Spring, 2006 ESE 601

Hybrid Systems

Homework 1

Hand-out : 6 Feb 2006, Hand-in : 20 Feb 2006

Problem 1 (20 pts)
Consider a linear autonomous system as: $\dot{x} = Ax \;\; x \in \Re^n$ Let V be subspace of $\Re^n$. The following suggests a recursive algorithm for finding the largest subspace $U \subseteq V$ such that $AU \subseteq U$

- $U_0 = V$
- $U_{i+1} = U_i \cap A^{-1}U_i$ where $A^{-1}U = \{x \mid Ax \in U\}$

Prove that:

1. The iteration terminates after finite step means that there exist N such that for all $i \geq N$ $U_{i+1} = U_i$. Hint: Prove that $U_i, i \geq 1$, are linear spaces and consider their dimension.
2. The fix point of the iteration is invariant under A means that $AU_N \subseteq U_N$
3. U_N is the largest invariant subspace i.e for all subspaces $W \subseteq V$, if $AW \subseteq W$ then we have $W \subseteq U_N$

Problem 2 (15 pts)
Consider three languages L_1, L_2 and L_3 and assume that L_1 does not contain the empty string. Show that if $L_2 = L_1L_2 \bigcup L_3$ then $L_2 = L_1^*L_3$

Problem 3 (20 pts)
Let $D = (Q, A, \rightarrow, Q_0, Q_m)$ be a deterministic finite state automaton where $Q_0 = \{q_0\}$. Given $a = a_0a_1...a_{n-1} \in L(D)$ and execution of D as $q_0a_0q_1a_1...q_n$ we define $\delta(q_0, a) = q_n$. We call a state $p \in Q$ is reachable iff there is some string $w \in L(D)$, such that $\delta(q_0, w) = p$, i.e., there is some path from q_0 to p in D. Consider the following method for computing the set Q_r of reachable states (of D): define the sequence of sets $Q_r^i \subseteq Q$, where

$Q_r^0 = \{q_0\}$,

$Q_r^{i+1} = \{q \in Q \mid \exists p \in Q_r^i, \exists a \in \Sigma,\; q = \delta(p, a)\}$.

(i) Prove by induction on i that Q_r^i is the set of all states reachable from q_0 using paths of length i (where i counts the number of transitions).

Show that it is generally false that there is an index i_0, such that $Q_r^{i_0+1} = Q_r^{i_0}$, by giving a counter-example.

(ii) Show that $Q_r^{i_0} = Q_r$ for some i_0 is generally false, by giving a counter-example.

(iii) Change the inductive definition of Q_r^i as follows:

$$Q_r^{i+1} = Q_r^i \cup \{q \in Q \mid \exists p \in Q_r^i, \exists a \in \Sigma,\ q = \delta(p,a)\}.$$

Prove that there is a smallest integer i_0 such that

$$Q_r^{i_0+1} = Q_r^{i_0} = Q_r.$$

Problem 4(20 pts)
Consider the discontinuous differential equation

$$\dot{x_1} = -sgn(x_1) + 2sgn(x_2)$$
$$\dot{x_2} = -2sgn(x_1) - sgn(x_2)$$

where $x(0) \neq (0,0)$, and

$$sgn(z) = \begin{cases} 1 \ \ if\ z > 0 \\ -1 \ \ if\ z < 0 \\ undefined \ \ otherwise \end{cases}$$

This system defines a hybrid automaton with four discrete modes having invariants corresponding to the four quadrants.
(a) Specify a deterministic hybrid automaton modeling the system.
(b) Prove that H has Zeno execution for every initial state, and compute the Zeno time as a function of the initial condition $(x_{1,0}, x_{2,0})$.

Problem 5(25 pts)
Consider the discontinuous differential equation

$$\dot{x_1} = -sgn(x_1)$$
$$\dot{x_2} = -x_2$$

(a) Show that this system has a livelock.
(b) Fix an initial condition $(x_{1,0}, x_{2,0}) = (1,1)$. Simulate the execution of the system with forward Euler method (see 1.2.7 in (R1)), for time $0 \leq t \leq 5$ with time steps $h = 0.1,\ 0.05,\ 0.01$. Provide a plot showing the results of all three simulations.
(c) We introduce a sliding mode in the system by redefining

$$sgn(z) = \begin{cases} 1 \ \ if\ z > 0 \\ -1 \ \ if\ z < 0 \\ 0 \ \ if\ z = 0 \end{cases}$$

Compute the execution of the system by hand for the given initial condition. Give a comment on the relation between this solution, and the results of the simulations in (b).
TOTAL: 100 points.