ESE601: Hybrid Systems

Reachability and Safety Analysis

Spring 2006

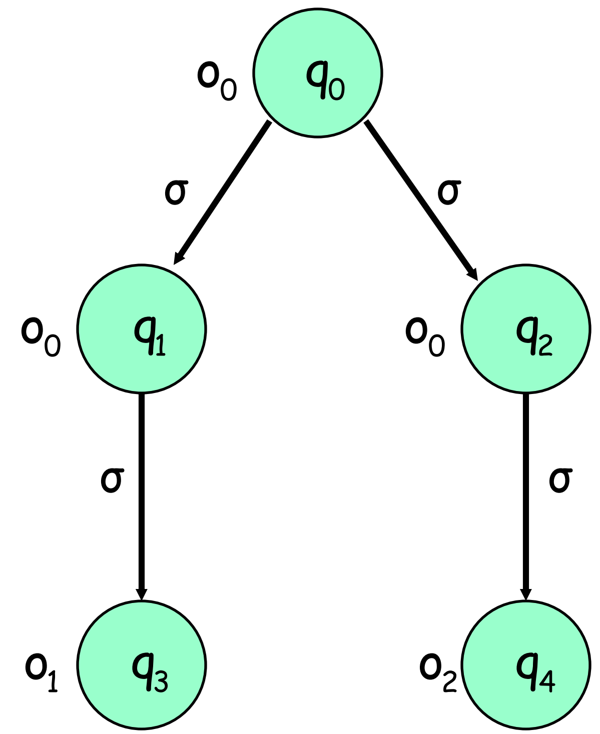
Transition Systems

A transition system

$$T = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$$

consists of

- A set of states Q
- A set of events Σ
- A set of observations O
- The transition relation $q_1 \xrightarrow{\sigma} q_2$
- The observation map $\langle q_1 \rangle = o_0$

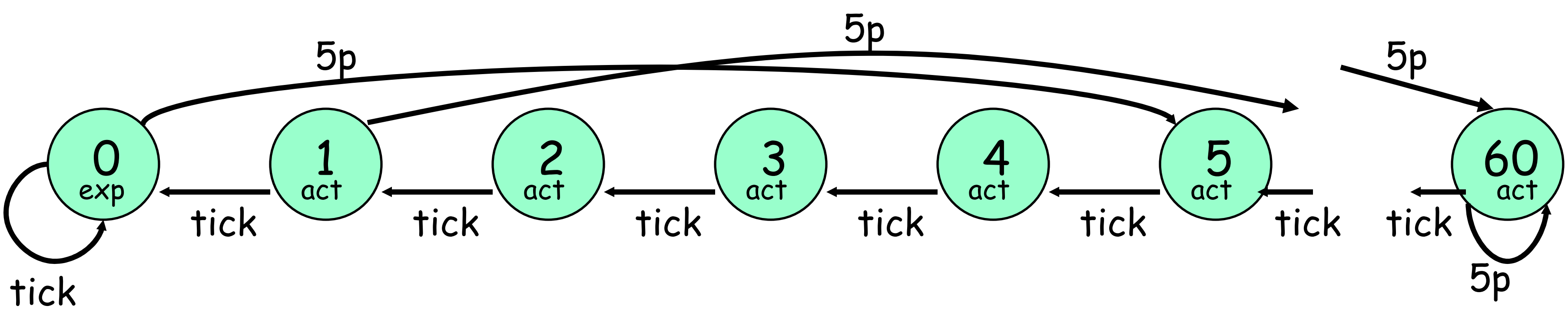


Initial or final states may be incorporated

The sets Q, Σ, and O may be infinite

A painful example

The parking meter



States Q ={0,1,2,…,60}

Events {tick,5p}

Observations {exp,act}

A possible string of observations (exp,act,act,act,act,act,exp,…)

A familiar example

$$T^{\Delta} = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$$

$$\Delta \quad \begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

Transition System T^{Δ}

State set $Q = X = R^n$

Label set $\Sigma = U = R^m$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times U \times X$

$$x_1 \xrightarrow{u} x_2 \Leftrightarrow x_2 = Ax_1 + Bu$$

Transition Systems

A region is a subset of states $P \subseteq Q$

We define the following operators

$$Pre_\sigma(P) = \{q \in Q \mid \exists p \in P \quad q \xrightarrow{\sigma} p\}$$

$$Pre(P) = \{q \in Q \mid \exists \sigma \in \Sigma \quad \exists p \in P \quad q \xrightarrow{\sigma} p\}$$

$$Post_\sigma(P) = \{q \in Q \mid \exists p \in P \quad p \xrightarrow{\sigma} q\}$$

$$Post(P) = \{q \in Q \mid \exists \sigma \in \Sigma \quad \exists p \in P \quad p \xrightarrow{\sigma} q\}$$

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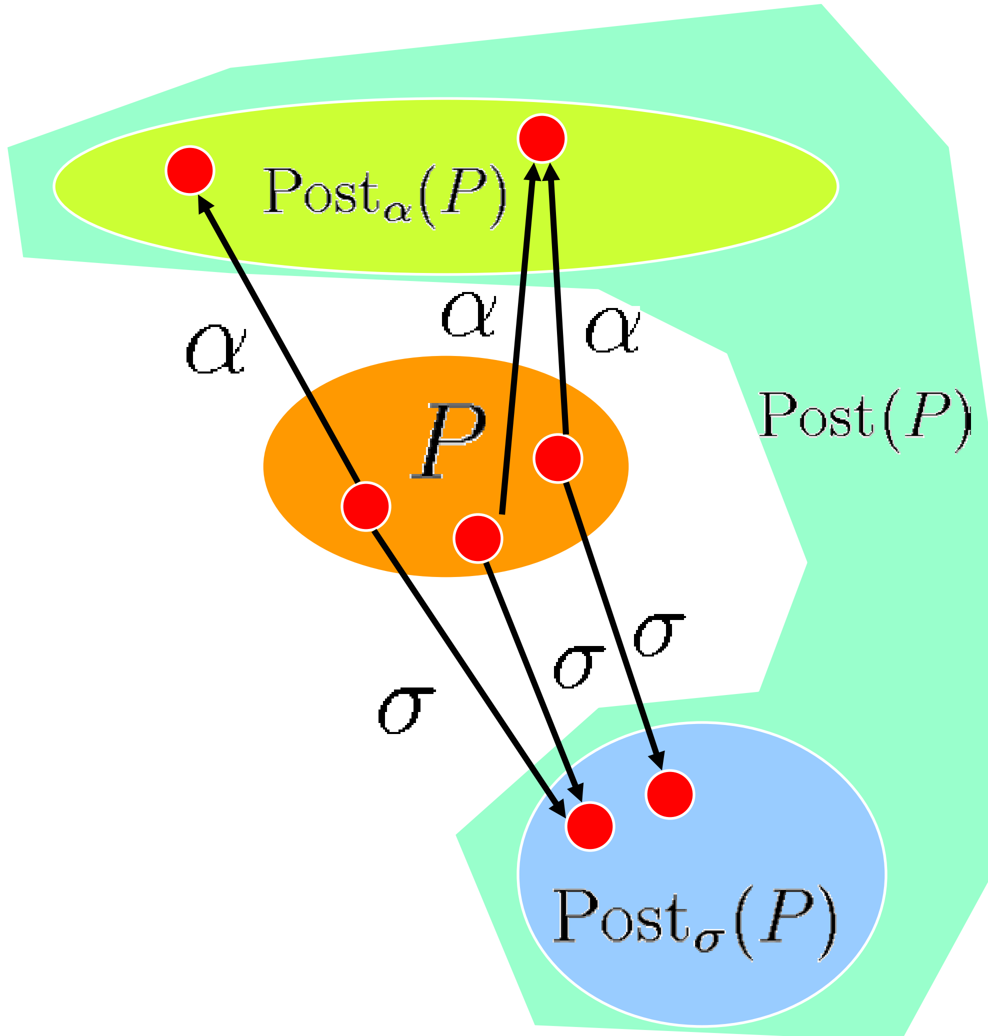
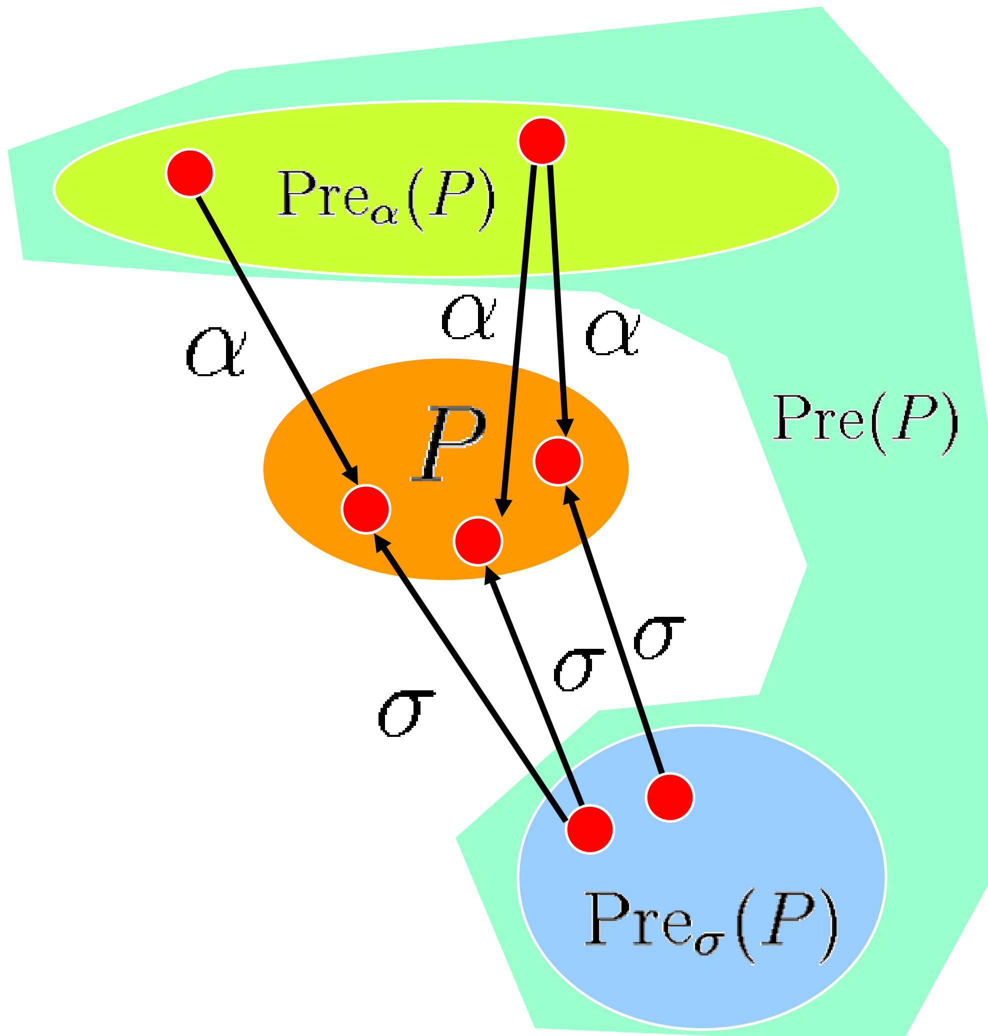
Pre and Post operator



Pre and Post operator



Transition Systems

We can recursively define

$$Pre_\sigma^1(P) = Pre_\sigma(P)$$

$$Pre_\sigma^n(P) = Pre_\sigma(Pre_\sigma^{n-1}(P))$$

Similarly for the other operators. *Also*

$$Pre^*(P) = \bigcup_{n \in N} Pre^n(P)$$

$$Post^*(P) = \bigcup_{n \in N} Post^n(P)$$

Basic safety problems

Given transition system T and regions P, S determine

Forward Reachability

$$Post^*(P) \cap S \neq \emptyset$$

Backward Reachability

$$P \cap Pre^*(S) \neq \emptyset$$

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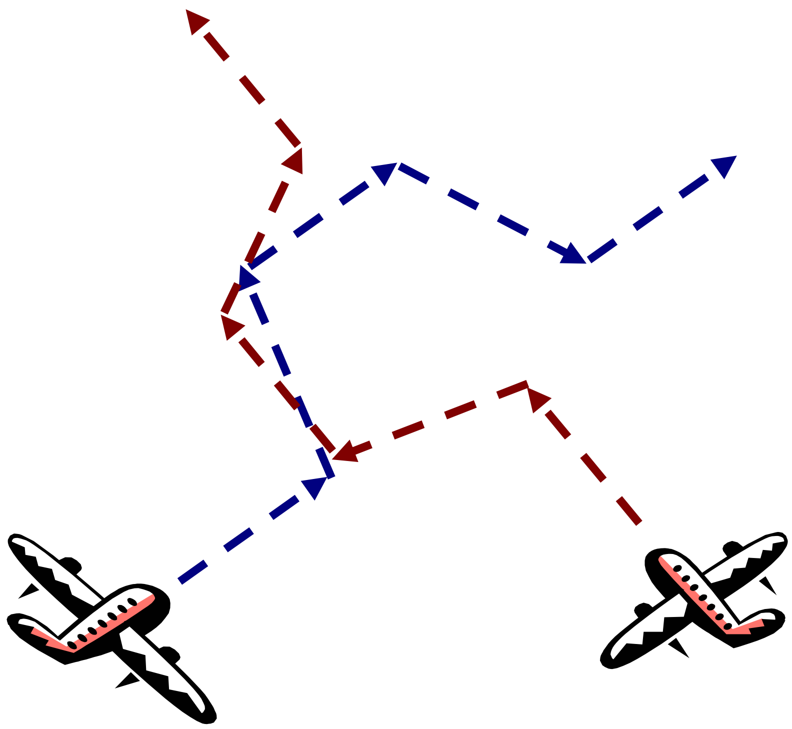
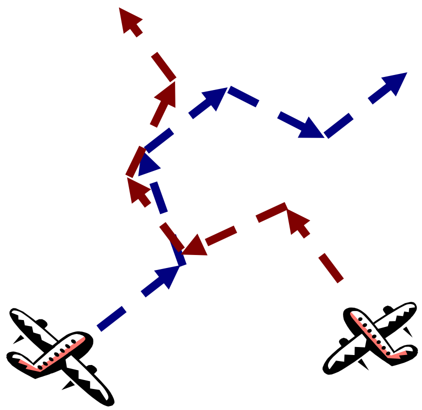
Conflict Resolution in ATM*



See paper (R8) on the website

Conflict Resolution Protocol



1. Cruise until a_1 miles away
2. Change heading by $\Delta\Phi$
3. Maintain heading until lateral distance d
4. Change to original heading
5. Change heading by $-\Delta\Phi$
6. Maintain heading until lateral distance -d
7. Change to original heading

Is this protocol safe ?

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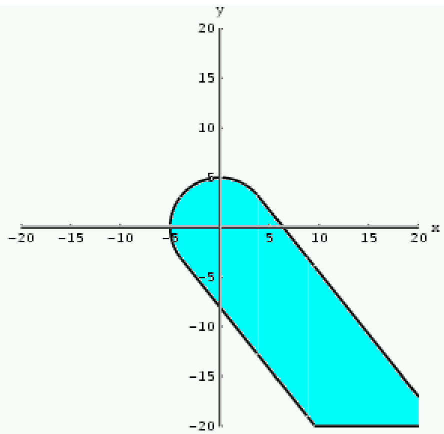
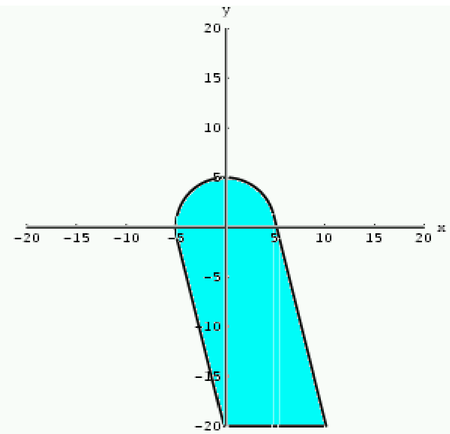
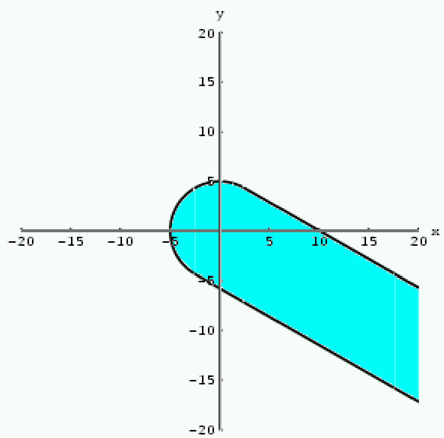
Safe Sets



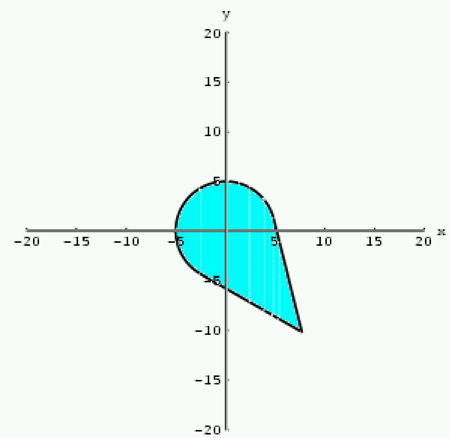
(a) unsafeCruise



(b) unsafeLeft

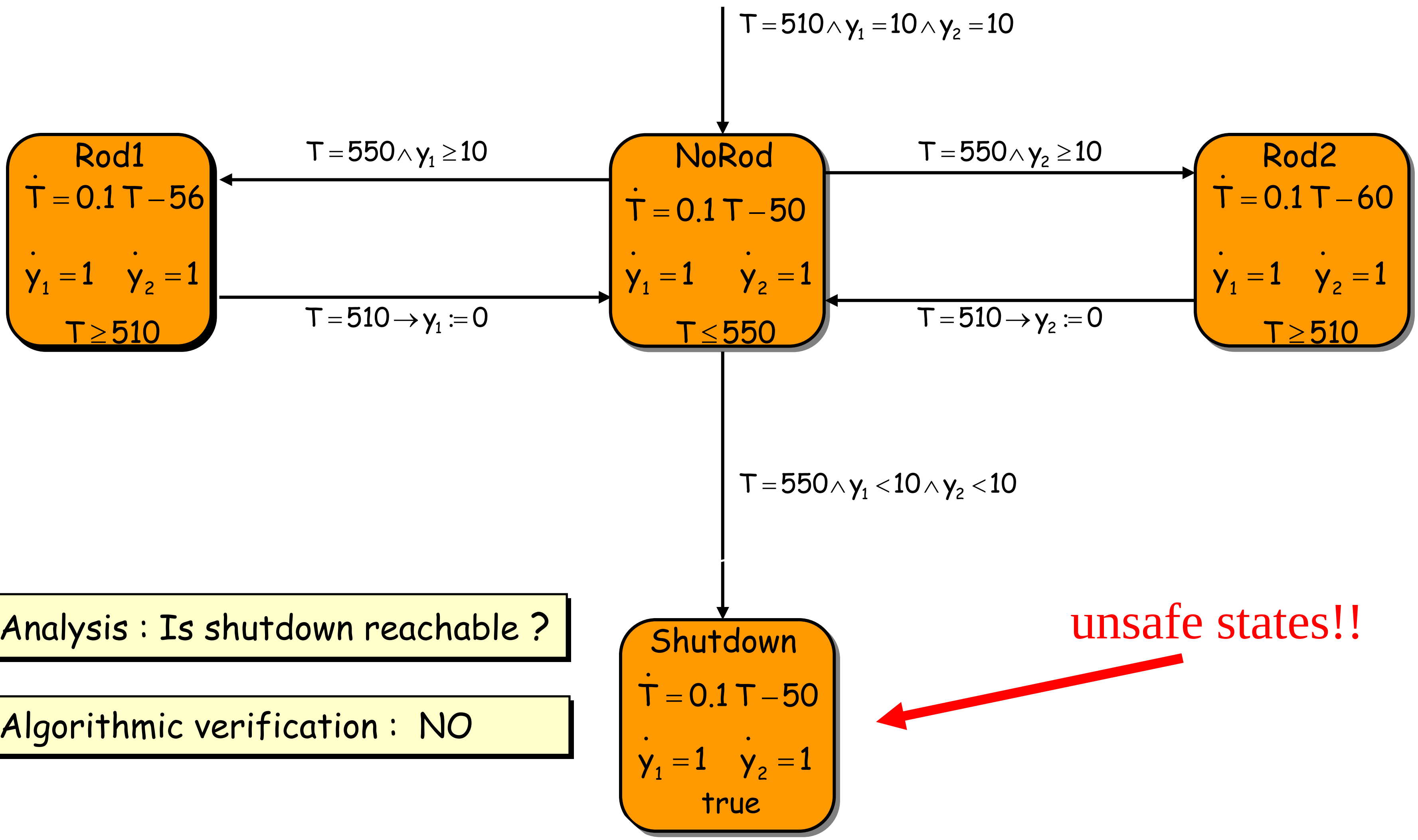


(c) unsafeRight



(d) unsafeCruise ∧ unsafeLeft ∧ unsafeRight

Hybrid model of nuclear reactor



Analysis : Is shutdown reachable ?

Algorithmic verification : NO

Forward reachability algorithm

Forward Reachability Algorithm

initialize $R := P$
while **TRUE** **do**
  if $R \cap S \neq \emptyset$ **return UNSAFE ; end if;**
  if $Post(R) \subseteq R$ **return SAFE ; end if;**
  $R := R \cup Post(R)$
end while

If T is finite, then algorithm terminates (decidability).

Complexity : $O(n_I + m_R)$

initial states

reachable transitions

Backward reachability algorithm

Backward Reachability Algorithm

initialize $R := S$

while TRUE do

  if $R \cap P \neq \emptyset$ return UNSAFE ; end if;

  if $Pre(R) \subseteq R$ return SAFE ; end if;

  $R := R \cup Pre(R)$

end while

If T is infinite, then there is no guarantee of termination.

Algorithmic issues

Representation issues

- Enumeration for finite sets
- Symbolic representation for infinite (or finite) sets

Operations on sets

- Boolean operations
- Pre and Post computations (closure?)

Algorithmic termination (decidability)

- Guaranteed for finite transition systems
- No guarantee for infinite transition systems

Continuous Dynamical Systems

$$\dot{x}(t) = f(x(t), u(t))$$
$$x(t) \in R^n, x(0) \in I, u(t) \in U$$

S

$x(t)$

Continuous dynamics given differential equations

- Input $u(t)$ is internal (disturbance rather than control)
- Dimension (scale) of the system: n
- Classification:
  - Linear system $f(x,u)=Ax+Bu$
  - Deterministic system $f(x,u)=f(x)$

Reachable Set

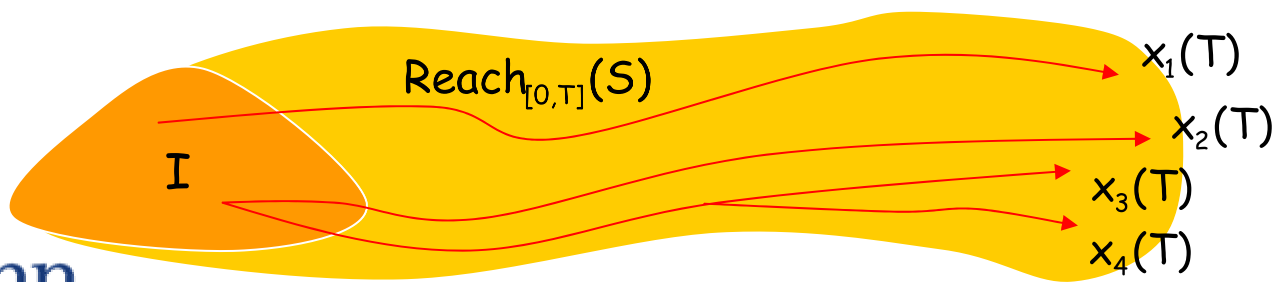
$$\dot{x}(t) = f(x(t), u(t))$$
$$x(t) \in R^n,\ x(0) \in I,\ u(t) \in U \qquad S$$

$\xrightarrow{x(t)}$

x_f is reachable (at time t) if
$\exists\ x_0 \in I,\ \exists\ u:[0,t] \to U$, such that $x(t) = x_f$.

- The reachable set is $Reach(S)=\{x_f \mid x_f \text{ is reachable}\}$
- The timed reachable set is

$$Reach_{[0,T]}(S) = \{x_f \mid x_f \text{ is reachable at time } t \le T\}$$



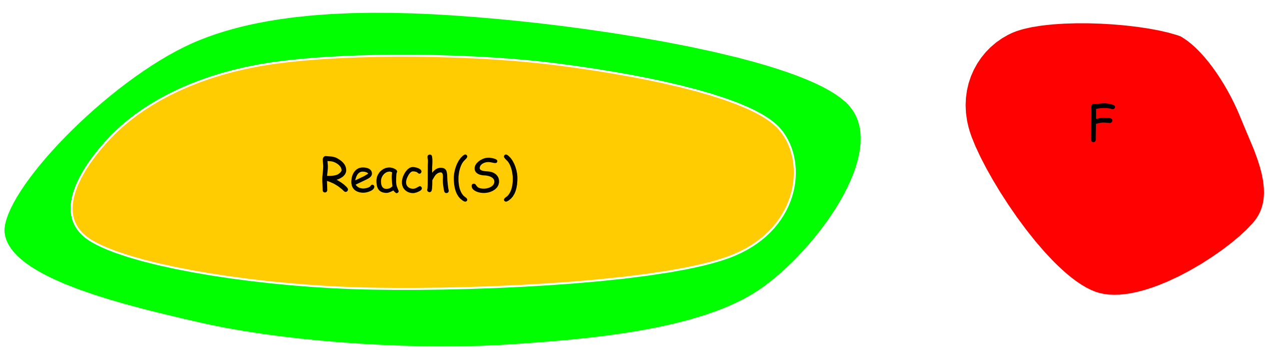
Reachability Problem

Given a set $F \subseteq R^n$, evaluate the expressions

$$Reach_{[0,T]}(S) \cap F = \emptyset$$

$$Reach(S) \cap F = \emptyset$$

- Safety verification (F is an unsafe set)
- Exact computation difficult (impossible for most systems):
  -> Compute over-approximations the reachable set



-> S is safe

Several Approaches

1. Numerical Analysis:
   - Level Sets Methods
     [Tomlin, Mitchell and Bayen]
   - Flow Pipes
     [Krogh; Dang; Kurzhanski and Varaiya; Girard]
2. Convex Optimization:
   - Barrier Certificates
     [Prajna and Jadbabaie]
3. Computer Science:
   - Discrete Abstractions of Continuous Dynamics
     [Alur, Dang and Ivancic; Tiwari, Belta]
   - Hybridization
     [Henzinger; Asarin, Dang and Girard; Frehse]

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Several Tools

- Level Sets Toolbox [Mitchell]
- Checkmate [Chutinan and Krogh]
- d/dt [Dang and Maler]
- Ellipsoidal Toolbox [Kurzhanski and Varaiya]
- Sostools [Parillo and Prajna]
- Hytech [Henzinger]
- PHAVer [Frehse]
- …

Check the Hybrid Systems Tools Wiki Page !

http://wiki.grasp.upenn.edu/~graspdoc/hst/

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Flow Pipe Approximation

Main observation:

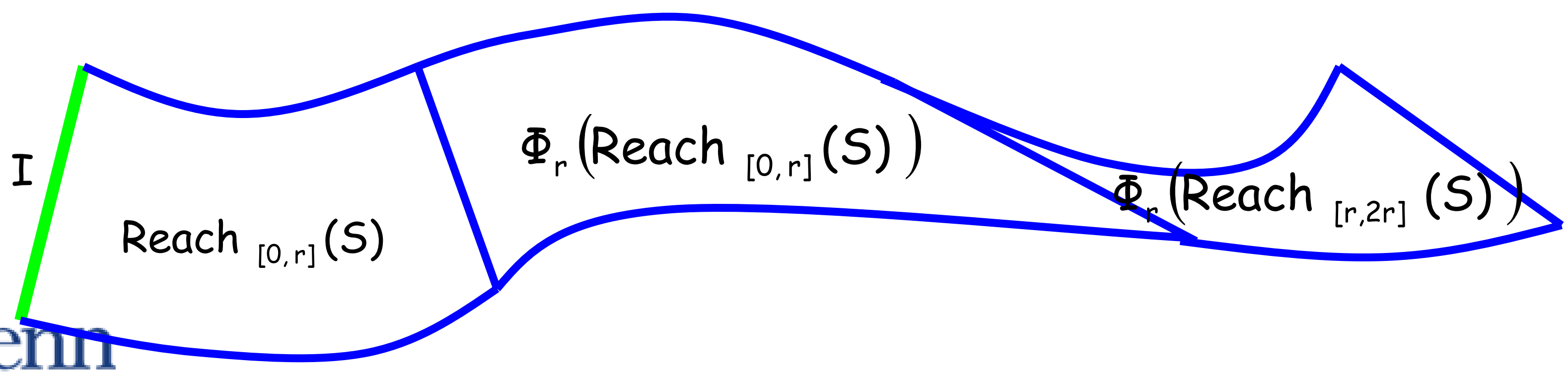
Given a time step r and T = Nr,

$$\mathrm{Reach}_{[0,T]}(S) = \bigcup_{k=0}^{k=N-1} \mathrm{Reach}_{[kr,(k+1)r]}(S).$$

and

$$\mathrm{Reach}_{[kr,(k+1)r]}(S) = \Phi_r\left(\mathrm{Reach}_{[(k-1)r,kr]}(S)\right)$$

where $\Phi_r(X)$ is the set of points reachable at time r from X.



Flow Pipe Approximation

Choice of the representation of the reachable set:

- closed under linear maps and Minkowski sums
  - -> Polytopes [Krogh; Dang]
  - -> Accurate
  - -> Not scalable (exponential complexity)
- scalable representations
  - -> Ellipsoids [Kurzhanski and Varaiya]
  - -> Oriented rectangular hull [Krogh]
  - -> Not closed under linear maps and Minkowski sums
  - -> Additional computations and approximations
- Zonotopes [Girard]
  - -> Closed under linear maps and Minkowski sums
  - -> Accurate
  - -> Scalable (polynomial complexity/dimension)

Flow Pipe Approximation

Algorithm:

- Over-approximate $Reach_{[0,r]}(S)$
- Propagate the reachable set using Φ_r

- For linear systems ($f(x,u)=Ax+Bu$):

$$\Phi_r(X) = \Phi_r X \oplus V \text{ where } \Phi_r = e^{rA}$$

- After initialization, only requires
  - linear transformations
  - Minkowski sums

Linear Systems

Linear systems: $\dot{x} = Ax + Bu$.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

If $u(t) \in \mathbb{R}^m$: $\mathrm{im}[B \; AB \; \cdots \; A^{n-1}B]$

If $\|u(t)\|_\infty \leq \mu$,

$$\left\| \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau \right\|_\infty \leq \int_0^t \left\|e^{(t-\tau)A}Bu(\tau)\right\|_\infty d\tau$$

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Linear Systems

$$\int_0^t \left\| e^{(t-\tau)A} B u(\tau) \right\|_\infty d\tau \leq \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty \| u(\tau) \|_\infty d\tau$$

$$\leq \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty \mu d\tau$$

$$= \mu \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty d\tau$$

$$= \mu \cdot \beta$$

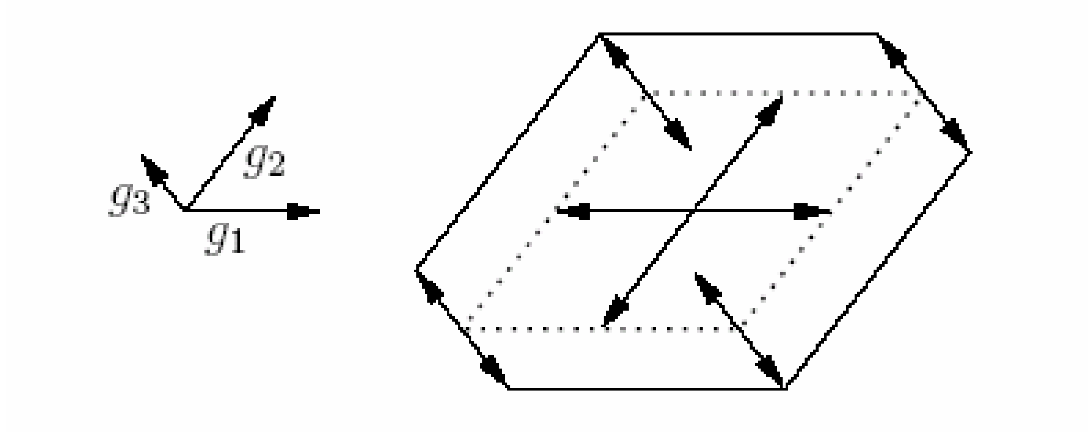
Thus $\int_0^t e^{A(t-\tau)} B u(\tau) d\tau$ is approximated with a rectangle.

What is a zonotope?

- Zonotope: Minkowski sum of a finite number of segments.

$$Z = \left\{ x \in R^n, x = c + \sum_{i=1}^{i=p} x_i g_i, -1 \le x_i \le 1 \right\}.$$

- c is the center of the zonotope, $\{g_1,\dots,g_p\}$ are the generators. The ratio p/n is the order of the zonotope.



Two dimensional zonotope with 3 generators

Some Properties of Zonotopes

- The encoding of a zonotope has a polynomial complexity with the dimension.

- The set of zonotopes is closed under linear transformation

$$Z = (c, < g_1, \ldots, g_p >), LZ = (Lc, < Lg_1, \ldots, Lg_p >).$$

- The set of zonotopes is closed under the Minkowski sum

$$Z_1 = (c_1, < g_1, \ldots, g_p >), Z_2 = (c_2, < h_1, \ldots, h_q >).$$

$$Z_1 \oplus Z_2 = (c_1 + c_2, < g_1, \ldots, g_p, h_1, \ldots, h_q >).$$

- Exactly what we need for our reachability algorithm

Reachability Algorithm with Zonotopes

APPROXIMATE $\text{Reach}_{[0,T]}(S)$

1. $r = \frac{T}{N}$
2. Obtain a zonotope Q_0 as an over approximation of $\text{Reach}_{[0,r]}(S)$.
3. $R_0 = Q_0$.
4. $Q_{i+1} = e^{Ar} Q_i \oplus \Box$.
5. $R_{i+1} = R_i \cup Q_{i+1}$.
6. $\text{Reach}_{[0,T]}(S) \approx R_N$

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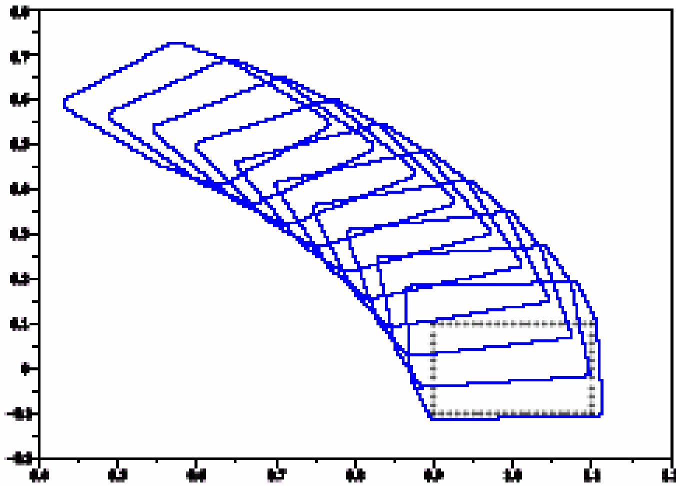
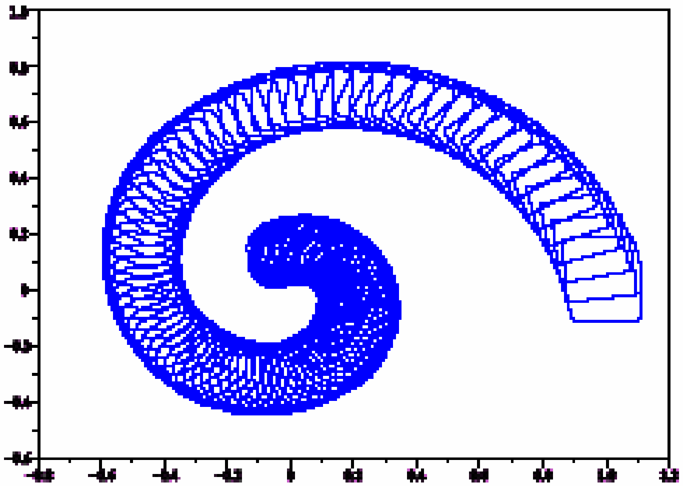
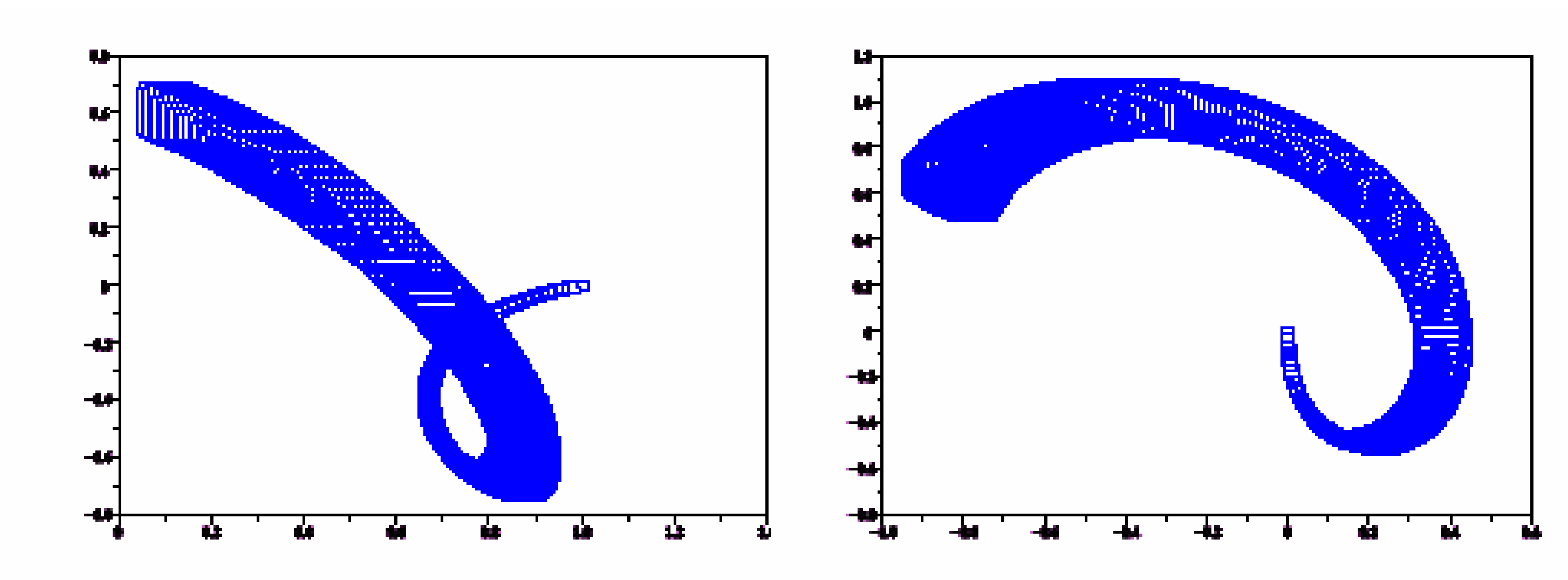
Two dimensional example





Reachable set on the interval [0,2],
100 iterations.

Five dimensional example



Projections of the reachable set on the interval [0,1], 200 iterations.

Flow Pipe Approximation

Advantages:

- Using a time step small enough, can approximate the reachable set at any desired accuracy.
- Suitable for the verification of large scale systems.

- Drawbacks:
  - Not so good for nonlinear systems.
  - Needs further work for hybrid systems

Ellipsoidal techniques

- Approximate the reachable set of

$$\dot{x} = A(t)x + B(t)u, 0 \leq t \leq T$$

with $u(t)$ contained in an ellipsoid

$$(u - q)^T Q(u - q) \leq 1$$

- q is the center of the ellipsoid, Q is a positive definite matrix.
- The reachable set is approximated from the inside and outside with ellipsoids.

Application to Hybrid Systems

- This method can be used for checking reachability within each location.
- To be applied to hybrid systems, we need to be able to detect intersection of the reachable set and the guard.
- Assuming that the guard is given by a hyperplane $d^T x = e$, the intersection of the reachable sets with the guard can be detected by detecting the intersection of Q_i and the guard.
- Note that intersection of a hyperplane and a zonotope is not necessarily a zonotope.