)$. Simulate the execution of the system with forward Euler method (see 1.2.7 in (R1)), for time $0 \leq t \leq 5$ with time steps $h = 0.1$, $0.05$, $0.01$. Provide a plot showing the results of all three simulations.

(c) We introduce a sliding mode in the system by redefining

$$\text{sgn}(z) = \begin{cases} 
1 & \text{if } z > 0 \\
-1 & \text{if } z < 0 \\
0 & \text{if } z = 0
\end{cases}$$

Compute the execution of the system by hand for the given initial condition. Give a comment on the relation between this solution, and the results of the simulations in (b).

TOTAL: 100 points.