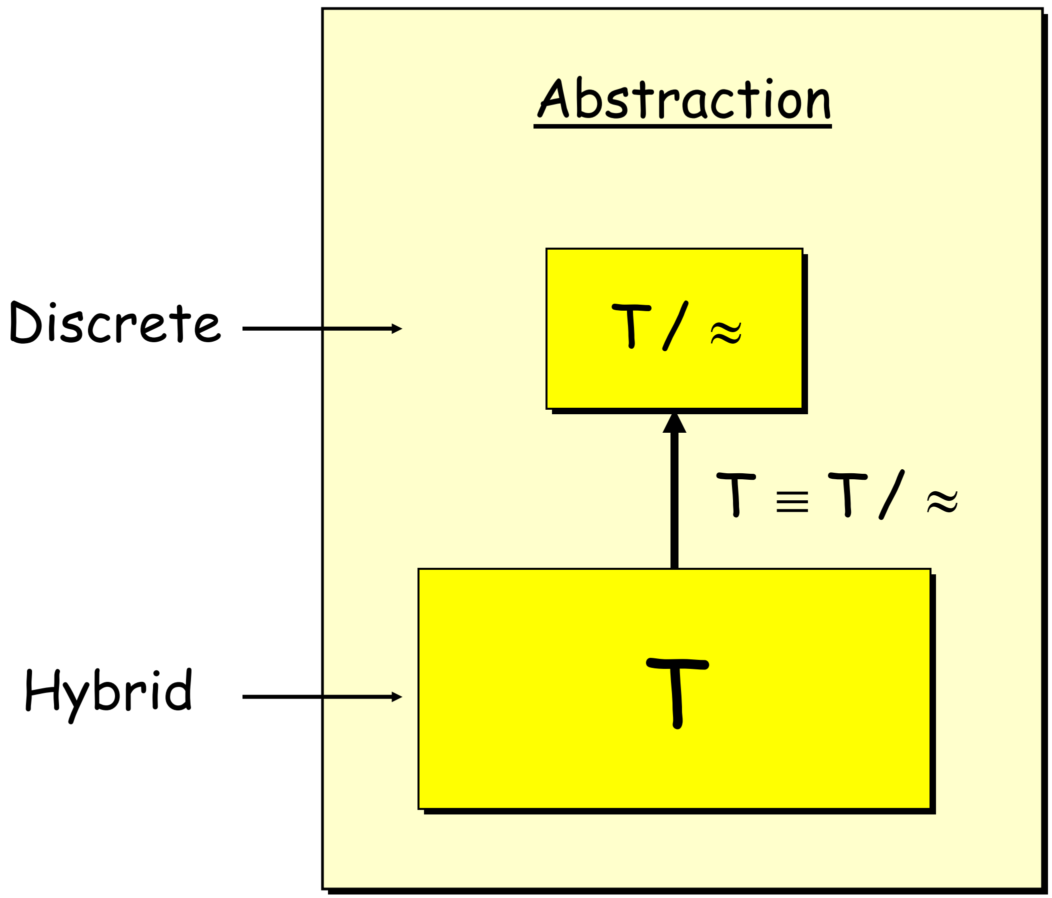
ESE601: Hybrid Systems

Bisimulation of hybrid and continuous systems

Penn
University of Pennsylvania

Spring 2006

Hybrid to discrete abstraction



Goal : Finite quotients of hybrid systems

Hybrid System Model

A hybrid system $H = (V, \Re^n, X_0, F, Inv, R)$ consists of

- V is a finite set of states
- $\Re^n$ is the continuous state space
- $X = V \times \Re^n$ is the state space of the hybrid system
- $X_0 \subseteq X$ is the set of initial states
- $F(l,x) \subseteq \Re^n$ maps a diff. inclusion to each discrete state
- $Inv(l) \subseteq \Re^n$ maps invariant sets to each discrete state
- $R \subseteq X \times X$ is a relation capturing discontinuous changes

Define $E = \{(l,l')| \ \exists x \in Inv(l), x' \in Inv(l') \ ((l,x),(l',x')) \in R\}$

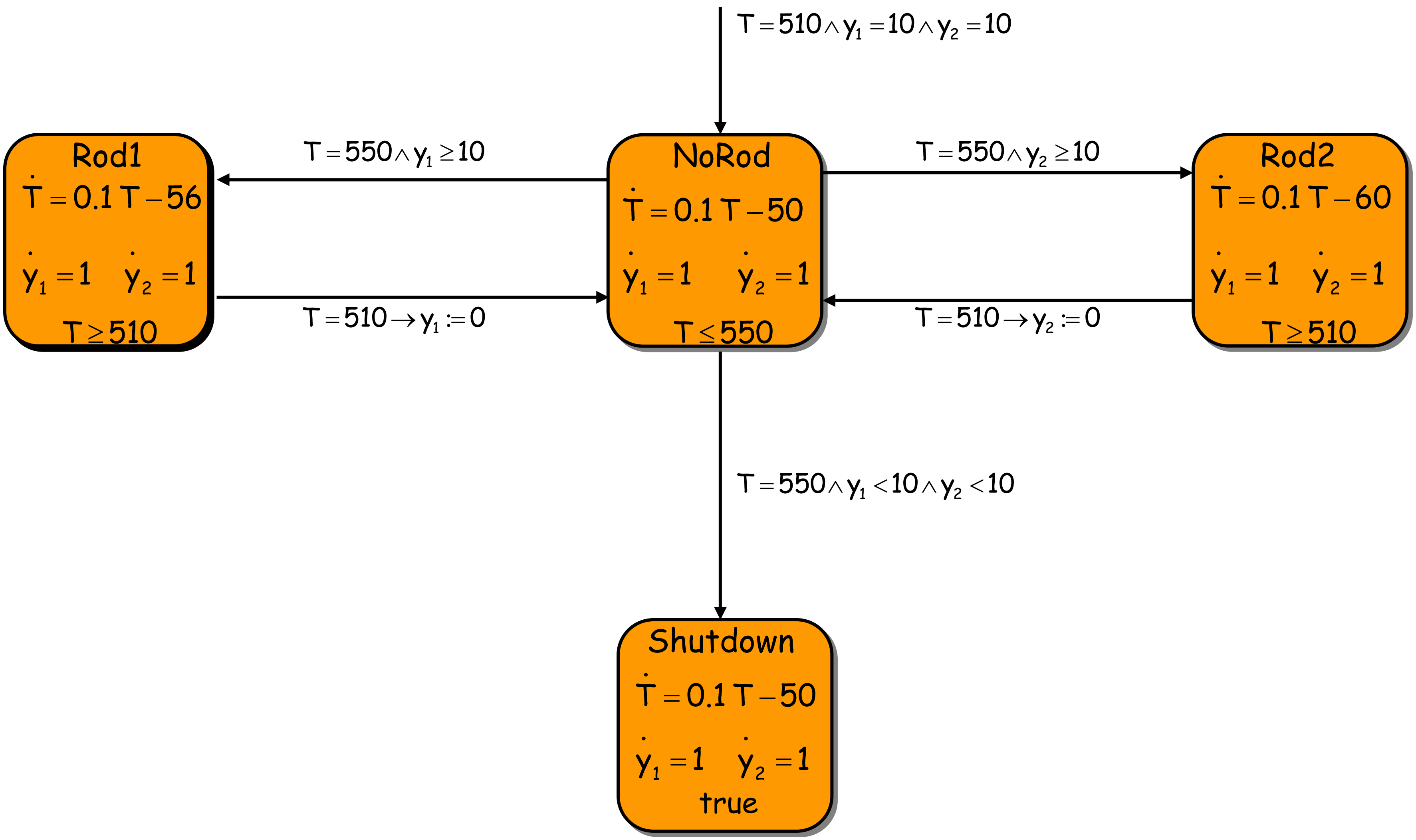
$$Init(l) = \{x \in Inv(l) \mid (l,x) \in X_0\}$$

$$Guard(e) = \{x \in Inv(l)| \ \exists x' \in Inv(l') \ ((l,x),(l',x')) \in R\}$$

$$Reset(e,x) = \{x' \in Inv(l')| \ ((l,x),(l',x')) \in R\}$$

Penn
University of Pennsylvania

An example



$T = 510 \wedge y_1 = 10 \wedge y_2 = 10$
Rod1
$\dot{T} = 0.1\,T - 56$
$\dot{y}_1 = 1 \quad \dot{y}_2 = 1$
$T \geq 510$
$T = 550 \wedge y_1 \geq 10$
$T = 510 \rightarrow y_1 := 0$
NoRod
$\dot{T} = 0.1\,T - 50$
$\dot{y}_1 = 1 \quad \dot{y}_2 = 1$
$T \leq 550$
$T = 550 \wedge y_2 \geq 10$
$T = 510 \rightarrow y_2 := 0$
Rod2
$\dot{T} = 0.1\,T - 60$
$\dot{y}_1 = 1 \quad \dot{y}_2 = 1$
$T \geq 510$
$T = 550 \wedge y_1 < 10 \wedge y_2 < 10$
Shutdown
$\dot{T} = 0.1\,T - 50$
$\dot{y}_1 = 1 \quad \dot{y}_2 = 1$
true

Transitions of Hybrid Systems

Hybrid systems can be embedded into transition systems

$$H = (V, \Re^n, X_0, F, Inv, R) \longrightarrow T_H = (Q, Q_0, \Sigma, \rightarrow, O, < \cdot >)$$

$$Q = V \times \Re^n$$
$$Q_0 = X_0$$
$$\Sigma = E \cup \{\tau\}$$
$$\rightarrow \subseteq Q \times \Sigma \times Q$$

Observation set and map depend on desired properties

Discrete transitions

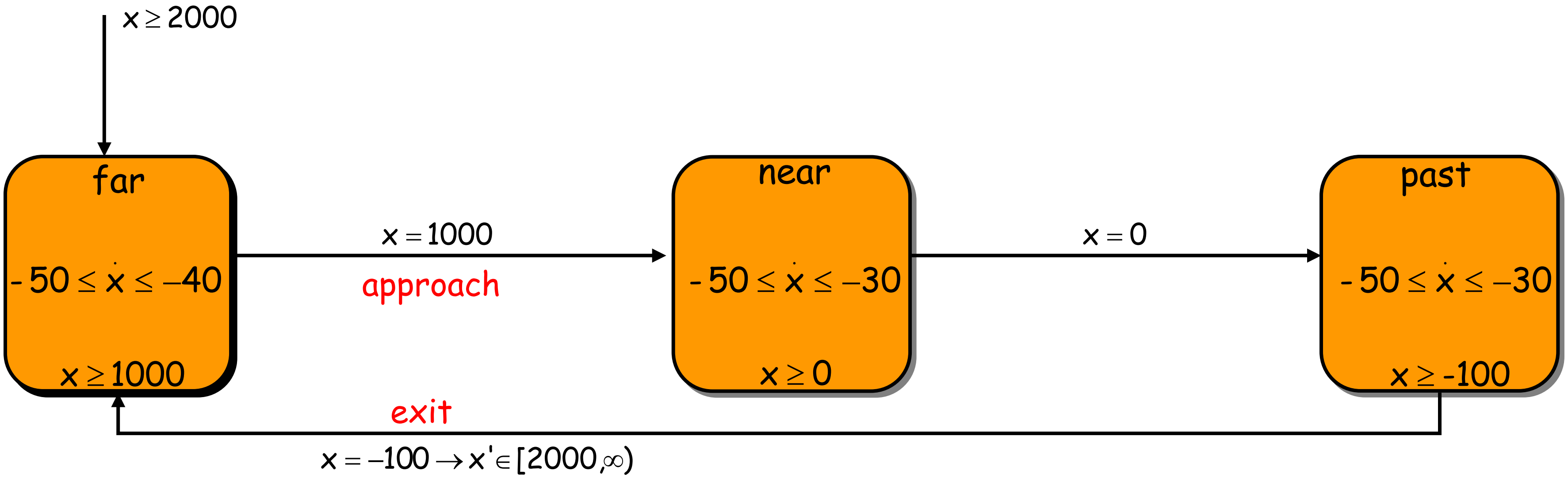
$$(l_1, x_1) \xrightarrow{e} (l_2, x_2) \quad \text{iff} \quad x_1 \in Guard(e), x_2 \in Reset(e, x_1)$$

Continuous (time-abstract) transitions

$$(l_1, x_1) \xrightarrow{\tau} (l_2, x_2) \quad \text{iff} \quad l_1 = l_2 \quad \text{and} \quad \exists \delta \geq 0 \quad x(\cdot) : [0, \delta] \rightarrow \Re^n$$
$$x(0) = x_1, x(\delta) = x_2, \quad \text{and} \quad \forall t \in [0, \delta]$$
$$\dot{x} \in F(l_1, x(t)) \quad \text{and} \quad x(t) \in Inv(l_1)$$

Rectangular hybrid automata

Rectangular sets : $\bigwedge_i x_i \sim c_i \quad \sim \in \{<, \leq, =, \geq, >\}, c_i \in Q$

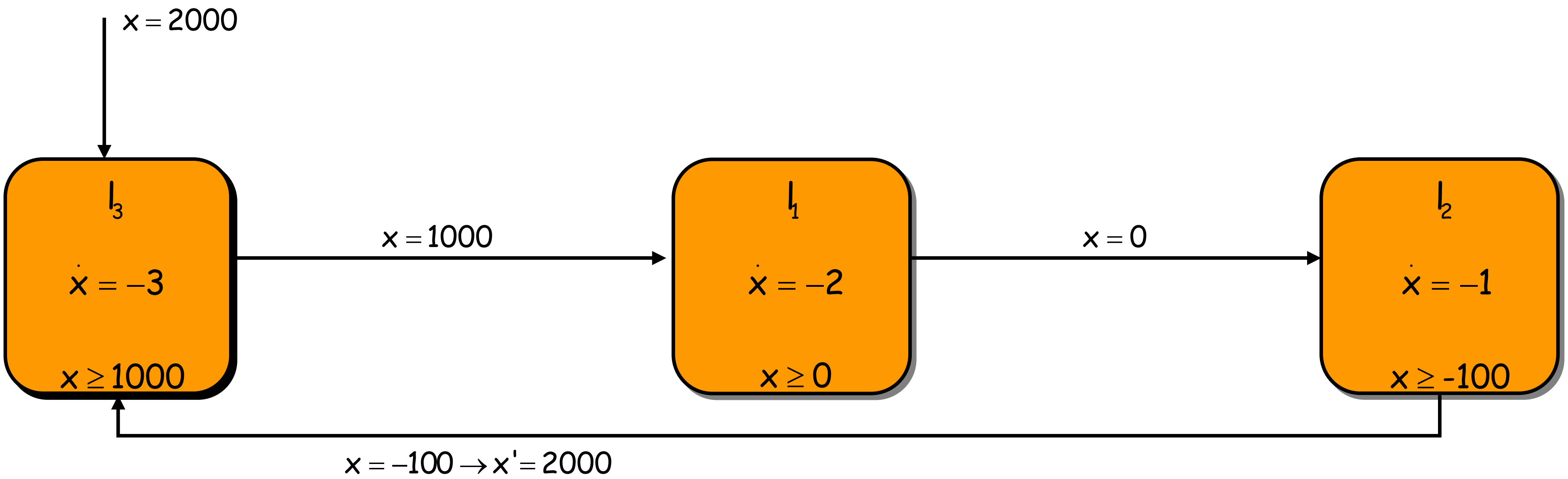


Rectangular hybrid automata are hybrid systems where

$$Init(l), Inv(l), F(l,x), Guard(e), Reset(e,x)_i$$

are rectangular sets

Multi-rate automata

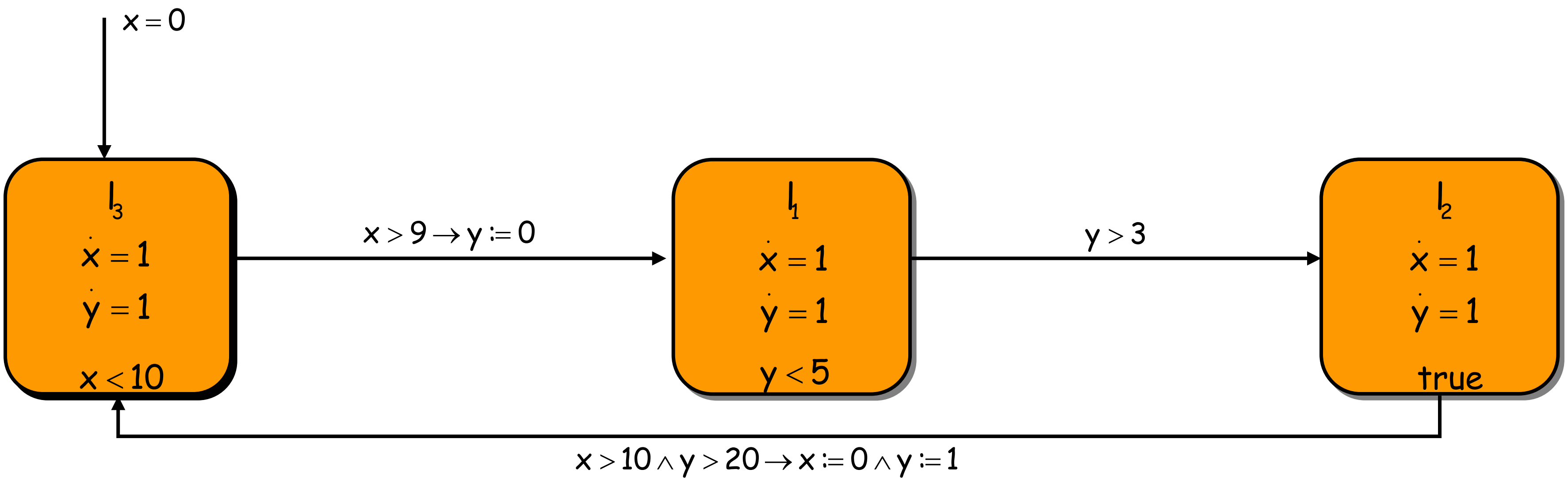


Multi-rate automata are rectangular hybrid automata where

$$Init(l), F(l,x), Reset(e,x)_i$$

are singleton sets

Timed automata



Timed automata are multi-rate automata where

$$F(l, x_i) = 1$$

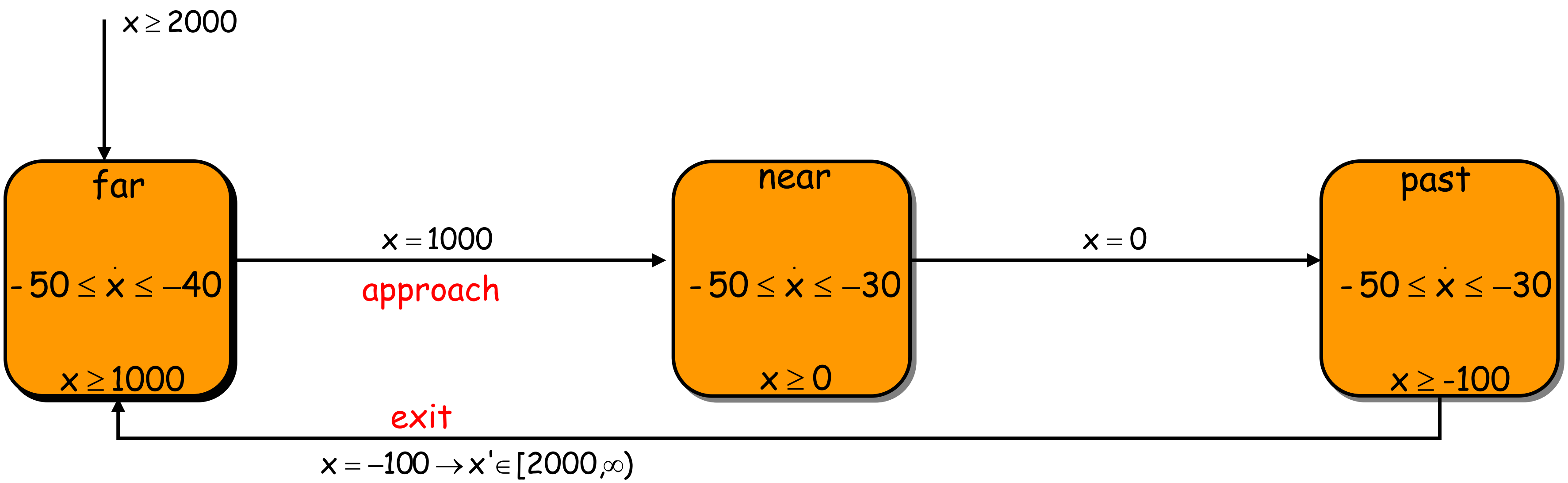
for all locations l and all variables.

Initialized automata

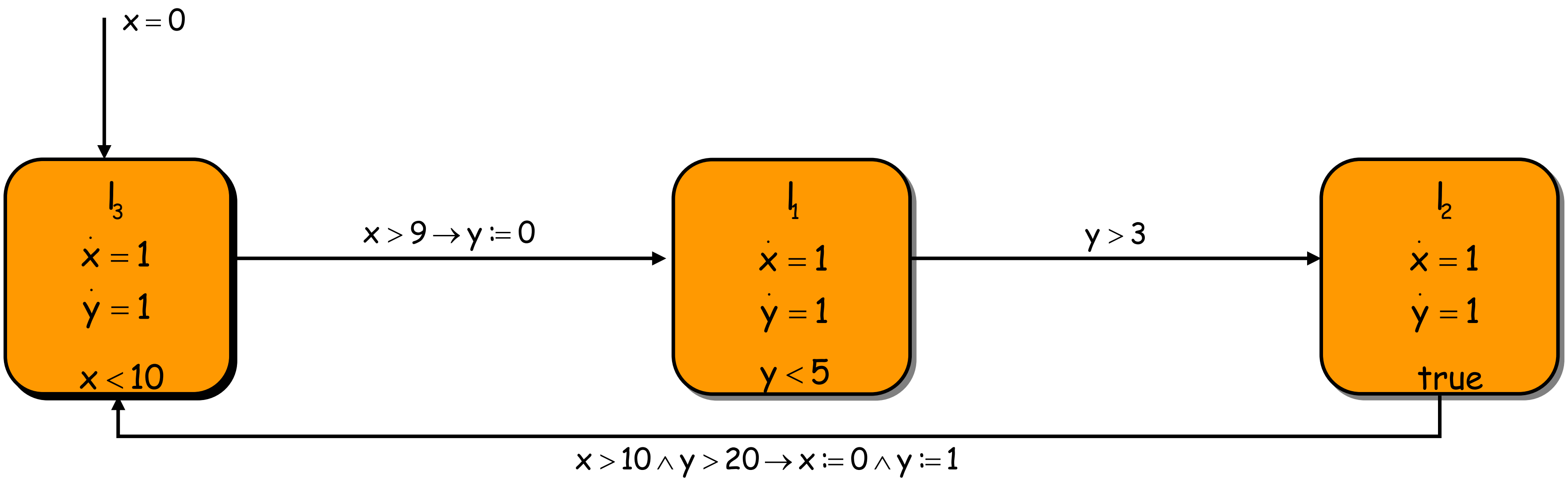
Rectangular hybrid automata are **initialized** if the following holds:

After a discrete transition, if the differential inclusion (equation) for a variable changes, then the variable must be reset to a fixed interval.

Timed automata are always initialized.



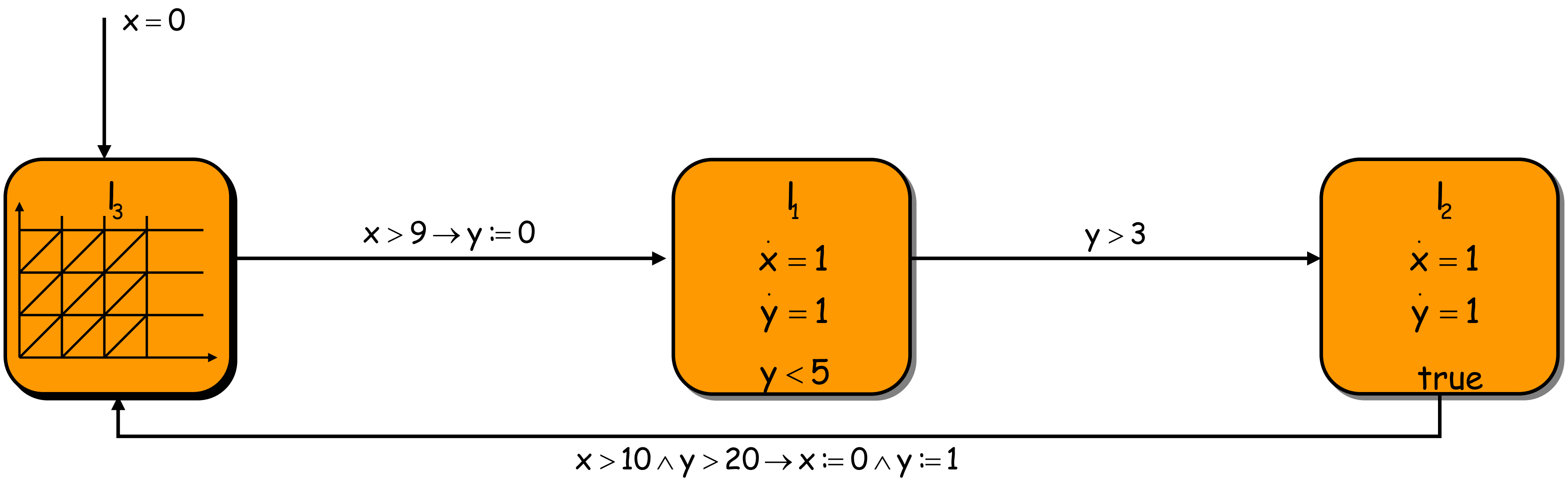
Timed automata



All timed automata admit a finite bisimulation

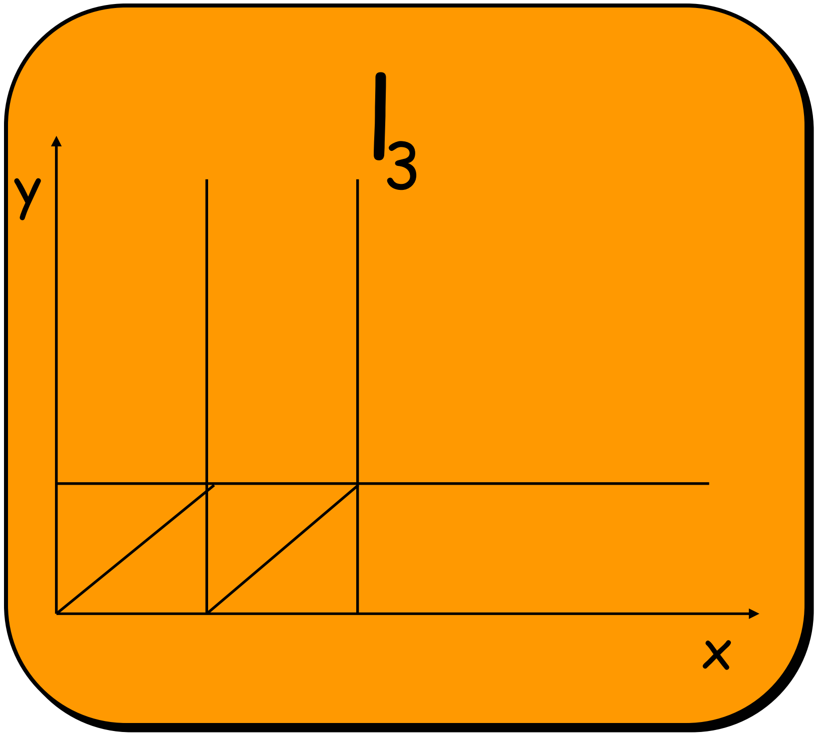
Hence CTL* model checking is decidable for timed automata

Timed automata



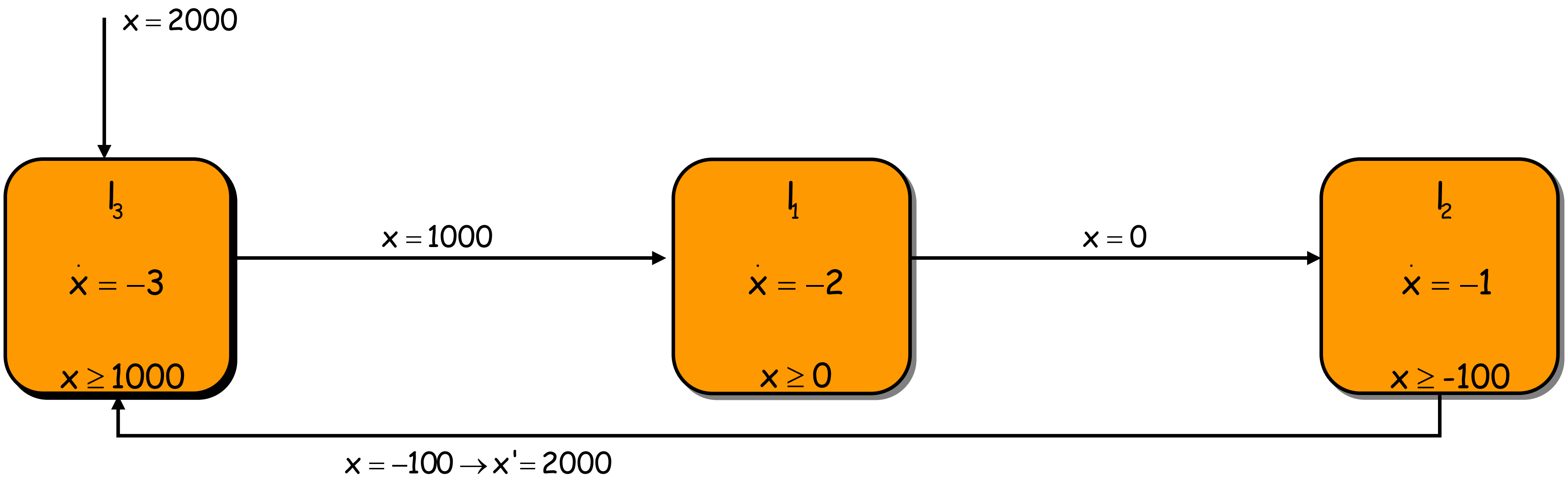
Approach : Discretize the clock dynamics using region equivalence

Region equivalence



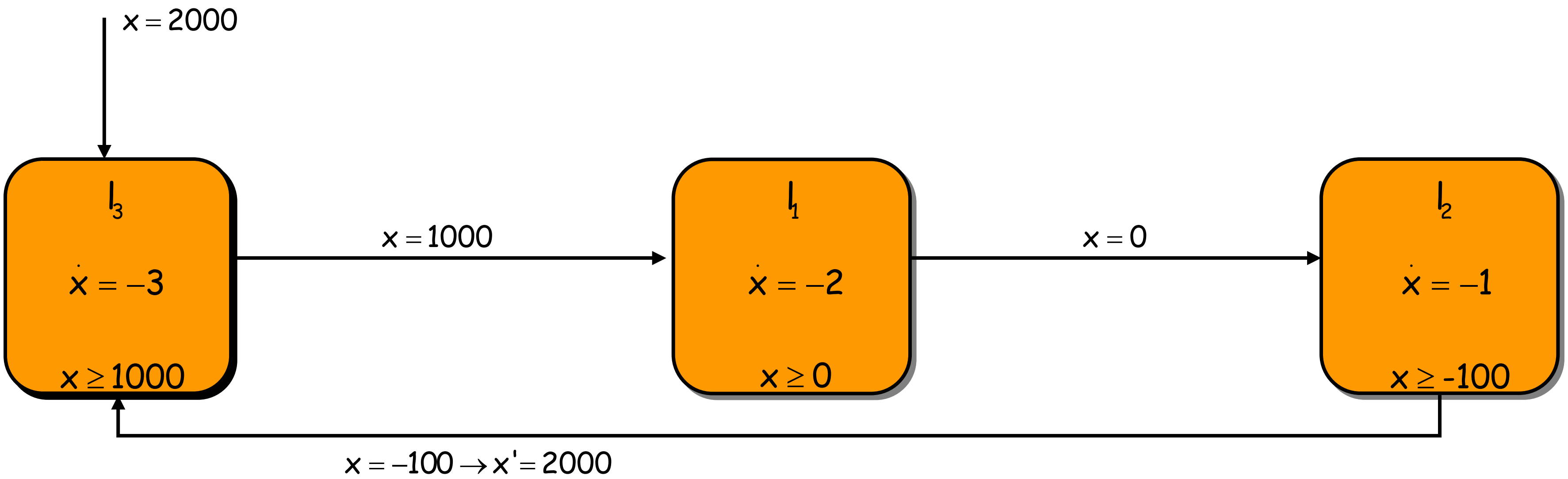
Equivalence classes : 6 corner points
14 open line segments
8 open regions

Multi-rate automata



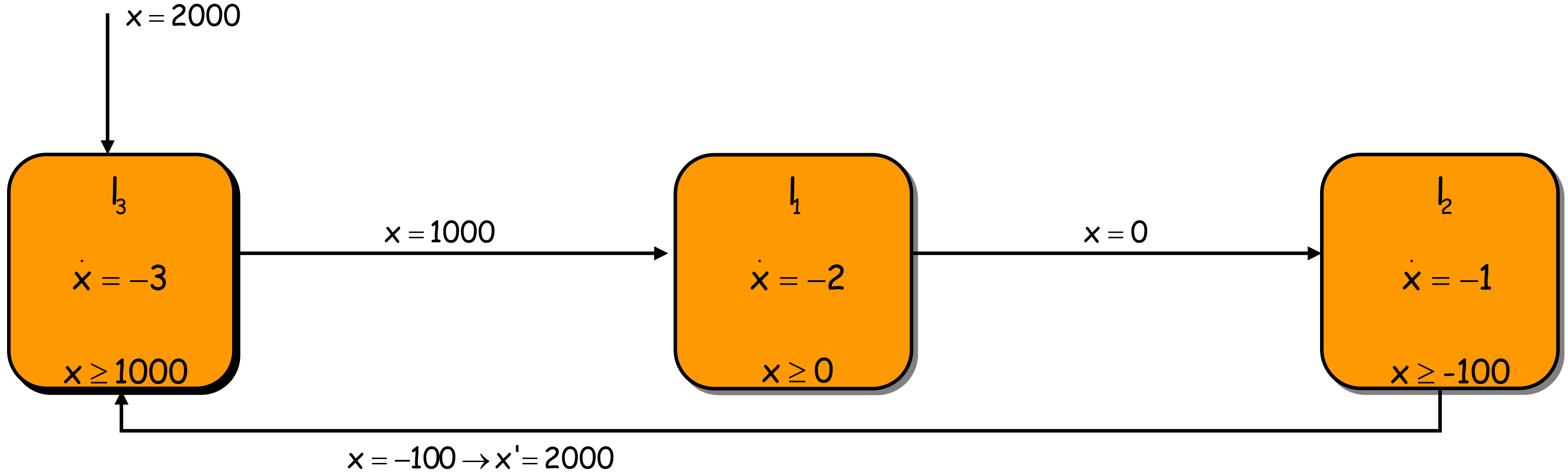
All initialized multi-rate automata admit a finite bisimulation

Rectangular automata



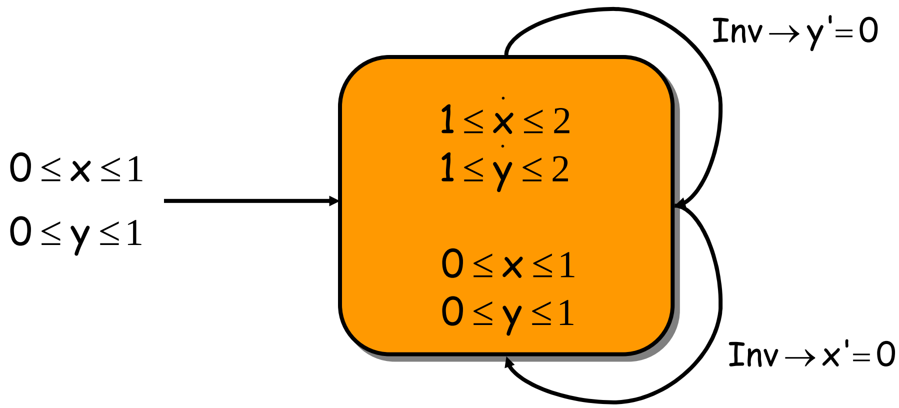
All initialized rectangular automata admit a finite bisimulation

Rectangular automata



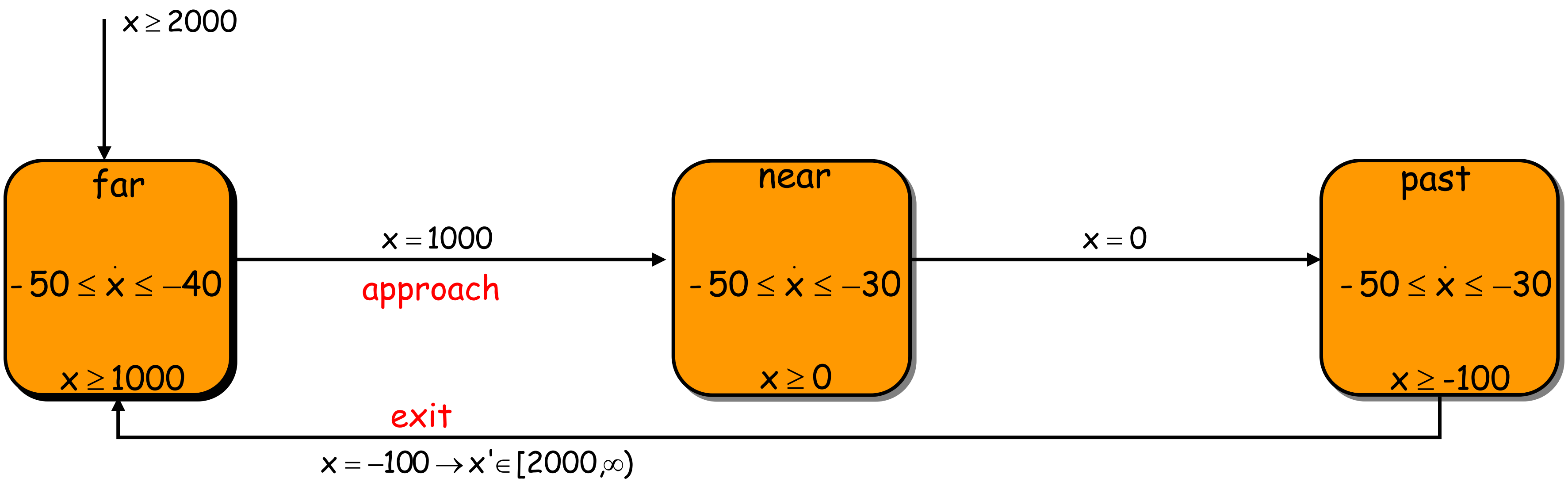
~~**All initialized rectangular automata admit a finite bisimulation**~~

No finite bisimulation



Bisimulation algorithm never terminates

but…



All initialized rectangular automata admit a finite language equivalence quotient which can be constructed effectively.

LTL model checking of rectangular automata is decidable.

Bad news

Undecidability barriers

Consider the class of uninitialized multi-rate automata with n-1 clock variables, and one two slope variable (with two different rates).

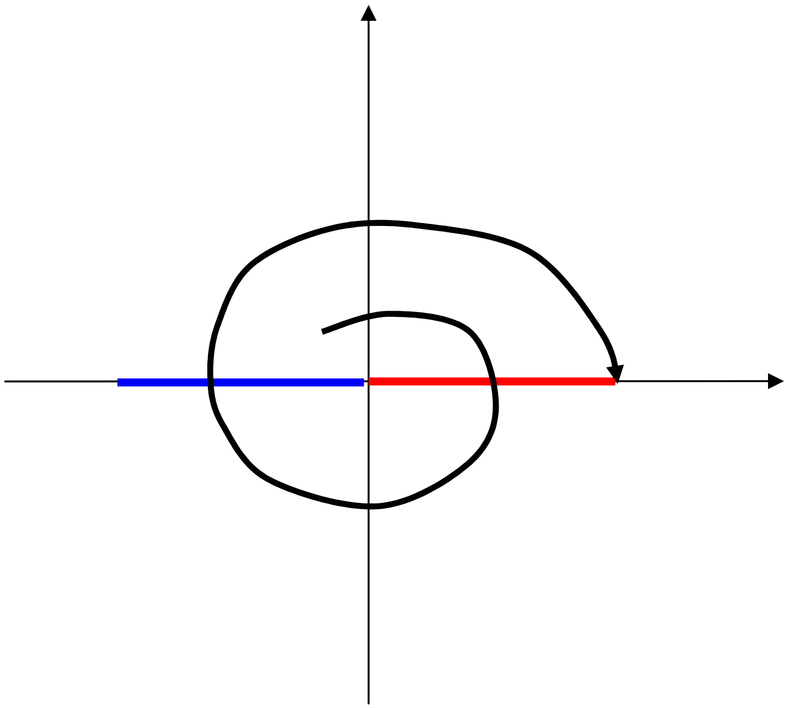
The reachability problem is undecidable for this class.

No algorithmic procedure exists.

Model checking temporal logic formulas is also undecidable

Initalization is necessary for decidability

More complicated dynamics?



Sets

$$P_1 = \{(x,0) \mid 0 \le x \le 4\}$$

$$P_2 = \{(x,0) \mid -4 \le x < 0\}$$

$$P_3 = R^2 \setminus (P_1 \cup P_2)$$

Dynamics

$$\dot{x}_1 = 0.2x_1 + x_2$$

$$\dot{x}_2 = -x_1 + 0.2x_2$$

Bisimulation algorithm never terminates !!

Basic problems

Finite bisimulations of continuous dynamical systems

Given a vector field F(x) and a finite partition of R^n

1. Does there exist a finite bisimulation ?
2. Can we compute it ?

Penn
University of Pennsylvania

Reminder

Representation issues

- Symbolic representation for infinite sets
- Rectangular sets ? Semi-linear ? Semi-algebraic ?

Operations on sets

- Boolean (logical) operations
- Can we compute Pre and Post ?
- Is our representation closed under Pre and Post ?

Algorithmic termination (decidability)

- No guarantee for infinite transition systems
- We need “nice” alignment of sets and flows
- Globally finite properties

First-order logic

Every theory of the reals has an associated language

$$(\Re, <, +, -, 0, 1)$$

Universe Relation Functions Constants

Variables : $x_1, x_2, x_3, \ldots$

TERMS : Variables, constants, or functions of them

$$x_1 - x_2 + 1, 1 + 1, -x_3$$

ATOMIC FORMULAS : Apply the relation and equality to the terms

$$x_1 + x_2 < -1, 2x_1 = 1, x_1 = x_3$$

(FIRST ORDER) FORMULAS : Atomic formulas are formulas

If φ_1, φ_2 are formulas, then $\varphi_1 \vee \varphi_2, \neg\varphi_1, \forall x.\varphi_1, \exists x.\varphi_1$

Penn
University of Pennsylvania

First-order logic

Useful languages

$$(\Re, <, +, -, 0, 1) \qquad \forall x \forall y (x + 2y \geq 0)$$

$$(\Re, <, +, -, \times, 0, 1) \qquad \exists x . ax^2 + bx + c = 0$$

$$(\Re, <, +, -, \times, e^x, 0, 1) \qquad \exists t . (t \geq 0) \wedge (y = e^t x)$$

A theory of the reals is **decidable** if there is an algorithm which in a finite number of steps will decide whether a formula is true or not

A theory of the reals admits **quantifier elimination** if there is an algorithm which will eliminate all quantified variables.

$$\exists x . ax^2 + bx + c = 0 \equiv b^2 - 4ac \geq 0$$

First-order logic

Theory	Decidable ?	Quant. Elim. ?
$(\Re, <, +, -, 0, 1)$	YES	YES
$(\Re, <, +, -, \times, 0, 1)$	YES	YES
$(\Re, <, +, -, \times, e^x, 0, 1)$	?	NO

Tarski's result : Every formula in $(\Re, <, +, -, \times, 0, 1)$ can be decided

1. Eliminate quantified variables
2. Quantifier free formulas can be decided

O-Minimal Theories

A definable set is $Y = \{(x_1, x_2, ..., x_n) \in \Re^n \mid \varphi(x_1, ..., x_n)\}$

A theory of the reals is called **o-minimal** if every definable subset of the reals is a finite union of points and intervals

Example: $Y = \{(x) \in \Re \mid p(x) \geq 0\}$ for polynomial p(x)

Recent o-minimal theories

$$(\Re, <, +, -, 0, 1)$$

$$(\Re, <, +, -, \times, 0, 1)$$

$$(\Re, <, +, -, \times, e^x, 0, 1)$$

Penn
University of Pennsylvania

Basic answers

Finite bisimulations of continuous dynamical systems

Consider a vector field X and a finite partition of R^n where

1. The flow of the vector field is definable in an o-minimal theory
2. The finite partition is definable in the same o-minimal theory

Then a finite bisimulation always exists.

Corollaries

$(\Re, <, +, -, 0, 1)$

Consider continuous systems where

- Finite partition is polyhedral
- Vector fields have linear flows (timed, multi-rate)

Then a finite bisimulation exists.

$(\Re, <, +, -, \times, 0, 1)$

Consider continuous systems where

- Finite partition is semialgebraic
- Vector fields have polynomial flows

Then a finite bisimulation exists.

Corollaries

$(\Re, <, +, -, \times, e^x, 0, 1)$

Consider continuous systems where

- Finite partition is semi-algebraic
- Vector fields are linear with real eigenvalues

Then a finite bisimulation exists.

$(\Re, <, +, -, \times, \hat{f}, 0, 1)$

Consider continuous systems where

- Finite partition is sub-analytic
- Vector fields are linear with purely imaginary eigenvalues

Then a finite bisimulation exists.

Penn
University of Pennsylvania

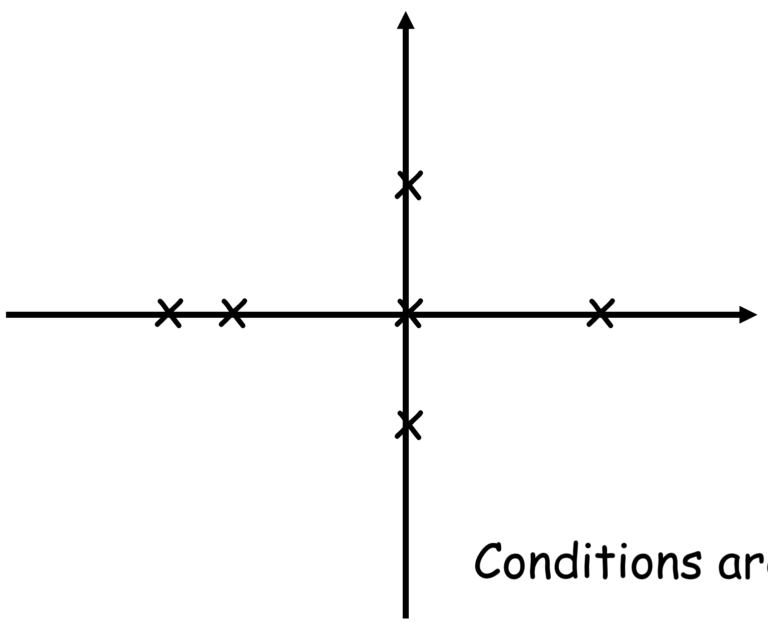
Corollaries

$(\Re, <, +, -, \times, \hat{f}, e^x, 0, 1)$

Consider continuous systems where

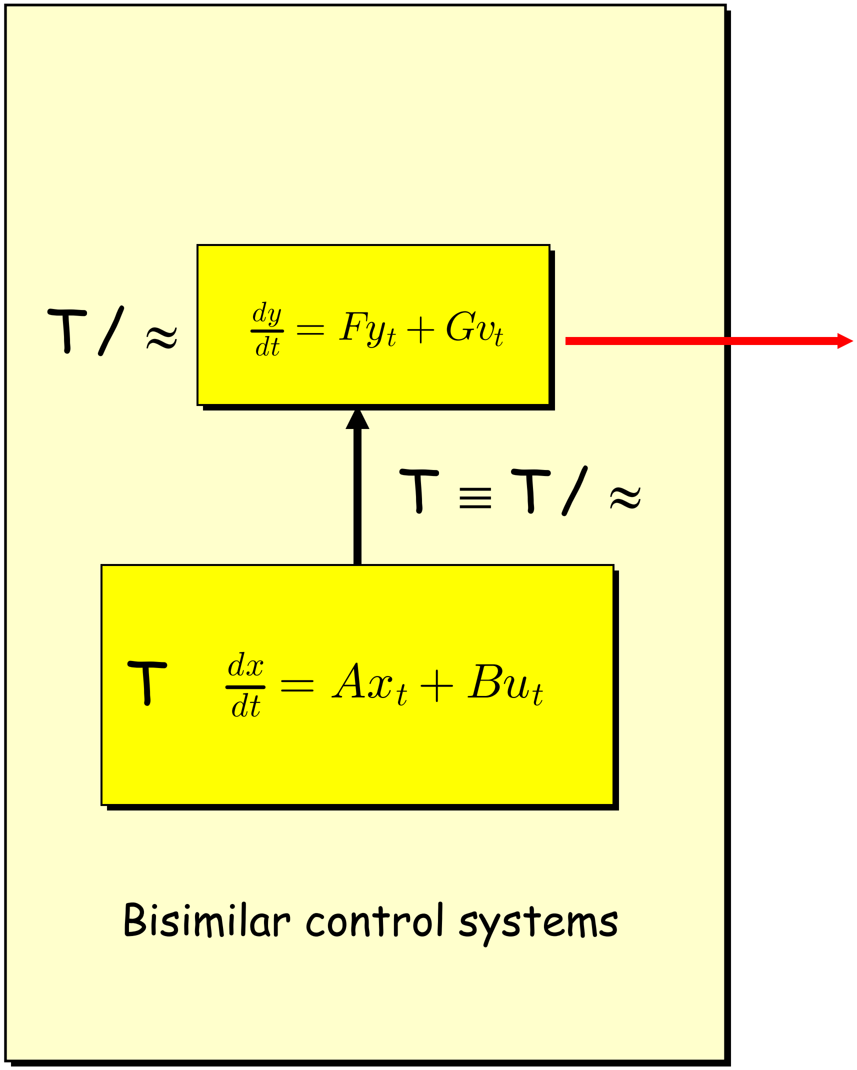
- Finite partition is semi-algebraic
- Vector fields are linear with real or imaginary eigenvalues

Then a finite bisimulation exists.



Conditions are sufficient but tight

Bisimulation of linear systems



quotient system is a linear system

Linear system to transition system
Keep continuous time....

$T_{\Sigma}^{\Delta} = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

embedding

$\Delta \quad \frac{dx}{dt} = Ax_t + Bu_t$

$y = Cx$

Transition System $T_{R_+}^{\Delta}$

State set $Q = X = R^n$

Label set $\Sigma = R_+$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times R_+ \times X$

$$x_1 \overset{t}{\rightarrow} x_2 \Leftrightarrow \exists u_{[0,t]} \text{ with } x_2 = e^{At}x_1 + \int_0^t e^{A(t-s)}Bu(s)ds$$

Linear system to transition system
Time abstract

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

embedding

Δ

$$\frac{dx}{dt} = Ax_t + Bu_t$$
$$y = Cx$$

Transition System T_τ^Δ

State set $Q = X = R^n$

Label set $\Sigma = \{\tau\}$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times \{\tau\} \times X$

$$x_1 \xrightarrow{\tau} x_2 \Leftrightarrow \exists t \text{ and } \exists u_{[0,t]} \text{ with } x_2 = e^{At}x_1 + \int_0^t e^{A(t-s)}Bu(s)ds$$

H based partitioning

Two states are equivalent iff

$$x_1 \approx x_2 \Leftrightarrow Hx_1 = Hx_2 \Leftrightarrow x_1 - x_2 \in Ker(H)$$

for some surjective map z=Hx. Simulation S=(x,Hx)

Partition is observation preserving iff

Linear observations :

$$Ker(H) \subseteq Ker(C)$$

Penn

University of Pennsylvania

Timed, continuous transitions

Consider the time-abstract transition system $T^{\Delta}_{R_+}$

$$x_1 \xrightarrow{t} x_1'$$

$$\approx \qquad\qquad\qquad \approx$$

$$x_2 \xrightarrow{t} x_2'$$

Proposition* : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H) + R(A,B)$$

$$R(A,B) = \mathrm{im}[B\ AB \cdots A^{n-1}B]$$

Untimed, continuous transitions

Consider the time-abstract transition system T_{T}^{Δ}

$$\begin{array}{ccc} x_1 & \xrightarrow{T} & x_1' \\ \approx & & \approx \\ x_2 & \xrightarrow{T} & x_2' \end{array}$$

Proposition* : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H) + R(A,B)$$

$$R(A,B) = \mathrm{im}[B \; AB \cdots A^{n-1}B]$$

Coarsest Bisimulation

Find map z=Hx which abstracts as much as possible.

Thus Ker(H) must be maximal but also…

Preserves observations

$$Ker(H) \subseteq Ker(C)$$

Preserves transitions of T_T^{Δ}

$$A\,Ker(H) \subseteq Ker(H) + R(A,B)$$

Other variations for other embeddings…

Penn
University of Pennsylvania

Coarsest Bisimulation Algorithm

Maximal controlled invariant subspace computation

$$V_0 = Ker(C)$$

$$V_{k+1} = V_{k-1} \cap A^{-1}(V_{k-1} + R(A,B))$$

Then $V^* = V_n$ is the maximal desired subspace

Once V* is computed, then pick map z=Hx such that

Ker(H)=V*