STOCHASTIC HYBRID SYSTEMS

Note Title 4/4/2006

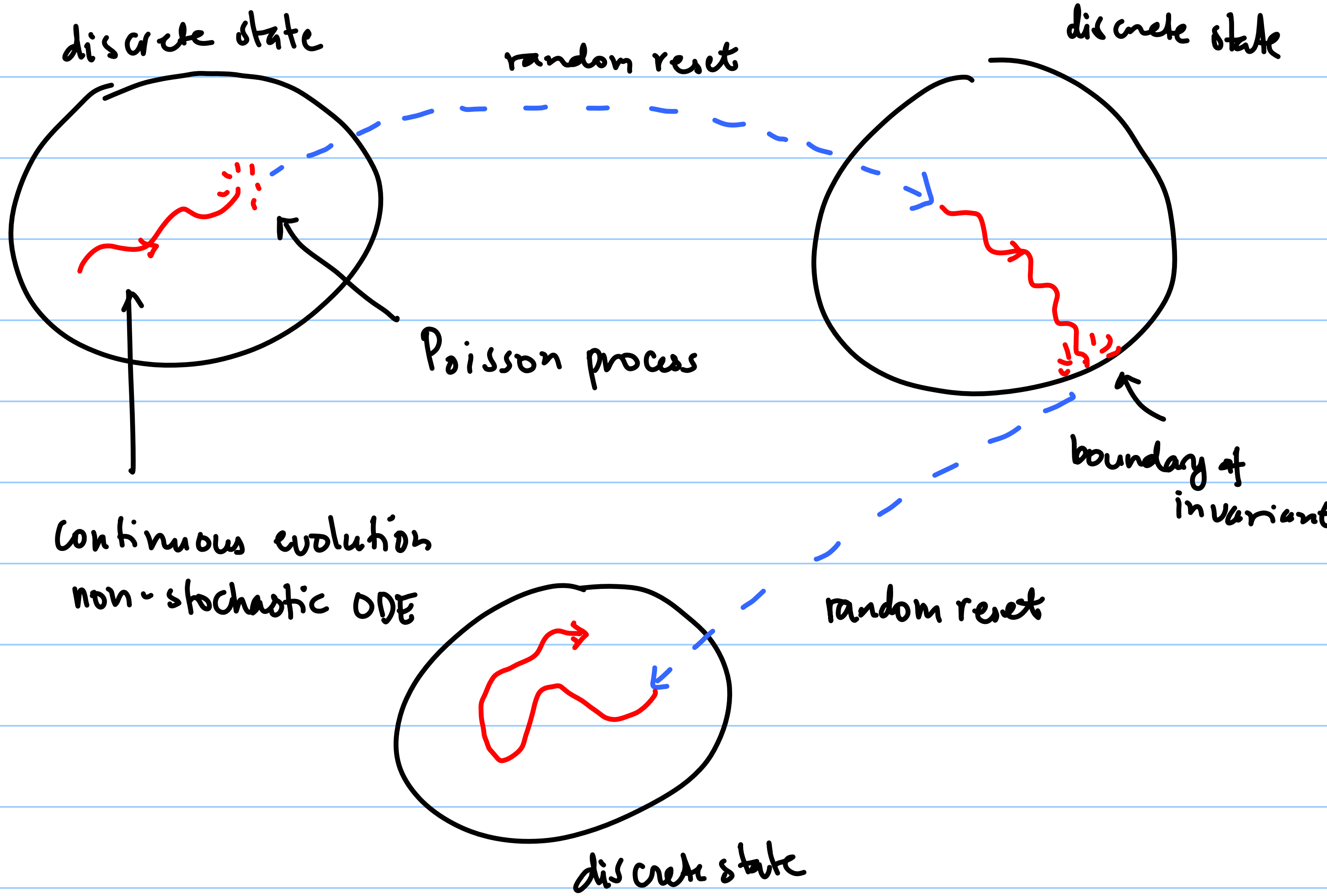
Piecewise Deterministic Markov Process (PDMP)

A PDMP is a hybrid stochastic process. The continuous dynamics is non-stochastic. Stochasticity arises when the system performs a transition.

A transition can be caused by:

- a Poisson process
- the continuous state hits the boundary of an invariant

When a transition occurs, the continuous state is reset according to a probabilistic distribution.


discrete state
random reset
discrete state
Poisson process
boundary of invariant
continuous evolution
non-stochastic ODE
random reset
discrete state

Formally:

A PDMP is: $H = (Q, d, X, f, \text{Init}, \lambda, R)$

Q is the set of discrete states/locations

$d : Q \to \mathbb{N}$ is the dimension of the continuous state space at each location.

$X \subset \mathbb{R}^d$ is the invariant of the location

We then define the hybrid state space as

$$D = \bigcup_{i \in Q} \{i\} \times X(i)$$

$f : D \to \mathbb{R}^{d(i)}$ is the vector field defining the continuous dynamics of location i

$$\dot{x} = f(x)$$

Init is the distribution of the initial state.

$\lambda : D \to \mathbb{R}_+$ is the rate of the Poisson process

R is the transition probability. For any hybrid state $d \in D$, $R(\cdot, d)$ is a probability measure in D, such that for any $\Gamma \subset D$,

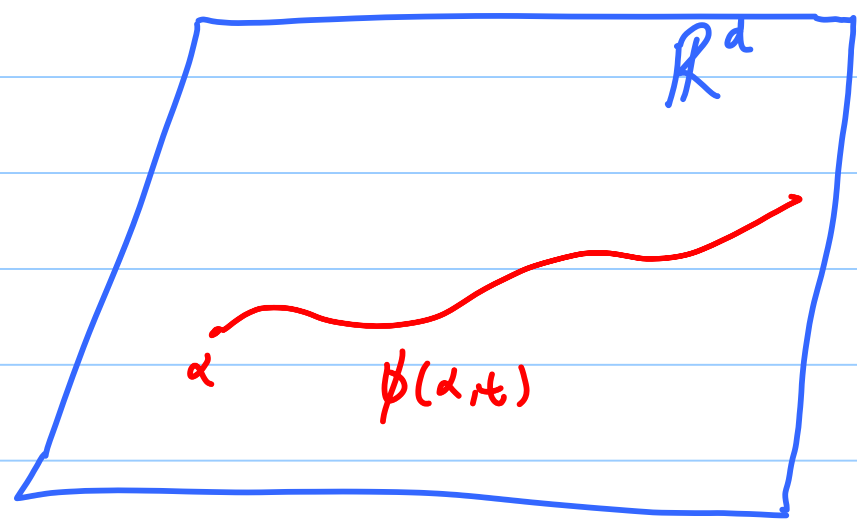
$$P(\text{the state is reset to } \Gamma \text{ from } d) = R(\Gamma, d)$$

Poisson process with variable rate:

$\dot{x} = f(x)$, and the rate λ is a function of x.

Assume that the invariant is $\mathbb{R}^d$.

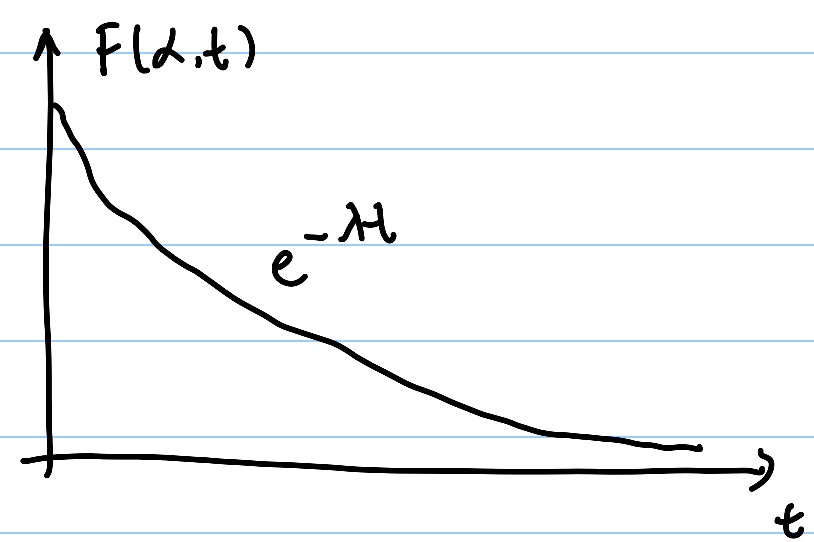
Define $\phi(\alpha, t)$ to be the solution of $\dot{x} = f(x)$, with initial condition $x(0) = \alpha$. Thus: $\dfrac{\partial \phi}{\partial t} = f(\phi)$.

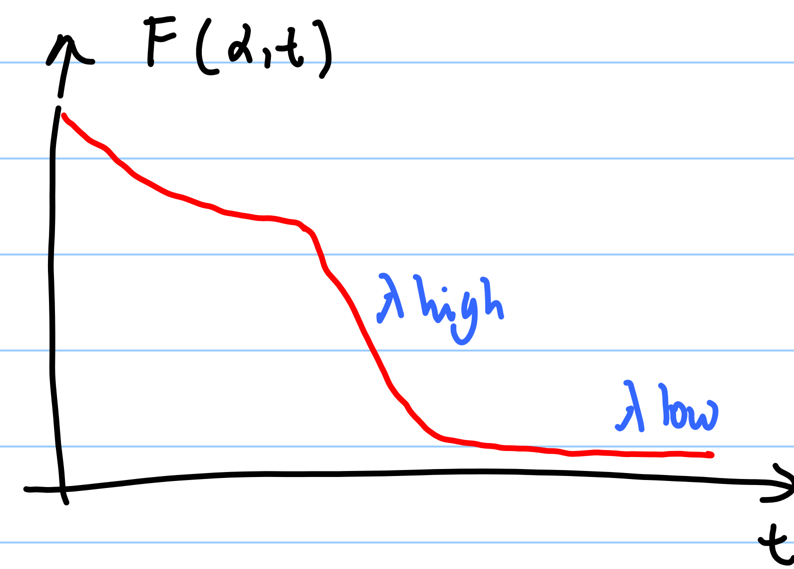


Define: $F(\alpha, t) = \exp\left(- \int_0^t \lambda(\phi(\alpha, \tau))\, d\tau \right)$

Interpretation: $F(\alpha, t)$ is the prob. that starting the state at α, the Poisson process has not generated any event until time t.

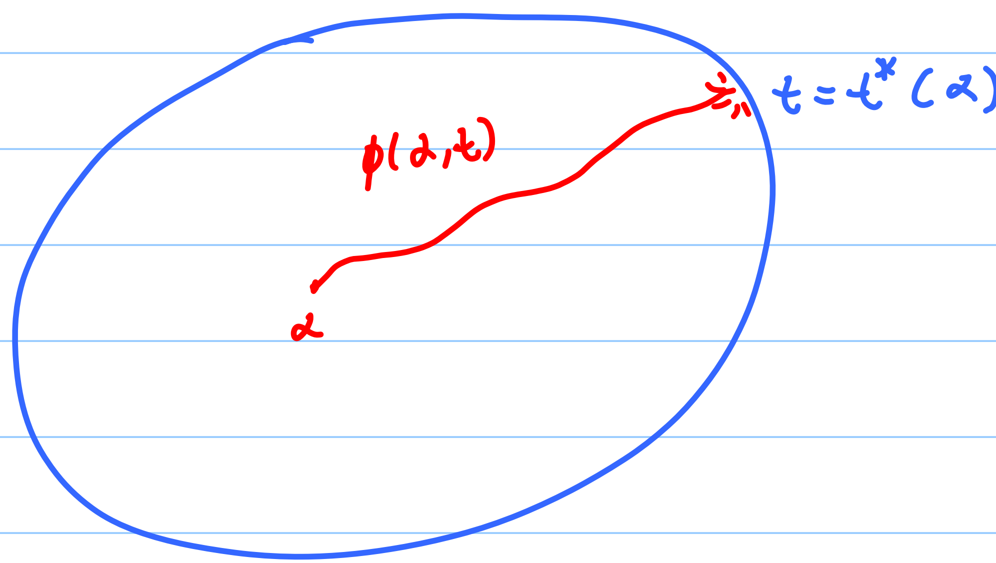
If λ is constant: $F(\alpha, t) = e^{-\lambda t}$





Generally, if the invariant is not unbounded:

Define $t^*(\alpha) = \inf_{\tau} \{ t > 0 \mid \phi(\alpha, \tau) \notin X \}$



To simulate: generate a random variable τ from the distribution given by $F(\alpha, \tau)$. If $\tau < t^*(\alpha)$, the Poisson process occurs at time τ and the state jumps, otherwise it jumps at time $t^*(\alpha)$.

Infinitesimal generator:

For any $\xi : D \to \mathbb{R}$, the generator of the stochastic process $\xi(d_t)$, where d_t is the stochastic process corresponding to hybrid state is given b

$$A\xi(d) = \lim_{t \downarrow 0} \frac{E(\xi(d_t) \mid d_0 = d) - \xi(d)}{t}$$

Dynkin's formula:

$$E(\xi(d_\tau) \mid d_0 = d) = \xi(d) + E\left[\int_0^\tau A\xi(d_t)\, dt \,\middle|\, d_0 = d \right]$$

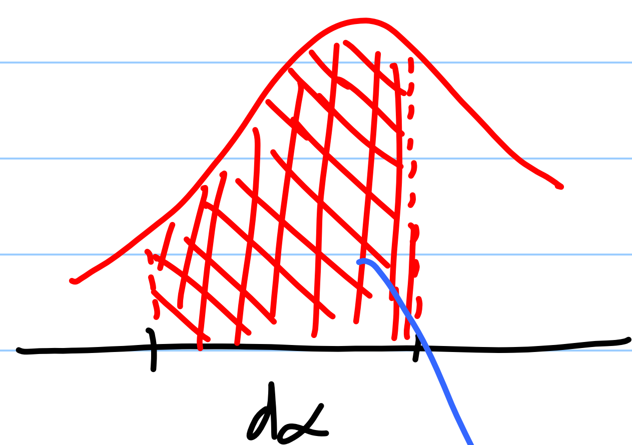
The infinitesimal generator of a PDMP is given by:

$$A\xi(d) = \frac{\partial \xi}{\partial x} \cdot f(d) + \lambda(d) \int_D R(d\alpha, d)\,(\xi(\alpha) - \xi(d))$$

infinitesimal measure

$$\int_D p(d,\alpha)\,(\xi(\alpha) - \xi(d))\,d\alpha$$

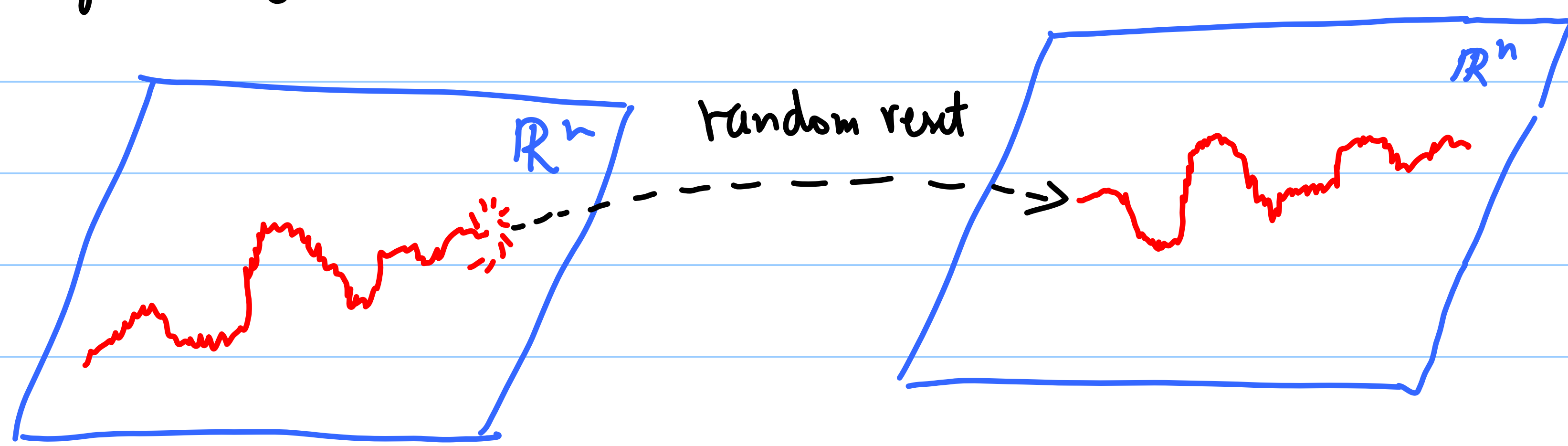
density



$$R(d\alpha, d) = \int p(d,\alpha)\,d\alpha$$

Switching Diffusion Process (SDP)

An SDP is a hybrid stochastic process, where the continuous state evolves with a stochastic differential equation, and transitions between discrete states are triggered by Poisson processes.
There is no invariant, so there is no jump associated with boundary hitting.



Formally, an SDP is

$$H = (Q, X, f, Init, \sigma, \lambda) \text{ where}$$

Q is the set of discrete states / locations

$X = \mathbb{R}^n$, the continuous state space

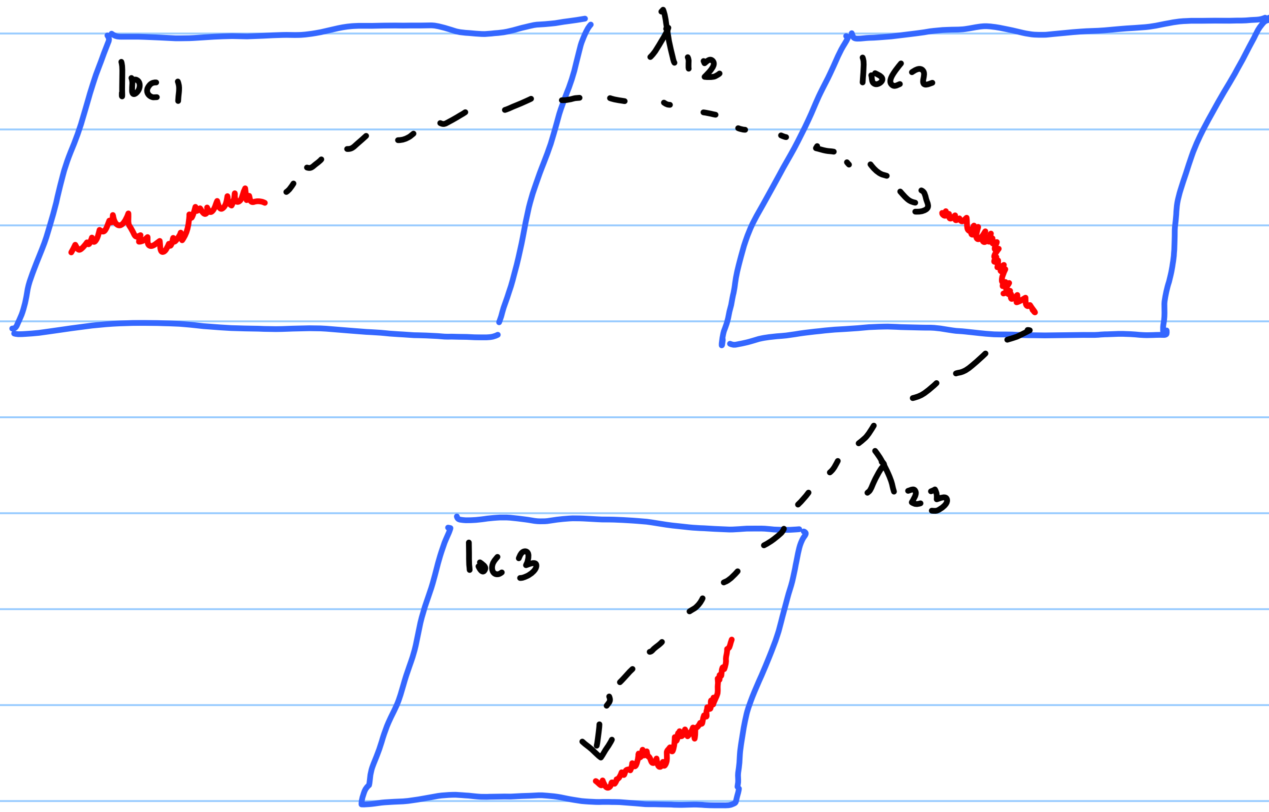
$f : Q \to \mathbb{R}^n$ a vector field that characterizes the continuous dynamics

$Init$: is the distribution of the initial state.

$\sigma : Q \times X \to \mathbb{R}^{n \times n}$

$\lambda_{ij} : X \to \mathbb{R}_+$ is the state dependent transition rate (Poisson process) from location i to location j, $i \neq j$

$$\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$$



Continuous states are not reset.

The stochastic differential equation in location q is given by:

$$dx_t = \underbrace{f(q, x_t)}_{\text{drift}}\, dt + \underbrace{\sigma(q, x_t)}_{\text{diffusion}}\, dW_t$$

W_t is a vector of n independent Brownian motion,

The infinitesimal generator of an SDP is given by:

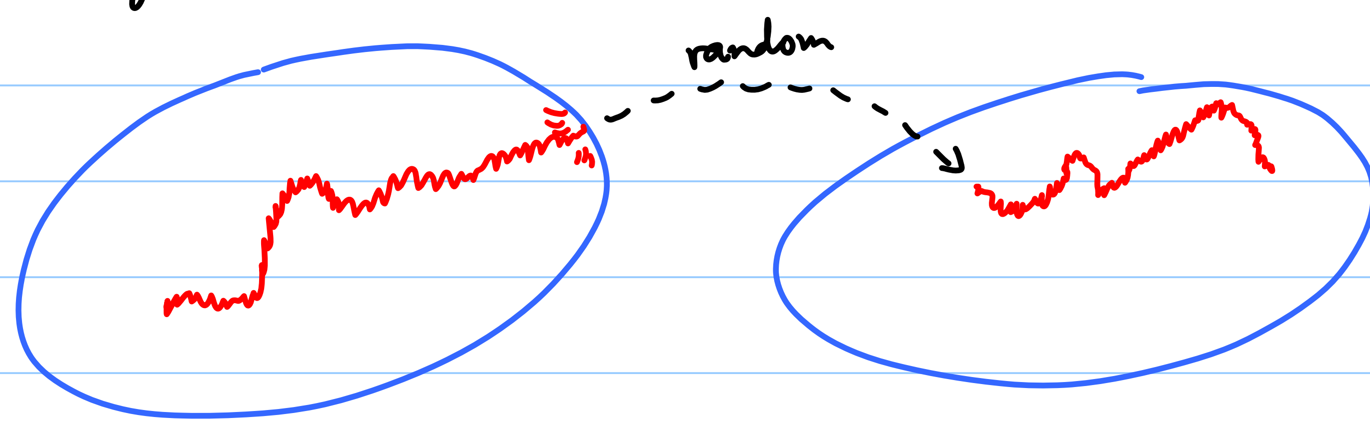
$d = (i, x)$

$$A\xi(d) = \frac{\partial \xi}{\partial x} \cdot f(d) + \frac{1}{2} \mathrm{tr}\left(\sigma(d)\sigma^T(d) \frac{\partial^2 \xi}{\partial x_i \partial x_j}\right)$$

$$+ \sum_{j \in Q} \lambda_{ij}(x)\left(\xi(j, x) - \xi(i, x)\right)$$

Stochastic Hybrid Systems (SHS)

In a stochastic hybrid system (SHS), the continuous state obeys a stochastic differential equation.
Transition between locations happen when the continuous trajectory hits the boundary of the invariant.
When a transition occurs, the continuous state is reset according to a probability distribution.



Formally, an SHS is:

$$H = (Q, X, \mathrm{Dom}, f, g, \mathrm{Init}, G, R), \text{ where}$$

Q is the set of locations

$X = \mathbb{R}^n$ is the continuous state space

$\mathrm{Dom}(q)$ is the invariant of location q

$f, g : Q \times X \to \mathbb{R}^n$ are vector fields

Init is the distribution of the initial state

$G : Q \times Q \to 2^X$ is the guard set

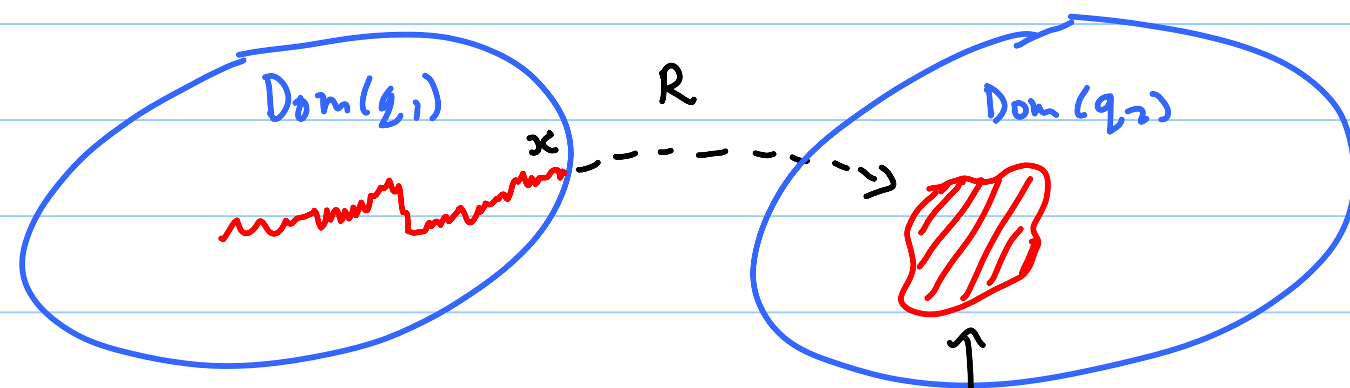
$$G(q_1, q_2) \subset \partial \mathrm{Dom}(q_1) \to \text{boundary}$$

$$G(q_1, q_2) \cap G(q_1, q_3) = \emptyset, \quad q_2 \neq q_3$$

$$\bigcup_{q_i \in Q} G(q_1, q_i) = \partial \mathrm{Dom}(q_1)$$

R is the reset distribution

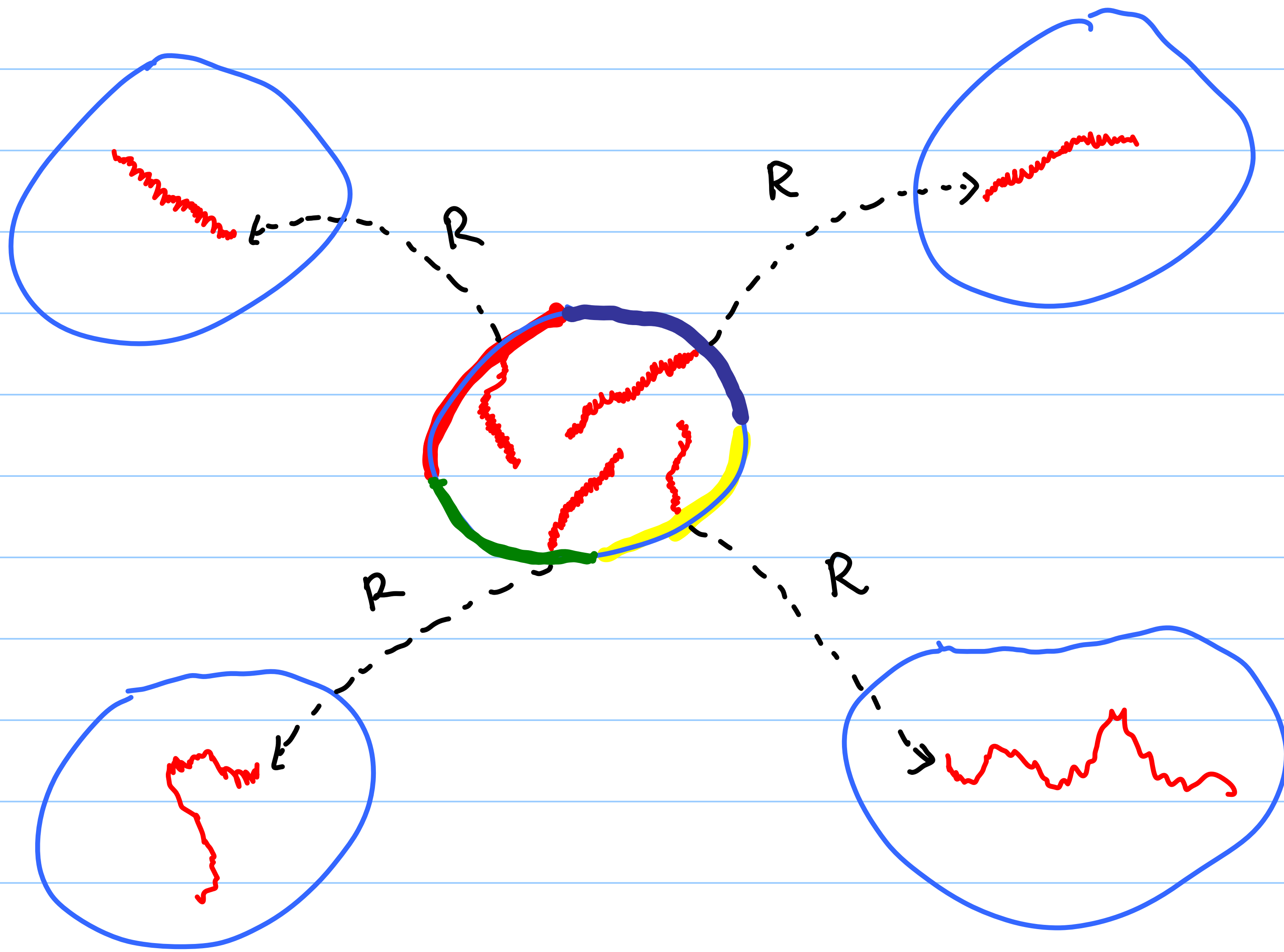
$R(q_1, q_2, x)$, $x \in G(q_1, q_2)$ is the probability measure in $\mathrm{Dom}(q_2)$.



the probability is given by $R(q_1, q_2, x)$

The continuous dynamics in each location is given by:

$$dx_t = f(q, x_t)\,dt + g(q, x_t)\,dW_t$$

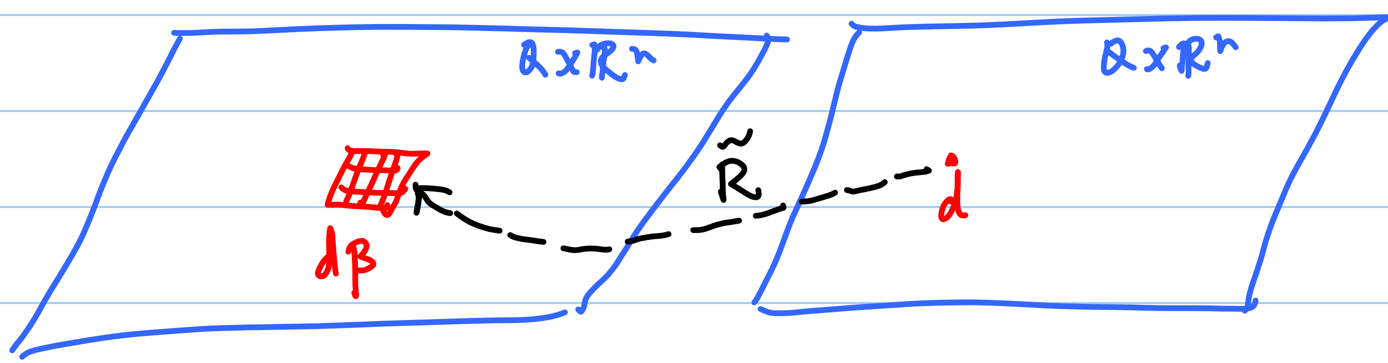

R
R
R
R

The infinitesimal generator of an SHS is given by:

$d = (q, x)$

$$L\xi(d) = \frac{\partial \xi}{\partial x} \cdot f(d) + \frac{1}{2} \operatorname{tr}\left(g(d)\, g^T(d) \frac{\partial^2 \xi}{\partial x_i \partial x_j} \right) + \int_{Q \times \mathbb{R}^n} \left(\xi(\beta) - \xi(d) \right) \tilde{R}(d\beta, d)$$

$\tilde{R}$ is the probability measure corresponding to the reset.



Summary :

	Diffusion	Poisson	Boundary
PDMP	✗	✓	✓
SDP	✓	✓	✗
SHS	✓	✗	✓

Polynomial Stochastic Hybrid Systems (PSHS)

A polynomial stochastic hybrid system has a hybrid state space, where the evolution of the continuous states follows an SDE, while transitions between locations are triggered by Poisson processes.

A PSHS is comparable to an SDP.

In PSHS, the vector fields, reset maps, and transition rates are polynomial functions.

Recall the infinitesimal generator of SDP:

The infinitesimal generator of an SDP is given by:

$d = (i, x)$

$$A\xi(d) = \frac{\partial \xi}{\partial x} \cdot f(d) + \frac{1}{2} \operatorname{tr}\left(\sigma(d)\sigma^T(d) \frac{\partial^2 \xi}{\partial x_i \partial x_j}\right)$$

$$+ \sum_{j \in Q} \lambda_{ij}(x) \left(\xi(j, x) - \xi(i, x) \right)$$

Assume that f, σ, λ are polynomials.

If ξ is also a polynomial, $A\xi$ is a polynomial.

$$\frac{d}{dt} \underbrace{E[\xi(d_t)]}_{\text{polynomial}} = E\underbrace{[A\xi(d_t)]}_{\text{polynomial}}$$

Thus if we define ξ as the infinite vector of monomials, the time evolution of $E(\xi(d_t))$ is linear infinite dimensional diff. equation.

Infinite dimensionality makes computation impossible, however, there are techniques that can be used to truncate the higher order moments (monomials)

The idea: the higher order moments are approximated by a (possibly nonlinear) function of the lower order ones.